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COLLEGIATE MATHEMATICS

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THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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MENASHA, WIS., AND CHICAGO, ILL.

OUR RETIRING SECRETARY-TREASURER

Twenty-seven years have now passed since the Mathematical Association of America was founded. During this time its membership has increased to considerably more than twice the eleven hundred charter members; meetings are held in twenty-two different sections throughout the country; and an ambitious program of periodical and book publication, and of sponsorship of other periodicals and of prize competitions, has come to fruition. Throughout this period of increasing usefulness the now retiring Secretary-Treasurer, WILLIAM DEWEESE CAIRNS has been an outstanding leader in the determination of policy and in the conduct of all of the Association's affairs. In the extent and value of his services to the Association, he is rivalled only by Herbert Ellsworth Slaught (1861–1937), also an enthusiastic and constant worker for the interests of the Association from the time of its founding. To these two men far more than to any others the Association owes its success and its present strong position.

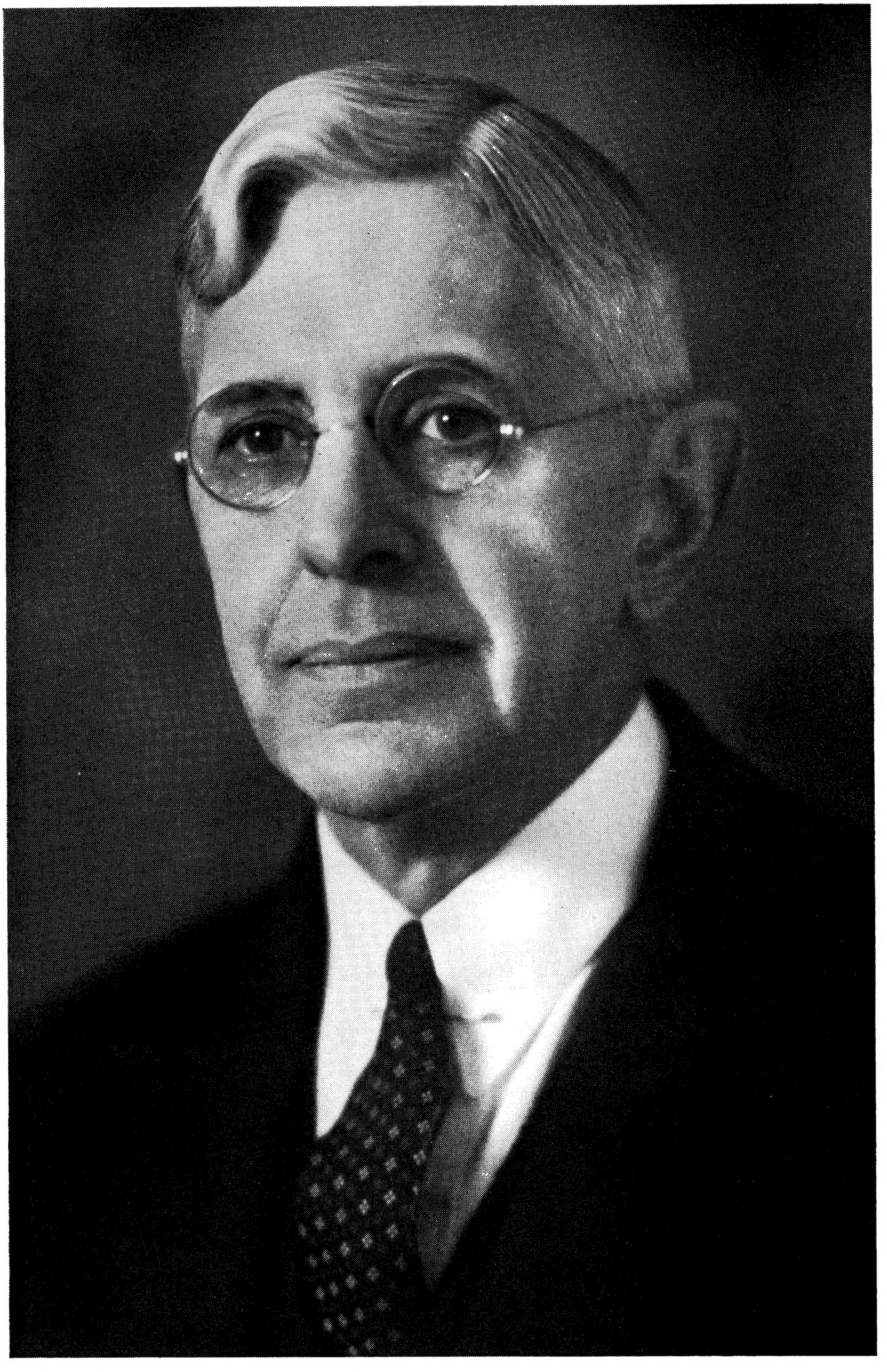
It must be the sincere hope of all who have the best interests of the Association at heart that Professor Cairns may long continue to contribute, by his influence and from the fullness of his wisdom, to her interests. For many decades to come his high achievement, his constructive leadership, and his great devotion will be remembered, and will serve as an inspiration to younger men who desire to advance the cause of mathematics in America.

EARL RAYMOND HEDRICK

November 1942

Since these words were written Professor Cairns has been elected to the presidency of the Association. It is a fortunate circumstance for us that he will guide the destinies of our organization during the difficult days that lie ahead.

L.R.F.



WILLIAM DEWEESE CAIRNS

WHAT IS DIMENSION?

KARL MENGER, University of Notre Dame

1. Solids, surfaces, and lines. Strictly speaking, all material objects are 3-dimensional. Yet, only such objects as a metal sphere, a wooden block, or a rock are considered to be typical representatives of 3-dimensional entities (solids). A piece of sheet-iron, paper, and a membrane approach what we mean when we speak of 2-dimensional objects (surfaces). Wire, threads, and streaks of chalk represent our idea of 1-dimensional entities (lines).

What is the difference between objects of different dimensions? Originally, mathematicians believed it to be a difference in quantity, in the sense that a surface contains more points than a line and less points than a solid. Now primarily the words "more," "less," and "equally many" are restricted to finite sets while surfaces, as well as lines and solids, contain infinitely many points. But Georg Cantor extended their use to all sets. We say that two sets—finite or infinite—contain equally many elements if we can establish a one-to-one correspondence between their elements. Cantor found that two infinite totalities do not necessarily contain equally many elements. For instance, among geometrical objects a straight line segment contains more points than some dispersed infinite sets, *e.g.*, the set of all points on a straight line whose distances from a certain point are integers. However, a straight line segment, a square, and a cube do contain equally many points [1]. Since these objects are of different dimensions, it follows that dimension is not a quantitative property.

Later, geometers thought that the difference between a 1-dimensional and a higher-dimensional object lay in the fact that the former, but not the latter, can be traversed by a continuously moving point. Indeed, lines on a paper or a blackboard are drawn, *i.e.*, traversed by the point of a pencil or chalk. However, Peano found that a continuously moving point can traverse a square surface or a solid cube though nobody would call these objects 1-dimensional. On the other hand, 1-dimensional objects were found which cannot be traversed by a continuously moving point [2]. The fact that an object is the path of a point is interesting in itself, but has no bearing on the question of the dimension of the object [3].

When one-to-one, as well as continuous, mappings had proved to be inadequate bases for the definition of dimension, mathematicians attempted to characterize dimension of a totality T as the least number of real numbers required to describe topologically (in a one-to-one and bi-continuous way) the elements of T. Each point of our ordinary space can be topologically characterized by three, but not less than three, real numbers, *e.g.*, its Cartesian or spherical coordinates; each point of a simple surface by two, but not less than two, real numbers, *e.g.*, the points of a sphere by longitude and latitude; each point of a simple line by one number. Thus, by the last definition, our space is 3-dimensional, simple surfaces are 2-dimensional, simple lines are 1-dimensional. Similarly, each color sensation of a normal eye can be topologically characterized by

three, but not less than three, real numbers, *viz.*, the quantities of three standard colors whose mixture produces an identical sensation. Hence, the totality of color sensations of a normal eye is 3-dimensional while the corresponding totalities for a partially or totally color blind eye are but 2- and 1-dimensional, respectively. In the same way, a totality of all mixtures of four ingredients which cannot be obtained by mixing less than four of them is called four-dimensional. In fact, in this direction lies our only elementary analytical approach to the fourth dimension and higher-dimensional spaces.

Unfortunately, however, the last definition applies only to very simple spatial entities, *viz.*, to those which can be obtained by means of a very simple transformation from a straight segment, a square, or a cube. Such entities are called arcs, discs, and topological spheres. In our space and in the plane, arcs and discs form only a small part of the lines and surfaces studied by modern geometry. Even if we admit objects which are sums of a finite number of arcs and discs our domain is still very restricted. For instance, the line mentioned above, which cannot be traversed by a continuously moving point [2], does not belong to this domain since it is not a sum of a finite number of arcs. In fact, it is the sum of infinitely many arcs, but all sets which are sums of infinitely many arcs cannot possibly be called 1-dimensional since the square and the cube are sums of infinitely many straight segments. [4].

To formulate the intuitive difference between lines, surfaces, and solids one can devise a simple experiment whose outcome depends upon the dimension of the object to which it is applied [5]. We cut out from the object a piece surrounding a given point. If the object is a solid we need a saw to accomplish this, and the cutting is along surfaces. If the object is a surface a pair of scissors suffices, and the cuts are along curves. If we deal with a curve we may use a pair of pliers and have to pinch the object in dispersed points. Finally, in a dispersed object no tool is required to perform our experiment, since nothing needs to be dissected. This characterization of dimension leads from n-dimensional to $(n-1)$-dimensional objects. It ends with dispersed sets, naturally called 0-dimensional, and, beyond these, with "nothing," in set theory called the "vacuous set." It is, therefore, convenient to consider the latter as -1-dimensional.

2. The definition of dimension. To make this idea precise we need only two simple auxiliary concepts: neighborhood and boundary. In our space we call a set N a *neighborhood* if each point of N is center of a sphere (though perhaps a very small sphere) all of whose points belong to N. The interior of a cube is a neighborhood, whereas a cube with its faces is not. For even the smallest sphere about a point of a face contains points not belonging to the cube. Nor is a plane a neighborhood in our space. For each sphere about each point of a plane contains points not belonging to the plane. The *boundary* of a neighborhood N is the set of all points which do not belong to N but are centers of arbitrarily small spheres which contain some points of N. For the interior of the cube the boundary obviously consists just of the six faces.

In terms of these concepts the result of our recursive dimension experiment can be explained as follows: A set S of points of our space is *at most n-dimensional* if each point of S lies in arbitrarily small neighborhoods whose boundaries have at most $(n-1)$-dimensional intersections with S. The set S is *n-dimensional* if it is at most n-dimensional but not at most $(n-1)$-dimensional. That S is not at most $(n-1)$-dimensional means that S contains at least one point at which S is at least n-dimensional, that is to say, a point which does not lie in arbitrarily small neighborhoods whose boundaries have at most $(n-2)$-dimensional intersections with S; the boundaries of all sufficiently small neighborhoods of such a point have at least $(n-1)$-dimensional intersections with S. The vacuous set, called -1-dimensional, is the starting point of the recursive definition [6].

By this definition, a set S is 0-dimensional if it is not vacuous, and each point of S lies in arbitrarily small neighborhoods whose boundaries have -1-dimensional, *i.e.*, vacuous, intersections with S—in other words, no points in common with S. A set S is 1-dimensional if it is not 0-dimensional and each point lies in arbitrarily small neighborhoods whose boundaries have at most 0-dimensional intersections with S. But it should be clearly understood that a point of a 1-dimensional set S may also be contained in arbitrarily small neighborhoods whose boundaries have more than 0-dimensional intersections with S. For instance, each point of a straight line S is contained in arbitrarily small neighborhoods whose boundaries contain whole pieces of S. Such neighborhoods can be formed by adding two cubes of different size, one of which has a face passing through S.

Furthermore, it should be clear that we cannot always expect to find *simple* neighborhoods of a point of an n-dimensional set whose boundaries have at most $(n-1)$-dimensional intersections with S. One of the most interesting examples in this respect arises from the study of the following four sets whose sum incidentally exhausts our space:

the set S_0 of all points which have three irrational coordinates,
the set S_1 of all points which have one rational and two irrational coordinates,
the set S_2, of all points which have two rational and one irrational coordinate,
the set S_3 of all points which have three rational coordinates.

If a, b, c are any three rational constants, then the planes $x=a$, $y=b$, $z=c$ do not contain any points of S_0, and the planes $ax+by+cz=1$ do not contain any points of S_2. If α, β, γ are any three irrational constants, then the planes $x=\alpha$, $y=\beta$, $z=\gamma$ do not contain any points of S_3. Now, for $i=0, 2, 3$, each point of S_i is contained in arbitrarily small *cubes* whose faces are part of such planes which have no point in common with S_i. Hence, S_0, S_2, S_3 are 0-dimensional. So is S_1 but the proof of this fact is much more difficult [7]. For not only each plane meets S_1, but as Schreier noticed, each surface of the form $z=f(x, y)$ where f is a continuous function, has points in common with S_1 and the same is true for each surface $y=f(x, z)$ and $x=f(y, z)$. In fact, only recently S. G. Reed, Jr. and the author constructed [8] a neighborhood whose necessarily complicated boundary has no point in common with S_1.

Since dimension of the subsets of our space has been defined in terms of neighborhoods the definition is applicable to the subsets of all spaces in which neighborhoods are given. An example of such a space is the 4-dimensional euclidean space whose points are the quadruples of real numbers x, y, z, u and in which the sphere with radius r and center x_0, y_0, z_0, u_0 consists of the points x, y, z, u satisfying the inequality $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+(u-u_0)^2=r^2$. A set N is a neighborhood if each point of N is center of a sphere all of whose points belong to N.

3. Criteria for a satisfactory definition. Now let us examine the definition of dimension. Its objective is to make precise and to extend the ordinary usage of the words "1-dimensional," "2-dimensional," and "3-dimensional." *A good definition of a word must include all entities which are always denoted and must exclude all entities which are never denoted by the word.* For the word "1-dimensional" straight lines, ellipses, and lemniscates are objects of the former type; square surfaces, solid cubes, and finite sets of the latter type. *A good definition should extend the use of the word by dealing with objects not known or not dealt with in ordinary language. With regard to such entities, a definition cannot help being arbitrary.* In connection with the word "1-dimensional" consider the four sets S_i whose sum exhausts our space. A general definition of "1-dimensional" will imply for each of the sets S_0, S_0+S_1, $S_0+S_1+S_2$ whether or not it is 1-dimensional. Our definition implicitly assigns to these sets the dimensions 0, 1, and 2, respectively, which like each assignment is somewhat arbitrary since ordinary language does not assign to them any dimension. *A good definition must yield many consequences, in particular theorems which are aesthetically satisfactory by their generality and simplicity, and theorems connecting the defined concept with concepts of other theories.* It is these theorems which justify the unavoidable arbitrary element of the definition. Some of the theorems will extend statements which are true in the restricted domain of ordinary language to the extended domain of the definition. Other theorems will exhibit interesting exceptions or even correct erroneous habits of thinking.

The definition outlined in this paper has yielded an extensive dimension theory which, since its foundation in the early twenties, has developed into one of the central branches of topology. Since even an enumeration of the main results would surpass the limits of this paper we shall confine ourselves to a few illustrations of the general criteria of the previous paragraph. An example of the numerous statements extending to all sets a proposition known to hold for the simple objects of ordinary language, is the theorem [5] that an n-dimensional set S contains infinitely many points at which S is n-dimensional, and that these points form a set S' which is at least $(n-1)$-dimensional. Under certain conditions we can say that S' is n-dimensional. However, there are rather unexpected exceptions in which S' is only $(n-1)$-dimensional. One of the facts which justify our definition of 0-dimensionality is the simple and beautiful general theorem that each n-dimensional set is the sum of $n+1$ but not less than $n+1$ 0-dimen-

sional sets. If we had assigned to the sets S_0, S_0+S_1, $S_0+S_1+S_2$ other dimensions than we did, it would have been at the expense of a simple systematic theory.

4. Five properties of dimension. In concluding, I shall select five of the theorems of dimension theory which, as we shall see, are of a particular importance:

I. *The euclidean n-space is n-dimensional.* (This theorem is due to Brouwer.) The cases $n=1, 2, 3$ of this theorem show, in particular, that the definition of 1-dimensionality excludes square surfaces and solid cubes which ordinary language always excludes and which older definitions failed to exclude.

II. *The topological image of an n-dimensional set is n-dimensional.* In conjunction with theorem I this simple theorem shows that the concepts of 1-dimensional and 2-dimensional sets include arcs and discs which are always called 1-dimensional and 2-dimensional, respectively.

III. *Each part of an n-dimensional set is at most n-dimensional.* Natural and simple as this theorem is it does not hold for some other definitions of dimension [9].

IV. *A set S cannot be split into denumerably many* [4] *closed* [10] *summands each of which is of smaller dimension than S.* (This so-called sum-theorem which occupies a central role in dimension theory, as well as the simple theorems II and III are due to Urysohn and the author.)

V. *Each n-dimensional set can be topologically transformed into a subset of a compact* [10] *n-dimensional set.* (This theorem is due to Hurewicz.)

5. Further aspects of the problem. What is dimension? Have we answered this question? In one sense, we have. We have explained which sets are 1-dimensional, which are 2-dimensional, etc. In fact, with each subset of our space and with each subset of much more general spaces we have associated an integer, the dimension of the set. This is also expressed by saying that dimension is a set function. However, there are many other set functions. With each set in our space we may, for example, associate the number of pieces of which it consists, or its measure (in some sense). In this connection the question "What is dimension?" may be interpreted in the following sense: "Among the many set functions, by which properties is dimension characterized?"

So far this question has only been answered for the plane [11]. There dimension is characterized by the properties described in theorems I to V, that is to say: In the plane, dimension is the only set function with the following properties:

1) It assumes the values 2, 1, 0, -1 for the square, the straight line segment, the single point, and the vacuous set, respectively;

2) It assumes the same value for any two sets which can be obtained from each other by a topological transformation;

3) It never has a greater value for the part than for the whole;

4) No set can be split into denumerably many closed sets of smaller function value;

5) Each set can be topologically transformed into a part of a compact set of equal function value.

In the plane, therefore, this is another answer to the question, "What is dimension?"

Footnotes

1. Also some dispersed sets and a straight line segment contain equally many points, *e.g.*, the set of all points on a line whose distance from a certain point is irrational, or Cantor's so-called *discontinuum*.

2. *E.g.*, the so-called *sinusoid* consisting of the points (x, y) of the plane for which either $0<x\leqq 1$ and $y=\sin 1/x$ or $x=0$ and $-1\leqq y\leqq 1$.

3. In this connection we may mention a comparatively recent result of the theory of curves. If a set S as well as each subcontinuum of S can be traversed by a continuously moving point, then S is 1-dimensional in the sense defined in this paper. The converse of this theorem is not true.

4. One might think that 1-dimensional are the sets which are the sum of *denumerably many* arcs, *i.e.*, of as many arcs as there are integers. But this definition would still be too narrow while the class of entities which are sums of *non*-denumerably many arcs contains the square and the cube and thus is too wide.

5. See the author's book "Dimensionstheorie" 1928.

6. The history of this definition and the ensuing theory is outlined in the beautiful exposition of Hurewicz and Wallman, Dimension Theory, Princeton University Press, 1941.

7. See Hurewicz and Wallman, p. 19.

8. To be published in Issue 5 of the Reports of a Mathematical Colloquium, University of Notre Dame publication.

9. See the Appendix to Hurewicz and Wallman, Dimension Theory.

10. A set C is *closed* if its complement is a neighborhood, and hence C contains all cluster points of C, *i.e.*, all points of which each neighborhood has infinitely many points in common with C. A set C is called *compact* if for each infinite subset of C there exists a cluster point in C. It should be noted that theorem IV, would not hold if we omitted the word closed: Our 3-dimensional space can be split into a finite number of sets of smaller dimensions which are not closed, *e.g.*, into the four 0-dimensional sets S_0, S_1, S_2, S_3. Nor would theorem IV hold if we admitted splitting in more than denumerably many closed sets. Our 3-dimensional space is sum of infinitely many (but not denumerably many) closed 0-dimensional sets *e.g.*, of sets each of which consists of exactly one point.

11. Monatshefte f. Mathematik u. Physik, 36, 1929, p. 193.

DIVISORS OF ZERO IN POLYNOMIAL RINGS

ALEXANDRA FORSYTHE, Vassar College

This is an alternative proof to Theorem 2 of an article by N. H. McCoy* on divisors of zero. The definitions are those of Professor McCoy unless otherwise stated.

We postulate a commutative ring R and let $R[x]$ be the commutative extension ring obtained through the adjunction of an indeterminate x to the ring R. We shall say that an element $f(x)$ is a *divisor of zero* in $R[x]$ if there exists a non-zero element $g(x)$ of $R[x]$ such that the product $g(x)f(x)=0$.

* McCoy, N. H., Remarks on divisors of zero, this MONTHLY, vol. 49, May 1942, pp. 286–295.

THEOREM. (McCoy) *If $f(x)$ is a divisor of zero in $R[x]$, then there exists a non-zero element c of R such that $c \cdot f(x) = 0$.*

By definition of divisor of zero, there exists in $R[x]$ a non-zero element $g(x)$ of some degree N such that $g(x)f(x)$ equals zero. Now by the zero degree elements, that is those of the form ax^0, in $R[x]$ we mean exactly the elements of the ring R. Hence McCoy's theorem will be proved if we prove:

THEOREM A. *For every such element $g(x)$ (defined just above) of degree N, $N>0$, there exists another element $g'(x)$ in $R[x]$ satisfying exactly the same conditions as $g(x)$ and of degree less than or equal to $N-1$.*

Proof. We may suppose $f(x)=\sum_{i=0}^{i=M} a_i x^i$ and $g(x)=\sum_{j=0}^{j=N} b_j x^j$, where the a_i and b_j are elements of R such that a_M and b_N are not zero. By hypothesis $g(x)f(x)=0$. Now if $b_0=0$ we have

$$g(x)f(x) = \sum_{j=0}^{j=N} b_j x^j f(x) = x \sum_{j=0}^{j=N-1} b_{j+1} x^j f(x) = 0.$$

In this case of course $g'(x)$ may be taken to be $\sum_{j=0}^{j=N-1} b_{j+1} x^j$. This expression may easily be seen to be different from zero since its coefficients include b_N. Its degree is $N-1$ and $g'(x)f(x)=0$.

Otherwise $b_0 \neq 0$ and, by equating coefficients in the expression $g(x)f(x)=0$, one has $b_0 a_0 = 0$. Since the original ring R is commutative $a_0 b_0$ is also zero. Consider the product $a_0 g(x)$.

$$a_0 g(x)f(x) = a_0 \sum_{j=0}^{j=N} b_j x^j f(x) = a_0 x \sum_{j=0}^{j=N-1} b_{j+1} x^j f(x) = 0.$$

If $a_0 g(x)$ is not equal to zero, the product $a_0 \sum_{j=0}^{j=N-1} b_{j+1} x^j$ can easily be seen to be different from zero and may be taken as $g'(x)$.

If $a_0 g(x)=0$, look at $a_i g(x)$ $(0 \leqq i < M)$. If all these products are zero, b_0 satisfies the conditions of $g'(x)$ for, since $a_i g(x)=0$ $(0 \leqq i < M)$, it follows that $a_i b_0 = 0$ $(0 \leqq i < M)$. Then this together with the fact that $g(x)f(x)=0$, gives us that $b_0 a_M = a_M b_0 = 0$. Hence $b_0 f(x) = 0$.

If $a_0 g(x)=0$ and $a_i g(x)=0$ $(0 \leqq i < p' < M)$ but $a_{p'} g(x) \neq 0$, it follows in the same way as before that $a_i b_0 = 0 (0 \leqq i \leqq p')$. Then one has

$$a_{p'} g(x)f(x) = a_{p'} \sum_{j=0}^{j=N} b_j x^j f(x) = a_{p'} x \sum_{j=0}^{j=N-1} b_{j+1} x^j f(x) = 0.$$

We may take $a_{p'} \sum_{j=0}^{j=N-1} b_{j+1} x^j$ as $g'(x)$. Again this element is non-zero, it has degree $N-1$ and it satisfies the condition $g'(x)f(x)=0$. This completes the proof of Theorem A and hence of McCoy's theorem.

THE DERIVATIVES OF COMPOSITE FUNCTIONS

ARNOLD DRESDEN, Swarthmore College

1. Introduction. In Vol. I of the *Quarterly Journal of Mathematics*, Faa de Bruno gave an explicit formula for the nth derivative of a composite function.* It is the purpose of the present note to obtain this formula, in a slightly modified form which has proved to be convenient in certain applications and the proof of which presents some interest.

2. Notation and result. Let $u=f(y)$, $y=g(x)$ and $u=f[g(x)]=F(x)$, represent single-valued and indefinitely differentiable functions of their arguments when $a\leqq x\leqq b$. Then we shall prove that

$$F^{(n)}(x) = \sum_{0}^{n-1} A_k^n f^{(n-k)}[g(x)], \tag{1}$$

where

$$A_k^n = \sum_{Rn,k} c^{nk}_{p_1\cdots p_{k+1}} (g')^{p_1}(g'')^{p_2}\cdots (g^{(k+1)})^{p_{k+1}}; \tag{2}$$

the range $R_{n,k}$, $k=0, 1, \cdots, n-1$, of the summation extends over all sets of non-negative integers $p_1, \cdots, p_{k+1}$ which satisfy the conditions

$$R_{n,k}:\ \begin{cases} p_1 + p_2 + \cdots + p_{k+1} = n - k, \\ p_1 + 2p_2 + \cdots + (k+1)p_{k+1} = n. \end{cases} \tag{3}$$

Finally, for any solution $p_1, \cdots, p_{n+1}$ of (3), for given n and k, the coefficients $c^{nk}_{p_1\cdots p_{k+1}}$ are integers represented by the formula

$$c^{nk}_{p_1\cdots p_{k+1}} = \frac{n!}{p_1!\cdots p_{k+1}!(2!)^{p_2}\cdots[(k+1)!]^{p_{k+1}}}. \tag{4}$$

For example, the determination of $F^{(5)}$ requires the calculation of A_0^5, $A_1^5, \cdots, A_4^5$. To illustrate the procedure we shall calculate A_k^n, $k=0, \cdots, 4$.

For A_0^n, the equations (3) reduced to $p_1=n$, so that

$$c_n^{n0} = \frac{n!}{n!} = 1 \quad \text{and} \quad A_0^n = (g')^n. \tag{5}$$

For A_1^n, we have the conditions $p_1+p_2=n-1$, $p_1+2p_2=n$, so that the range $R_{n,1}$ consists of the single set $p_1=n-2$, $p_2=1$, and

$$c^{n1}_{n-2,1} = \frac{n!}{(n-2)!2!} = \binom{n}{2} \quad \text{and} \quad A_1^n = \binom{n}{2}(g')^{n-2}g''. \tag{5a}$$

The coefficient A_2^n depends on the conditions $p_1+p_2+p_3=n-2$, $p_1+2p_2+3p_3=n$, whence we find $p_2+2p_3=2$. The range $R_{n,2}$ consists of the sets $p_1=n-3$,

* Compare also Goursat, Cours d'Analyse, I, 1910, p. 81.

$p_2=0$, $p_3=1$ and $p_1=n-4$, $p_2=2$, $p_3=0$, so that

$$\begin{aligned} A_2^n &= \frac{n!}{(n-3)!3!}(g')^{n-3}g''' + \frac{n!}{(n-2)!2!(2!)^2}(g')^{n-4}(g'')^2 \\ &= \binom{n}{3}(g')^{n-3}g''' + 3\binom{n}{4}(g')^{n-4}(g'')^2. \end{aligned}$$

In similar manner, we find

$$A_3^n = \binom{n}{4}(g')^{n-4}g^{\mathrm{IV}} + 10\binom{n}{5}(g')^{n-5}g''g''' + 15\binom{n}{6}(g')^{n-6}(g'')^3,$$

$$\begin{aligned} A_4^n = \binom{n}{5}(g')^{n-5}g^{\mathrm{V}} &+ 10\binom{n}{6}(g')^{n-6}(g''')^2 + 15\binom{n}{6}(g')^{n-6}g''g^{\mathrm{IV}} \\ &+ 105\binom{n}{7}(g')^{n-7}(g'')^2g''' + 105\binom{n}{8}(g')^{n-8}(g'')^4. \end{aligned}$$

Thus we obtain

$$\begin{aligned} F^{(5)} &= f^{\mathrm{V}}(g')^5 + 10f^{\mathrm{IV}}(g')^3g'' + f'''[10(g')^2g''' + 15g'(g'')^2] \\ &\quad + f''[5g'g^{\mathrm{IV}} + 10g''g'''] + f'g^{\mathrm{V}}. \end{aligned}$$

This result is readily verified. We observe that $R_{n,n-1}$ consists of the single set $0, \cdots, 0, 1$, so that

$$c^{n,n-1}_{0,\cdots,0,1} = 1 \quad \text{and} \quad A^n_{n-1} = g^{(n)}. \tag{6}$$

3. Proof of formula. The proof of our formula (1) will be made by induction. Its validity for $n=1$ follows from (5); the example discussed above shows moreover that it holds for $n=5$.

By differentiating (1) with respect to x, we obtain

$$\begin{aligned} F^{(n+1)}(x) &= \sum_0^{n-1} A_k^n f^{(n-k+1)}(y)\cdot g' + \sum_0^{n-1}(A_k^n)'f^{(n-k)}(y) \\ &= A_0^n f^{(n+1)}g' + \sum_1^{n-1}(A_k^n g' + A_{k-1}^{n\prime})f^{(n-k+1)} + A_{n-1}^{n\prime}f'. \end{aligned}$$

Therefore, we have to show that

$$A_0^n g' = A_0^{n+1}, \quad A_{n-1}^{n\prime} = A_n^{n+1}, \quad \text{and} \tag{7}$$

$$A_k^n g' + A_{k-1}^{n\prime} = A_k^{n+1} \qquad \text{for } k = 1, \cdots, n-1. \tag{8}$$

The proof of (7) is contained in (5) and (6). In virtue of (2), the assertion in (8) is equivalent to

$$
(9)\qquad
\begin{aligned}
&\sum_{R_{n,k}} c^{n,k}_{p_1\cdots p_{k+1}}(g')^{p_1+1}(g'')^{p_2}\cdots(g^{(k+1)})^{p_{k+1}}\\
&\qquad+\sum_{R_{n,k-1}} c^{n,k-1}_{p_1\cdots p_k}\left[(g')^{p_1}\cdots(g^{(k)})^{p_k}\right]'\\
&\qquad=\sum_{R_{n+1,k}} c^{n+1,k}_{p_1\cdots p_{k+1}}(g')^{p_1}\cdots(g^{(k+1)})^{p_{k+1}},\qquad \text{for } k=1,\cdots,n-1.
\end{aligned}
$$

According to (3), the range $R_{n+1,k}$ occurring on the right of (9) is determined by the equations

$$
(10)\qquad
\begin{cases}
p_1+\cdots+p_{k+1}=n-k+1\\
p_1+2p_2+\cdots+(k+1)p_{k+1}=n+1.
\end{cases}
$$

In a set of non-negative integers which satisfy these conditions, we must have $p_{k+1}\leqq 1$, since it follows from equations (10) that $p_2+2p_3+\cdots+kp_{k+1}=k$; moreover if $p_{k+1}=1$, then $p_2=\cdots=p_k=0$, so that the only solution of (10) in which $p_{k+1}\neq 0$, is given by $p_1=n-k$, $p_2=\cdots=p_k=0$, $p_{k+1}=1$. Hence the only term on the right of (9), which actually involves $g^{(k+1)}$ is $c^{n+1,k}_{n-k,0,\cdots,0,1}(g')^{n-k}g^{(k+1)}$. The same considerations show that in $R_{n,k}$ the only term which involves $g^{(k+1)}$ is obtained for $p_1=n-k-1$, $p_2=\cdots=p_k=0$, $p_{k+1}=1$; hence in the first sum on the left, the only term which involves $g^{(k+1)}$ is $c^{n,k}_{n-k-1,0,\cdots,0,1}(g')^{n-k}g^{(k+1)}$. To obtain a term involving $g^{(k+1)}$ in the second sum on the left, we must have $p_k=1$ and hence $p_1=n-k$, $p_2=\cdots=p_{k-1}=0$; it follows that the only term of this type in the second sum is $c^{n,k-1}_{n-k,0,\cdots,0,1}(g')^{n-k}g^{(k+1)}$. Consequently, it remains to show that

$$
(11)\qquad c^{n,k}_{n-k-1,0,\cdots,0,1}+c^{n,k-1}_{n-k,0,\cdots,0,1}=c^{n+1,k}_{n-k,0,\cdots,0,1}\qquad \text{for } k=1,\cdots,n-1.
$$

In virtue of (4), this reduces to

$$
\frac{n!}{(n-k-1)!(k+1)!}+\frac{n!}{(n-k)!k!}=\frac{(n+1)!}{(n-k)!(k+1)!},
$$

which is the well-known relation $\binom{n}{k+1}+\binom{n}{k}=\binom{n+1}{k+1}$ between binomial coefficients. Thus we have shown that the terms on the two sides of (9) which actually contain $g^{(k+1)}$ have equal coefficients.

It remains to consider those terms in (9) in which $g^{(k+1)}$ does not occur, *i.e.* the sets in $R_{n,k}$ and in $R_{n+1,k}$ in which $p_{k+1}=0$, and the sets in $R_{n,k-1}$ in which $p_k=0$. Let $p_1,\cdots,p_k, 0$ be a set in the range $R_{n+1,k}$ so that the term $c^{n+1,k}_{p_1,\cdots,p_k,0}(g')^{p_1}\cdots(g^{(k)})^{p_k}$ occurs on the right of (9). Then $p_1-1, p_2,\cdots,p_k, 0$ will be a set in the range $R_{n,k}$ and hence $c^{n,k}_{p_1-1,p_2,\cdots,p_k,0}(g')^{p_1}\cdots(g^{(k)})^{p_k}$ will occur in the first sum on the left of (9). Furthermore, $p_1,\cdots,p_{i-1}, p_i+1, p_{i+1}-1, p_{i+2},\cdots,p_k$ will belong to the range $R_{n,k-1}$ for $i=1,\cdots,k-1$ and the terms $(p_i+1)c^{n,k-1}_{p_1,\cdots,p_{i-1},p_i+1,p_{i+1}-1,p_{i+2},\cdots,p_k}(g')^{p_1}\cdots(g^{(k)})^{p_k}$, $i=1,\cdots,k-1$ will occur in the second sum on the left, it being understood that any $c^{n,k}_{p_1,\cdots,p_{k+1}}$ in which one or more subscripts are negative, is to be replaced by zero. Con-

versely, to any set $p_1, \cdots, p_k, 0$ of $R_{n,k}$ correspond the set $p_1+1, p_2, \cdots, p_k, 0$ of $R_{n+1,k}$ and the sets $p_1, \cdots, p_{i-1}, p_i+1, p_{i+1}-1, p_{i+2}, \cdots, p_k$ of $R_{n,k-1}$. Therefore, it remains to show that, if $p_1, \cdots, p_k, 0$ is any set which belongs to $R_{n+1,k}$, then

$$(12) \qquad c^{n,k}_{p_1-1,p_2,\cdots,p_k,0} + \sum_{i=1}^{k-1} (p_i + 1) c^{n,k-1}_{p_1,\cdots,p_{i-1},p_i+1,p_{i+1}-1,p_{i+2},\cdots,p_k} = c^{n+1,k}_{p_1\cdots p_k,0}.$$

By means of (4), it is found that the left side of this relation is equal to

$$\frac{n!}{(p_1-1)!p_2!\cdots p_k!(2!)^{p_2}\cdots(k!)^{p_k}}$$
$$+ \sum_{i=1}^{k-1} \frac{(p_i+1)n!}{p_1!\cdots p_{i-1}!(p_i+1)!(p_{i+1}-1)!p_{i+2}!\cdots p_k!(2!)^{p_2}\cdots[(i-1)!]^{p_{i-2}} (i!)^{p_i+1}[(i+1)!]^{p_{i+1}-1}[(i+2)!]^{p_{i+2}}\cdots(k!)^{p_k}}$$
$$= \frac{n!}{p_1!\cdots p_k!(2!)^{p_2}\cdots(k!)^{p_k}}\left[p_1 + \sum_1^{k-1} \frac{p_{i+1}(i+1)!}{i!}\right]$$
$$= \frac{(n+1)!}{p_1!\cdots p_k!(2!)^{p_2}\cdots(k!)^{p_k}} = c^{n+1,k}_{p_1,\cdots,p_k,0}, \quad \text{by use of (10) and (4).}$$

This completes the proof of (9) and hence the proof of (1). That the coefficients $c^{n,k}_{p_1,\cdots,p_{k+1}}$ are integers follows from (11) and (12), in conjunction with (5), (5a) and (6), by means of induction.

We observe that the coefficient A^n_k, defined by (2) is equal to $n!/(n-k)!$ times the terms of weight n in the expansion of $(g'+g''/2!+\cdots+g^{(k+1)}/(k+1))^{(n-k)}$ the derivatives $g^{(i)}, i=1, \cdots, k+1$ being given weight i.

THE SPRING MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The eighteenth regular meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at the University of Pittsburgh, Pittsburgh, Pennsylvania, on April 25, 1942. The chairman of the Section, Dr. R. G. Sturm of the Aluminum Research Laboratories, presided at both morning and afternoon sessions.

The attendance was fifty-three, including the following twenty-nine members of the Association: O. F. H. Bert, J. O. Blumberg, R. C. Briant, Elizabeth F. Brown, A. M. Bryson, W. E. Buker, Helen Calkins, W. E. Cleland, H. B. Curry, L. L. Dines, J. D. Donaldson, H. L. Dorwart, F. A. Foraker, Beatrice L. Hagen, H. C. Hicks, M. L. Manning, L. T. Moston, David Moskovitz, J. H. Neelley, E. G. Olds, F. W. Owens, J. B. Rosenbach, E. M. Starr, R. G. Sturm, E. A. Saibel, J. S. Taylor, W. J. Wagner, E. A. Whitman, E. D. Wells.

The invitation of Pennsylvania State College, to hold the next meeting of the Section at State College, Pennsylvania, was accepted. This meeting is to be held on a Saturday during the month of October, 1942, the exact date to be determined by the Executive Committee.

After an address of welcome by Dean S. C. Crawford of the University of Pittsburgh, the following papers were read:

1. "A discussion of electric tabulating equipment as an aid to statistical work" by P. M. Walter, Jones and Laughlin Steel Corporation, introduced by the Secretary.

2. "Application of Lie theory to differential equations," by Dr. R. F. Clippinger, Carnegie Institute of Technology, introduced by Professor Dines.

3. "Continuity in structural members," by Marshall Holt, Aluminum Research Laboratories, introduced by Dr. Sturm.

4. "Exact values of the trigonometric ratios of certain angles," by Professor H. L. Dorwart, Washington and Jefferson College.

5. "Nomograms for tidal corrections of gravity," by T. A. Elkins, Gulf Research Laboratories, introduced by the Secretary.

6. "Clyde Shepherd Atchison, Ph.D., 1882–1941," by Professor O. F. H. Bert, Washington and Jefferson College.

7. "Symposium: The place of mathematics in the total war effort."

Abstracts of the papers follow:

1. Mr. Walter described a typical installation of electrically operated tabulating equipment consisting of a card punch, collator, horizontal sorter, reproducing punch, and a tabulating machine, and gave a number of examples of the use of this equipment in industrial work.

2. The purpose of Dr. Clippinger's paper was to illustrate for equations of first and second order the following theorem proved by Sophus Lie: If an rth order ordinary differential equation is invariant under the r infinitesimal transformations of an integrable group, its general solution may be found by r quadratures, provided the differential equation is independent of the group in a certain sense.

3. Mr. Holt considered a structural member subjected to direct stress and bending under assumptions which led to the differential equation

$$\frac{d^2y}{dx^2}+\frac{P_y}{EI}=-\frac{M_A}{EI}-\frac{Pe_0}{EI}\sin\frac{\pi x}{L}+\frac{M_B+M_A}{EIL}x.$$

He obtained expressions similar to the ordinary slope-deflection equations for the moments M_A and M_B by a procedure much the same as that outlined by Manderla in which the initial crookedness was not considered.

4. The paper of Professor Dorwart appeared in the Notes and Discussion section of this MONTHLY, May, 1942.

5. Mr. Elkins first defined tidal gravity and explained its value both in theoretical studies of the constitution of the earth and in geophysical prospecting. Nomograms devised by Dr. James B. Friauf for the solution of the astro-

nomical triangle were next discussed. Two types of nomograms for the routine computation of tidal gravity, which can be easily derived from the Friauf nomograms, were then explained.

6. Professor Bert's paper consisted of a biographical sketch of the late Clyde Shepherd Atchison, Ph.D., who had served for nearly thirty years as professor of mathematics at Washington and Jefferson College, together with an estimate of his character and personality as they had impressed themselves upon the author, who had been his colleague for many years. Professor Atchison was shown to have been a man of fine character, lofty ideals, deep convictions, and unswerving devotion. A tireless worker adhering to high standards, he attained great things in his service to his own students and to the cause of education in general. He was pictured as a person of such social graces as to commend him highly as good company, and his passing is widely mourned beyond the bounds of the scholarly and professional fields in which he moved. He was a charter member of the Mathematical Association of America.

7. Dr. Sturm conducted the symposium, and, in his opening remarks, suggested that in addition to the training of other mathematicians, it is possible for mathematicians to contribute to the total war effort by (1) direct service with the armed forces in making the computations involved in the use of ordnance, navigation, and troop movements; (2) the application of mathematical analysis to industrial problems in the manufacture of war materials; and (3) a distinct contribution to the general morale by exercise and stimulation of factual thinking such as is required in mathematics. This latter will be of extreme value in minimizing wishful thinking and susceptibility to propaganda.

Professor H. B. Curry of Pennsylvania State College, a member of the War Preparedness Committee of this Association and of the American Mathematical Society, then reviewed the history and set-up of this committee and urged the members to reread the articles of Chairman Morse and Professor Hart, published in the May and June-July numbers of this MONTHLY for 1941. Professor Curry stated that there had recently been a considerable decrease in the number of graduate students in mathematics so that it is anticipated that there will soon be an acute shortage of mathematicians and mathematics teachers. The shortage on the research level does not seem to be so serious. As the two outstanding needs, Professor Curry cited the training of large numbers of students in elementary mathematics, particularly in trigonometry, and the closing of the gap between pure and applied mathematicians.

Professor J. S. Taylor of the University of Pittsburgh referred to the letter of Admiral Nimitz in the March, 1942, issue of this MONTHLY, and commented on the use of mathematics by the Navy as a yardstick in selecting men for training. While describing courses now being given or in prospect at the University of Pittsburgh, he emphasized that while some college students and graduates have time to take courses on the regular semester basis, others have only one or two months available.

Professor E. G. Olds of Carnegie Institute of Technology stated that although

the present war is often called a mathematician's war, the mathematicians, as such, are not "in the saddle," but are in key positions only when they also happen to have had training in physics, engineering, meteorology, etc. He urged an open-mindedness among teachers of mathematics as regards short courses and emphasis on applications; and urged the formation of an organization to use the spare time of the mathematicians of the Pittsburgh district in industry this summer. Such a committee with Dr. Sturm as chairman was later authorized by the Section during the general discussion which followed these speakers.

H. L. DORWART, *Secretary*

THE ANNUAL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at St. Olaf College, Northfield, Minnesota, on Saturday, May 9, 1942. A morning session, held at 10:30 o'clock, was followed by luncheon and an afternoon session at 2:30 o'clock. Professor E. J. Camp of Macalester College presided at each session, being relieved by the regional governor, Professor Cornelius Gouwens, when giving his own paper.

Seventy persons attended the meeting, including the following thirty-three members of the Association: N. R. Amundson, R. W. Brink, L. E. Bush, W. H. Bussey, E. J. Camp, C. S. Carlson, S. Elizabeth Carlson, Sister M. Claudette, Paul Cramer, W. S. H. Crawford, Gladys Gibbens, C. H. Gingrich, Cornelius Gouwens, Clara L. Hancock, Dunham Jackson, J. S. Hickman, W. H. Kirchner, W. R. McEwen, W. D. Munro, J. D. Novak, Isaac Opatowski, F. J. Polansky, C. Grace Shover, Abraham Spitzbart, A. J. Strane, L. W. Swanson, F. J. Taylor, H. P. Thielman, Ella Thorp, H. L. Turrittin, A. L. Underhill, K. W. Wegner, Frantisek Wolf, and Sister Thomas à Kempis, institutional member representative.

At the business session officers were elected for the coming year as follows: Chairman: C. H. Gingrich, Carleton College; Secretary: A. L. Underhill, University of Minnesota; Executive Committee: G. C. Priester, University of Minnesota, H. P. Thielman, College of St. Thomas, K. W. Wegner, College of St. Catherine. Professors L. E. Bush and K. W. Wegner were chosen candidates for the position of regional governor for 1943 and 1944.

The following nine papers were presented:

1. "On the separation of parts of a natural boundary of an analytic function" by Dr. Frantisek Wolf, Macalester College.

2. "The discriminant function and its applications" by G. D. Kyle, University of Minnesota, introduced by Professor Underhill.

3. "Mathematical problems arising in the theory of airplane wings" by Professor Albert Gail, University of Minnesota, introduced by Professor Underhill.

4. "Legendre functions of the second kind and related functions" by Professor Dunham Jackson, University of Minnesota.

5. "On a certain functional equation" by Professor H. P. Thielman, College of St. Thomas.

6. "Some instruments for mathematics and physics" by R. M. Sutton, Lecturer in Physics, University of Minnesota, introduced by Professor Bussey.

7. "On the general solution of the nth order linear differential equation" by Professor E. J. Camp, Macalester College.

8. "Square roots from a table of cosines" by W. S. H. Crawford, University of Minnesota.

9. "On the Laplace transformation" by Dr. Isaac Opatowski, University of Minnesota.

Abstracts of these papers follow:

1. Given a function $u(z)$, harmonic in $|z|<1$ and an arc A of $|z|=1$. By B we denote the arc of $|z|=1$ complementary to A. Dr. Wolf gave a method to construct a function $v(z)$, harmonic in $|z|<1$ which is zero continuously on B and such that $u(z)-v(z)$ is zero continuously on A. Given $f(z)$, an analytic function in $|z|<1$, having at least part of $|z|=1$ as a natural boundary, then—applying the above result to the real part of it—he showed that for any A there is a $g(z)$ such that $g(z)$ is analytic on B and $f(z)-g(z)$ is analytic on A. Hence $g(z)$ takes over part of the natural boundary of $f(z)$ and its singular behavior, there, is exactly the same as that of $f(z)$. The proof uses some results from the theory of analytic functions and Parseval's theorem.

2. Mr. Kyle showed that a set of multiple measurements on samples taken from two or more classes may be used to develop a discriminant function, linear in the observations and having the property that better than any other linear function, it will discriminate between any specificed classes. Measurements, p in number, are made upon each object of each sample. Expressions for the difference between the means of the linear function in the specified classes and a quantity proportional to the variance of the linear function within classes are set up, and the ratio of the square of the former to the latter is maximized by variation of each of the unknown coefficients of the linear function independently. This operation leads to a set of p simultaneous linear equations in p unknowns, the unknowns being the desired coefficients of the discriminant function which is readily obtained. Such functions have numerous practical applications in the various branches of science and especially in confident selection, whether in the program of carrying on a war, in industry, or elsewhere.

3. Professor Gail demonstrated Frandtl's equation (for the spanwise circulation distribution of airplane wings) as an example of an integral equation that is commonly used for fundamental calculations in airplane design. He discussed the physical significance of that equation and called attention to its unexplored formal properties.

4. It was shown by Professor Jackson that solutions of the Legendre differential equation for arbitrary positive integral n can be calculated in a wholly elementary manner by means of recurrence relations when the equation has been solved for $n=0$. Alternative choices of a solution for $n=0$ lead to the

Legendre polynomials and to the Legendre functions of the second kind. A similar treatment is applicable to the differential equations of Hermite, Laguerre, Jacobi, and Bessel.

5. Professor Thielman considered the functional equation $f(x+1) = R(x)/F(x)$, where $R(x)$ is any rational function. He gave an explicit solution of this equation in terms of gamma functions, and stated that this was the only monotone and the only either convex or concave solution for large values of x. His derivations were based on, and were extensions of results of earlier articles. (F. John, *Acta Mathematica*, vol. 71, pp. 175–187; H. P. Thielman, *Bulletin of the American Mathematical Society*, vol. 47, pp. 118–120, 1941).

6. Six mechanisms and linkages were demonstrated by Mr. Sutton: (1) a device with spring roller for drawing families of confocal ellipses and hyperbolas, with provision for changing distance between foci; (2) a simple sextant with modified optical system for student use in trigonometry; (3) a combination of plumb-bob with protractor which, as an inclinometer, serves for blackboard use, and for measuring vertical angles in trigonometry and astronomy; (4) a compass using steel tape and rubber suction cup suitable for blackboard use and, when combined with the preceding instrument, for graphing polar coördinate figures; (5) a geometrical linkage based upon the trigonometric law of sines for demonstrating Snell's law of optical refraction; (6) a linkage for graphical solution and visualization of electric circuit problems, showing the interrelation of electromotive force, potential difference, internal resistance, external resistance, current, and power in both internal and external circuits.

7. The usual proof that every solution of the linear differential equation of nth order

$$\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots P_n(x)y = 0 \tag{1}$$

is expressible as a linear combination of n linearly independent solutions, depends on the general existence theorem for linear differential equations. Professor Camp, without making use of the existence theorems of solution of a linear differential equation, proved that if $y_1, y_2, \cdots y_n$ are n linearly independent solutions of equation (1), then every solution is expressible as a linear combination of them. Let $V(x)$ be any solution of equation (1) which is linearly independent of any $n-1$ of the functions $y_1, y_2, \cdots, y_n$. Then Wronski determinants formed from the n combinations of $V(x)$ with $n-1$ of the y_i give a set of linear equations in $V, V^1, \cdots, V^{(n-1)}$. On solving these equations for V, we prove that V is a linear combination of $y_1, \cdots, y_n$ with constant coefficients.

8. Mr. Crawford explained a method whereby the square root of any real number can easily be found by means of a little mental arithmetic and a table of cosines.

9. Some considerations and examples showing the usefulness of the Laplace transformation were given by Dr. Opatowski.

A. L. UNDERHILL, *Secretary*

AN ARITHMETIC FUNCTION ARISING FROM THE ϕ FUNCTION

HAROLD SHAPIRO, Princeton University

1. **Introduction.** In the following discussion we shall consider a rather curious concept which arises very naturally from the iteration of the Euler ϕ function. Since $\phi(x)$ is defined as the number of positive integers, not exceeding x, which are prime to x, it follows immediately that for $x>1$, $\phi(x)<x$. If we write $\phi^2(x)=\phi[\phi(x)]$, $\phi^3(x)=\phi[\phi^2(x)]$, $\cdots$, $\phi^n(x)=\phi[\phi^{n-1}(x)]$, then we have

$$\phi^n(x) < \phi^{n-1}(x), \tag{1}$$

for $\phi^{n-1}(x)>1$. It is well known that for $x=\prod p_i^{\alpha_i}$,

$$\phi(x) = \prod p_i^{\alpha_i-1}(p_i - 1), \tag{2}$$

from which it is easily seen that for $x>2$, $\phi(x)$ is even. Thus we see from (1) that by making n large enough we must always arrive at

$$\phi^n(x) = 2.* \tag{3}$$

When (3) holds we shall say that x is of class n, and write this as $C(x)=n$.

Now it is clear that every positive integer belongs to a uniquely defined class with the exception of 1 and 2. We define $C(1)=C(2)=0$.

It is our purpose, in what follows, to determine several properties of these classes, and properties of the numbers which belong to any one class. We will be able to determine bounds for the function $C(x)$, and to study the general structure of a class.

2. **The fundamental theorem.** First we seek the relationship between $C(xy)$, $C(x)$, and $C(y)$, for any two integers x and y. This is given by

THEOREM 1. *If either x or y is odd*

$$C(xy) = C(x) + C(y) \tag{4}$$

and if both x and y are even

$$C(xy) = C(x) + C(y) + 1. \tag{5}$$

It is obvious that to establish (4) and (5), in general, it is sufficient to prove them for the case when y is a prime. The proof of this will result from the following sequence of theorems.

LEMMA. *For two positive integers x and y, if $(x, y)=1$, then $\phi(xy)=\phi(x)\phi(y)$. On the other hand, if x is a multiple of y, then $\phi(xy)=y\phi(x)$.*

* The function $\phi^n(x)$ was considered by F. de Rocquigny who stated the formula

$$\phi^p(N^m) = \phi^{p-2}(N^{m-2})\phi^{p-1}[(N-1)^2],$$

N a prime, $m>2$, and $p>2 \cdots$ L. E. Dickson, "History of the Theory of Numbers," vol. 1, p. 134; F. de Rocquigny, Les Mondes-Revue Hebdom. des Sciences, 48, 1879, 327.

This relation is not true in general as may be seen by taking $N=5$, $m=3$, and $p=6$. Moreover, on the basis of the results of our discussion, we will show that with any prime $N>3$, and $m>2$, the formula does not hold for all p.

The truth of this Lemma follows immediately from (2). Note that for y a prime, the two possibilities considered in the lemma are the only two.

THEOREM 2. *For any odd integer* x, $C(2x)=C(x)$.

This is a consequence of the fact that for x odd, $\phi(2x)=\phi(x)$.

THEOREM 3. *For any even integer* x, $C(2x)=C(x)+1$.

Proof: Since x is even we have, by the lemma,

$$\phi(2x) = 2\phi(x),$$

and since $\phi(x)$ is even,

$$\phi^2(2x) = 2\phi^2(x).$$

In general, for any $m \leqq C(x)$

$$\phi^m(2x) = 2\phi^m(x),$$

and so taking $m=C(x)$, we have

$$\phi^m(2x) = 2\cdot 2 = 4,$$

$$\phi^{m+1}(2x) = 2.$$

Thus $C(2x)=m+1=C(x)+1$.

COROLLARY 1. *For* x *odd*, $C(2^\alpha x)=C(2^\alpha)+C(x)$.

COROLLARY 2. *For* x *even*, $C(2^\alpha x)=C(2^\alpha)+C(x)+1$.

We note here the reason for the *two* conditions, (4) and (5). When we multiply an odd number by 2, the class of the number remains unaltered. On the other hand, for x even, multiplying x by 2 changes its class by 1, and here 2 behaves as if it belonged to class 1, in the sense of (4).

THEOREM 4. *For any integer* x, $C(3x)=C(x)+C(3)$.

Proof: From the lemma we get

$$\phi(3x) = 3\phi(x) \quad \text{or} \quad 2\phi(x),$$

$$\phi^2(3x) = 3\phi^2(x) \quad \text{or} \quad 2\phi^2(x),$$

and in general for any $m \leqq C(x)$, we have

$$\phi^m(3x) = 3\phi^m(x) \quad \text{or} \quad 2\phi^m(x).$$

Then, taking $m=C(x)$ we have

$$\phi^m(3x) = 3\cdot 2 \quad \text{or} \quad 2\cdot 2 = 6 \quad \text{or} \quad 4.$$

and

$$\phi^{m+1}(3x) = 2.$$

Hence $C(3x)=m+1=C(x)+1=C(x)+C(3)$.

THEOREM 5. *For any odd prime p, $C(px)=C(p)+C(x)$.*

Proof: The theorem has been shown to be true for the case $p_1=3$. Assuming it true for the first $k-1$ odd primes, $p_1, \cdots, p_{k-1}$, we shall prove it true for the k-th prime, p_k, and thus complete the induction.

From the lemma we have

$$\phi(p_k x) = p_k\phi(x) \quad \text{or} \quad (p_k - 1)\phi(x). \tag{6}$$

Now, if $\phi(p_k x)=(p_k-1)\phi(x)$, we have $(p_k-1)=2^\alpha\prod q_i^{\alpha_i}$ where the q_i are odd primes less than p_k, and thus among the first $k-1$. Thus, by Theorem 3, Corollary 2, and our assumption,

$$C[(p_k - 1)\phi(x)] = C(p_k - 1) + C[\phi(x)] + 1.$$

Then,

$$1 + C[\phi(p_k x)] = [C(p_k - 1) + 1] + [C[\phi(x)] + 1]$$

or

$$C(p_k x) = C(p_k) + C(x),$$

which was to be proved.

Thus in (6) there remains to consider the case $\phi(p_k x)=p_k\phi(x)$. In this case we have

$$\phi^2(p_k x) = p_k\phi^2(x) \quad \text{or} \quad (p_k - 1)\phi^2(x).$$

Here the case $\phi^2(p_k x)=(p_k-1)\phi^2(x)$ leads to the required result by an argument similar to that applied to $(p_k-1)\phi(x)$, in (6). Again we are left only to consider

$$\phi^2(p_k x) = p_k\phi^2(x).$$

Continuing this process we see that at each step the only alternative from which our result is not immediate is

$$\phi^m(p_k x) = p_k\phi^m(x).$$

Then taking $m=C(x)$ we get

$$\phi^m(p_k x) = 2p_k,$$

so that $C(p_k x)=m+C(2p_k)=C(x)+C(p_k)$.

As was already mentioned, Theorem 5, and Theorem 3, Corollaries 1 and 2 are sufficient to establish Theorem 1.*

* With regard to de Rocquigny's formula we can now demonstrate its falsity for any $N>3$, $m>2$, and $3<p\leqq C(N^m)$. For if

$$\phi^p(N^m) = \phi^{p-2}(N^{m-2})\phi^{p-1}[(N - 1)^2],$$

then the class of the number on the left side of this equation is equal to the class of the number on

Utilizing the relationships (4) and (5) we can readily calculate the integers of the first few classes.*

Class	Numbers of This Class
0	1, 2,
1	3, 4, 6,
2	5, 7, 8, 9, 10, 12, 14, 18,
3	11, 13, 15, 16, 19, 20, 21, 22, 24, 26, 27, 28, 30, 36, 38, 42, 54,
4	17, 23, 25, 29, 31, 32, 33, 34, 35, 37, 39, 40, 43, 44, 45, 46, 48, 49, 50, 52, 56, 57, 58, 60, 62, 63, 66, 70, 72, 74, 76, 78, 81, 84, 86, 90, 98, 108, 114, 126, 162,
5	41, 47, 51, 53, 55, 59, 61, 64, 65, 67, 68, 69, 71, 73, 75, 77, 79, 80, 82, 87, 88, 91, 92, 93, 94, 95, 96, 99, 100, 102, 104, 105, 106, 109, 110, 111, 112, 116, 117, 118, 120, 122, 124, 127, 129, 130, 132, 133, 134, 135, 138, 140, 142, 144, 146, 147, 148, 150, 152, 154, 156, 158, 163, 168, 171, 172, 174, 180, 182, 186, 189, 190, 196, 198, 210, 216, 218, 222, 228, 234, 243, 252, 254, 258, 266, 270, 294, 324, 326, 342, 378, 486,
6	83, 85, 89, 97, $\cdots$
7	137, $\cdots$
8	257, $\cdots$

Brief as this table may be it furnishes us with an excellent basis for conjecture and inductive proof.

3. **Bounds for C(x).** Noting that the table above gives *all* the numbers of each of the first few classes, we now proceed to determine bounds for $C(x)$.

THEOREM 6. *The largest odd number in the class m is 3^m, and the largest even number in the class m is $2\cdot 3^m$.*

Proof: From the table above we see that the theorem is true for $m=0, 1, 2, 3$. Now we assume it to be true for all $m'<m$, and will prove it true for class m, thus completing the proof of the theorem.

First, for any odd prime p in class m we have $C(p-1)=m-1<m$, and therefore, by our assumption, $p-1\leqq 2\cdot 3^{m-1}$. Then $p\leqq 2\cdot 3^{m-1}+1\leqq 3^m$.

Next, for an odd prime p, if p^α, $\alpha>1$, is in class m, we have $C(p^\alpha)=\alpha C(p)=m$, and $C(p)=m/\alpha<m$. Hence, by our assumption $p\leqq 3^{m/\alpha}$ and $p^\alpha\leqq 3^m$.

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the right. This implies that either

(1) $mC(N)-p=(m-2)C(N)-(p-2)+2C(N-1)+1-(p-1)+1=mC(N)-2p+3$,

or

(2) $mC(N)-p=(m-2)C(N)-(p-2)$,

or

(3) $mC(N)-p=2C(N-1)-p+2=2C(N)-p$,

which give respectively that we must have either (1) $p=3$, (2) $C(N)=1$, which for prime N is true only if $N=3$, or (3) $m=2$.

* The completeness of this table, as given, can be verified from the tables of solutions of the equation $\phi(x)=n$, $\cdots$ Number-Divisor Tables, Br. Ass. Mathematical Tables, V. 8, $\cdots$ or by utilizing known methods of solving this equation as given by Wright, Theory of Numbers, and Carmichael, Amer. Journal of Math., 30, 1908, p. 394–400.

Now, if $n=2^{\alpha}\prod p_i^{\alpha_i}$* is any composite number such that $C(n)=m$, we know from Theorem 1 that

$$\left.\begin{aligned} C(n) &= \sum C(p_i^{\alpha_i}), \quad \text{for} \quad \alpha = 0; \\ &= \sum C(p_i^{\alpha_i}) + \alpha - 1, \quad \text{for} \quad \alpha \geqq 1. \end{aligned}\right\}$$

Then, since for each of these $p_i^{\alpha_i}$, $p_i^{\alpha_i} \leqq 3^{C(p_i^{\alpha_i})}$, $C(p_i^{\alpha_i}) \geqq \dfrac{\log p_i^{\alpha_i}}{\log 3}$ and we get

$$\left.\begin{aligned} C(n) &\geqq \sum \frac{\log p_i^{\alpha_i}}{\log 3}, \quad \text{for} \quad \alpha = 0; \\ &\geqq \sum \frac{\log p_i^{\alpha_i}}{\log 3} + \alpha - 1, \quad \text{for} \quad \alpha \geqq 1 \end{aligned}\right\}$$

$$\left.\begin{aligned} &\geqq \frac{\log n}{\log 3}, \quad \text{for} \quad \alpha = 0; \\ &\geqq \frac{\log n}{\log 3} - \frac{\log 2}{\log 3}, \quad \alpha \geqq 1. \end{aligned}\right\}$$

From this we see that for n odd, $n \leqq 3^{C(n)}$, and for n even, $n \leqq 2 \cdot 3^{C(n)}$. This completes the induction and the proof of our theorem.

COROLLARY: *The maximum value of x for which $C(x)=m$ is $2\cdot 3^m$. In other words* $x \leqq 2 \cdot 3^{C(x)}$.

THEOREM 7. *The smallest even number of class m is 2^{m+1}.*

Proof: We note from our table that the theorem is true for $m=0$, 1, 2, 3, and we now assume it true for all classes $m'<m$. Suppose next that $C(s)=m$, where $s=2^{\alpha}r$, r odd. Now if $\alpha>1$, $s=2\cdot 2^{\alpha-1}r$, $\alpha-1>0$, and $C(2^{\alpha-1}r)=m-1$. Then $2^{\alpha-1}r \geqq 2^m$, by our assumption, and hence $s=2^{\alpha}r \geqq 2^{m+1}$.

On the other hand, if $\alpha=1$, $s=2r$, and it remains to show $2r \geqq 2^{m+1}$. Suppose $2r<2^{m+1}$; then $r<2^m$, where r is odd, and $C(r)=m$. Thus $\phi(r)<r<2^m$, but $\phi(r)$ is even and $C[\phi(r)]=m-1$, so that by our assumption $\phi(r)\geqq 2^m$. From this contradiction we see that $2r \geqq 2^{m+1}$, and our proof is completed.

THEOREM 8. *The smallest odd number of class m is greater than 2^m.*

Proof: Let s be the smallest odd number of class m. Then, by Theorem 7, $2s>2^{m+1}$, and $s>2^m$.

COROLLARY. *The smallest number in class m is greater than 2^m. In other words* $x>2^{C(x)}$.

From the corollaries to Theorems 6, and 8, we have

$$2^{C(x)} < x \leqq 2\cdot 3^{C(x)}$$

* This represents the unique factorization of n as a product of powers of distinct primes.

and the bounds for $C(x)$ are

$$\frac{\log x}{\log 2} > C(x) \geqq \frac{\log x/2}{\log 3} \cdot$$

4. A lower bound for $\phi(x)$. In passing, it is interesting to note that we can derive a lower bound for $\phi(x)$, by utilizing the properties of classes which we have developed thus far. From Theorem 7, since $\phi(x)$ is even, $\phi(x) \geqq 2^{C(x)}$. Also, we know from Theorem 6 that for x odd, $x \leqq 3^{C(x)}$ or $C(x) \geqq \log x/\log 3$. Thus combining these two facts we get

$$\phi(x) \geqq 2^{C(x)} \geqq 2^{\log x/\log 3} = x^{\log 2/\log 3}, \tag{7}$$

for all odd x. Having obtained the inequality (7) for all odd x, the question naturally arises as to whether or not it holds also for even x. In this connection we shall digress slightly and prove that (1) The only solutions of the equation $\phi(x) = x^{\log 2/\log 3}$ are $x = 1$, and 3; and (2) For all integers x, except $x = 1, 2, 3, 4, 6, 10, 12, 18$, and 30, $\phi(x) > x^{\log 2/\log 3}$.

LEMMA 1. *If $M = M(x)$ stands for the number of distinct prime divisors of the integer x, then for all odd x, with the exception of $x = 1, 3, 5, 7, 9, 15$, and 21,*

$$\log x > \frac{(\log 2)^2}{\log 3/2} + M \log 3. \tag{8}$$

Proof: For all odd $x > 5$, which do not contain a factor 3, it is obvious that

$$x > 5^M. \tag{9}$$

Now for those odd $x > 5$, which are divisible by 3, (9) holds, except for $x = 15$, and 105. Thus from (9) we have that for x odd and greater than 5, $x \neq 15, 105$, $\log x > M \log 5$.

Next we show that

$$M \log 5 > \frac{(\log 2)^2}{\log 3/2} + M \log 3, \tag{10}$$

for $M \geqq 3$. This follows readily since (10) simplifies to

$$M > \frac{(\log 2)^2}{(\log 3/2)(\log 5/3)} = 2.3 \cdots .$$

Now for $M = 1, 2$, the only odd x not satisfying (8) are $x = 1, 3, 5, 7, 9, 15$, and 21. Also, $x = 105$ satisfies (8), so that the proof of the Lemma is completed.

LEMMA 2. *For all odd x, with the exception of $x = 1, 3, 5, 7, 9, 15$, and 21.*

$$x(2/3)^M > (2x)^{\log 2/\log 3}. \tag{11}$$

Proof: This is immediate since (8) may be reduced to (11).

From Lemma 2 we get immediately

$$\phi(x) \geqq x(2/3)^M$$

$$> (2x)^{\log 2/\log 3} \tag{12}$$

$$> x^{\log 2/\log 3} \tag{13}$$

for all odd $x \neq 1, 3, 5, 7, 9, 15, 21$. However, actual trial shows that of these seven integers 1 and 3 are the only ones for which $\phi(x) \not> x^{\log 2/\log 3}$. For $x=1$, and 3, $\phi(x)=x^{\log 2/\log 3}$, so that 1 and 3 are the only odd solutions of this last equation.

Furthermore $x=7$ and 21 also satisfy (12) so that the only exceptions there are $x=1, 3, 5, 9$, and 15.

Next we consider the even numbers which are of the form $x'=2^\alpha x$, $\alpha>0$, x odd and $\neq 1, 3, 5, 9$, or 15. Then

$$\begin{aligned} \phi(x') &= 2^{\alpha-1}\phi(x) > 2^{\alpha-1}.2^{\log 2/\log 3}.x^{\log 2/\log 3} \\ &= 2^{\alpha-1+\log 2/\log 3}.(x'/2^\alpha)^{\log 2/\log 3} \\ &= 2^{(\alpha-1)(1-\log 2/\log 3)}.(x')^{\log 2/\log 3} \\ &\geqq (x')^{\log 2/\log 3}, \end{aligned}$$

so that

$$\phi(x') > (x')^{\log 2/\log 3}.$$

There remains to be considered the cases

$$x' = 2^\alpha y, \qquad y = 1, 3, 5, 9, \text{ and } 15.$$

Here $\phi(x') > (x')^{\log 2/\log 3}$ implies

$$2^{\alpha-1}\phi(y) > (2^\alpha y)^{\log 2/\log 3}$$

$$(\alpha-1)(\log 2)(\log 3) + (\log 3)(\log \phi(y)) > \alpha(\log 2)^2 + (\log 2)(\log y)$$

$$\alpha > \frac{(\log 2)(\log 3) + (\log 2)(\log y) - (\log 3)(\log \phi(y))}{(\log 2)(\log 3) - (\log 2)^2}, \tag{14}$$

so we see that for a given y, when α is large enough our result must certainly follow. For $y=1, 3, 5, 9$, and 15, (14) gives that $\alpha \geqq 3, 3, 2, 2, 2$, respectively, is necessary and sufficient, so that the only x for which (13) does not hold are $x=1, 2, 3, 4, 6, 10, 12, 18$, and 30.

Also, since none of the even numbers among these satisfies $\phi(x)=x^{\log 2/\log 3}$, $x=1$, and 3, are the only solutions.

5. The structure of a class. We note from our discussion thus far that each class has, in a sense, a definite structure. We know that in class m we will always find the numbers 2^{m+1}, 3^m, and $2\cdot 3^m$. This plus the important role they play in each class, as indicated by the previous theorems, suggests that we divide class m into sections, as follows:—

$$(m):\quad 2^m < \underbrace{s,\cdots,}_{\text{I}}\ \underbrace{2^{m+1},\cdots,3^m,}_{\text{II}}\ \underbrace{\cdots,2\cdot 3^m}_{\text{III}}$$

From the theorems already proved we know immediately that the numbers of section

$$\begin{cases}\text{I are all odd.}\\ \text{II may be either odd or even.}\\ \text{III are all even.}\end{cases}$$

We now proceed to deduce several other properties peculiar to the integers lying in the same section of their class.

If we let $x=2^\alpha s$, s odd and >1, $\alpha\geqq 1$, and $C(x)=m$, we have $C(2^\alpha s)=C(2^\alpha)+C(s)=\alpha-1+C(s)=m$. We know that

$$s > 2^{C(s)}$$

and thus,

$$2^\alpha s > 2^{\alpha+C(s)} = 2^{m+1}. \tag{15}$$

Also, since s is odd,

$$s \leqq 3^{C(s)}$$

$$2^\alpha s \leqq 2^\alpha 3^{C(s)} < 3^{\alpha-1}3^{C(s)}, \qquad \alpha \geqq 3,$$

$$2^\alpha s < 3^{\alpha-1+C(s)} = 3^m. \tag{16}$$

From (15) and (16) and the already mentioned fact that the numbers in section III of any class are even, we have the following theorems.

THEOREM 9. *For any integer x in section III of its class, $x\equiv 0$ (mod 2), but $x\not\equiv 0$ (mod 8).*

COROLLARY. *If $\log x/\log 3>C(x)$, then $x\equiv 0$ (mod 2), and $x\not\equiv 0$ (mod 8).*

Theorem 9, though it is a simple result, gives us, approximately, the structure of section III of any class m. This section includes those integers formed as follows:

(a) 2 times any odd number $s>3^{m/2}$, for which $C(s)=m$.

(b) 4 times any odd number $s>3^m/4$, for which $C(s)=m-1$.

However, we may go farther, and give a more complete determination of the integers in section III of any class, by giving explicitly all the integers which come under (b), and by finding narrower bounds for the s which need be considered under (a). In order to do this we need to consider certain portions of section II of a class.

THEOREM 10. *There exist no odd x such that*

$$3^{m-2}\cdot 7 < x < 3^m, \tag{17}$$

where $C(x)=m$.

Proof: We note from the table that the theorem is true for $m=0, 1, 2, 3$, and proceeding by induction, assume it true for all $m'<m$, $m\geqq 3$. Now let us suppose that for an odd integer x, $C(x)=m$, and we have

$$3^{m-2}\cdot 7 < x < 3^m. \tag{18}$$

If x is a prime, $C(x-1)=m-1$, and $x-1\leqq 2\cdot 3^{m-1}$, but $x-1>3^{m-2}\cdot 7-1>2\cdot 3^{m-1}$ for $m\geqq 3$. Thus x cannot be a prime, and it certainly is not of the form 3^a, so that we may write $x=st$, $3^{C(s)}>s>1$, $3^{C(t)}\geqq t>1$. Then from (18) we get

$$3^{C(s)} > s > 3^{C(s)+C(t)-2}\cdot 7/t \geqq 3^{C(s)-2}\cdot 7. \tag{19}$$

But $C(s)<m$, and s is odd, so that (19) contradicts our original assumption. Hence, the assumption that an odd x satisfying (17) exists, is false, and our induction is complete.

THEOREM 11. *There exists no odd x such that*

$$\cdots \qquad 3^{m-3}\cdot 19 < x < 3^{m-2}\cdot 7, \tag{20}$$

where $C(x)=m$.

Proof: Proceeding by induction as in the proof of Theorem 10 we note that the theorem is true for $m=0, 1, 2, 3$, and assume it true for all $m'<m$, $m>3$. Again we suppose the existence of an odd integer x of class m such that (20) holds. As in Theorem 10, x cannot be a prime since then $x-1\leqq 2\cdot 3^{m-1}$, and at the same time $x-1>3^{m-3}\cdot 19-1>2\cdot 3^{m-1}$, for $m>3$. Thus since x is composite and not of the form 3^a, we may write $x=st$, $s>1$, $t>1$, where, by Theorem 10, we may assume $s\leqq 3^{C(s)-2}\cdot 7$. Now we cannot have $s=3^{C(s)-2}\cdot 7$, and $t=3^{C(t)}$, for then $x=st=3^{C(s)+C(t)-2}\cdot 7=3^{C(x)-2}\cdot 7$. Hence, either $s<3^{C(s)-2}\cdot 7$, or $t\leqq 3^{C(t)-2}\cdot 7$. Also, we cannot have $s=7\cdot 3^{C(s)-2}$ and $t=7\cdot 3^{C(t)-2}$, for then $x=st=49\cdot 3^{C(x)-4}=3^{m-4}\cdot 49<3^{m-3}\cdot 19$. Thus we may assume $s<3^{C(s)-2}\cdot 7$, and from (20) we get

$$3^{C(s)-2}\cdot 7 > s > 3^{C(s)+C(t)-3}\cdot 19/t > 3^{C(s)-3}\cdot 19. \tag{21}$$

But $C(s)<m$, and s is odd, so that (21) contradicts our original assumption. Hence the assumption that an odd x satisfying (20) exists, is false, and the proof of the theorem is complete.

COROLLARY. *The three largest odd numbers of class m, $m\geqq 3$, are in order of magnitude, $19\cdot 3^{m-3}$, $7\cdot 3^{m-2}$, and 3^m.*

THEOREM 12. *The numbers of section II of class m are*

(a) $2\cdot 3^{m-3}\cdot 19$, $2\cdot 3^{m-2}\cdot 7$, $2\cdot 3^m$; *and 2 times any odd number s, $C(s)=m$, such that $3^m/2<s<19\cdot 3^{m-3}$;*

(b) $4\cdot 3^{m-3}\cdot 7$, *and* $4\cdot 3^{m-1}$.

Proof: From the discussion immediately following Theorem 9, we already have the integers of section III of class m separated into two sets (a) and (b).

In (a) we have 2 times any odd number $s>3^m/2$ for which $C(s)=m$. Thus by the Corollary to Theorem 11 we have that these numbers are $2\cdot 3^{m-3}\cdot 19$, $2\cdot 3^{m-2}\cdot 7$, $2\cdot 3^m$; and 2 times any odd number s, $C(s)=m$, such that $3^m/2<s<19\cdot 3^{m-3}$. Since $3^m/4>19\cdot 3^{m-4}$, the only odd s fitting the description under (b), on page 25, are $7\cdot 3^{m-3}$, and 3^{m-1}. Thus the only numbers of section III of class m which are divisible by 4, are the two given in the theorem.

In addition to giving us this more complete determination of the integers of section III of any class, Theorems 10, and 11 reflect upon the prime numbers which can occur in certain parts of section II of any class. We will see that restricting a prime number to certain parts of its class, limits its form very definitely.

THEOREM 13. *A prime number p $(\neq 5)$ satisfies the inequality*

$$p > 43\cdot 3^{C(p)-4}, \tag{22}$$

if and only if $p=2\cdot 3^{C(p)-1}+1$.

Proof: For $p=2\cdot 3^{C(p)-1}+1$, it is obvious that p satisfies (22). On the other hand, if any prime p satisfies (22), then it remains for us to prove that it must equal $2\cdot 3^{C(p)-1}+1$.

We will assume $C(p)>4$, which is valid since an inspection of the table shows the theorem to be true for the first four classes, (excluding the class 0). Then

$$p-1 > 42\cdot 3^{C(p)-4} > 3^{C(p)-1} = 3^{C(p-1)}$$

so that $p-1$ is in section III of its class. Hence $p-1=2x'$, or $4x'$, x' odd. If $p-1=4x'$, then

$$x' = (p-1)/4 > (42/4)\cdot 3^{C(p)-4} = (42/36)\cdot 3^{C(p)-2} > 3^{C(x')},$$

which is impossible. On the other hand, if $p-1=2x'$,

$$x' = (p-1)/2 > (42/2)\cdot 3^{C(p)-4} = (42/6)\cdot 3^{C(p)-3} = 7 . 3^{C(p)-3},$$

so that by Theorem 10, $x'=3^{C(p)-1}$, and $p=2\cdot 3^{C(p)-1}+1$.

THEOREM 14. *A prime number p satisfies the inequalities*

$$43\cdot 3^{C(p)-4} > p > 39\cdot 3^{C(p)-4} \tag{23}$$

if and only if $p=2\cdot 7\cdot 3^{C(p)-3}+1$.

Proof: Here again we may assume $C(p)>4$. Also, since a prime of the given form obviously satisfies (23), it remains only to prove the converse. If a prime p satisfies (23), then

$$p-1 > 39\cdot 3^{C(p)-4} - 1 > 3^{C(p)-1} = 3^{C(p-1)}$$

so that $p-1$ is in section III of its class. Thus $p-1=2x'$, or $4x'$, x' odd. If $p-1=4x'$, $x'=p-1/4>(38/4)\cdot 3^{C(p)-4}>3^{C(p)-2}$, which is impossible. For

$p-1=2x'$, $x'=(p-1)/2>(38/2)\cdot 3^{C(p)-4}=19\cdot 3^{C(p)-4}$ so that by Theorems 10, and 11, $x'=7\cdot 3^{C(p)-3}$, or $3^{C(p)-1}$. However, if $x'=3^{C(p-1)}$, then we would have $p>43\cdot 3^{C(p)-4}$, so that we must have $x'=7\cdot 3^{C(p)-3}$, or $p=2\cdot 7\cdot 3^{C(p)-3}+1$.

6. Section I of a class. We next consider some properties of the integers of section I of any class.

THEOREM 15. *If an integer x is in section I of the class to which it belongs, then every divisor of x is in section I of its class.*

Proof: Since x is in section I of its class, we know that x is odd. Now, our theorem is obviously true for x an odd prime, so that we need only consider the case where x is composite. Then, if $x>d>1$, where $x=ds$, we have

$$2^{C(x)+1} = 2^{C(ds)+1} > x > 2^{C(ds)} = 2^{C(x)}$$

and $ds<2^{C(ds)+1}=2^{C(d)+C(s)+1}$, since both d and s are odd. If $d>2^{C(d)+1}$, *i.e.* if d were not in section I of its class, since $s>2^{C(s)}$, we would have $ds>2^{C(d)+C(s)+1}$ which is a contradiction. From this we have the truth of our theorem.

It is now reasonable to ask whether the converse of Theorem 15 is true; *i.e.* if all the proper divisors of a number x are in section I of their respective classes, is x in section I of its class? The answer is obviously, no, since if we consider any two odd numbers s and t such that

$$\begin{aligned} 2^{C(s)}\sqrt{2} &< s < 2^{C(s)+1} \\ 2^{C(t)}\sqrt{2} &< t < 2^{C(t)+1}. \end{aligned} \tag{24}$$

Then $st>2^{C(s)+C(t)+1}=2^{C(st)+1}$, and thus st is in section II of its class. That numbers s and t satisfying (24) actually exist is easily verified.

In connection with the existence of numbers for which every proper divisor is in section I of its class whereas the number itself is in section II of its class, we have the following interesting theorem.

THEOREM 16. *If P is any number of the form 2^n-1 which is in section I of its class, and s is any number also in section I of its class; then, if $C(s)\leqq C(P)$, $x=sP$ is in section II of its class.*

Proof: From our hypothesis

$$2^{C(s)} < s < 2^{C(s)+1} \tag{25}$$

and

$$P = 2^{C(P)+1} - 1. \tag{26}$$

Also,

$$C(s) \leqq C(P) < C(P) + 1,$$

$$2^{C(P)+1} - 1 > 2^{C(s)},$$

$$2^{C(s)+C(P)+1} + 2^{C(P)+1} - 1 - 2^{C(s)} > 2^{C(s)+C(P)+1},$$

$$(2^{C(s)} + 1)(2^{C(P)+1} - 1) > 2^{C(sP)+1}.$$

Then from (25)

$$s(2^{C(P)+1} - 1) \geqq (2^{C(s)} + 1)(2^{C(P)+1} - 1) > 2^{C(sP)+1},$$

and combining this with (26) we get

$$sP > 2^{C(sP)+1},$$

so that sP is in section II of its class.

COROLLARY. *If x is any number in section I of its class, then*

$$x \not\equiv 0 \pmod{P^2}, \tag{27}$$

where P is any number of the form 2^n-1.

Proof: If P is not in section I of its class, then we have by Theorem 15, $x \not\equiv 0 \pmod{P}$ which certainly implies (27). If P is in section I of its class, letting $s=P$, we see that the conditions of Theorem 16 are satisfied, and thus P^2 is in section II of its class. Then again by Theorem 15, $x \not\equiv 0 \pmod{P^2}$, which completes the proof of the corollary.

We note that in discussing numbers of the form $P=2^n-1$ which are in section I of their class, Theorem 16 and its corollary also include the special case where P is a Mersenne prime. However, with respect to Mersenne primes the hypothesis that they be in section I of their class is a strong limitation. In fact, in the following we determine all such primes.

LEMMA. $C(2^x+1)=x$ *if and only if* 2^x+1 *is a Fermat prime.*

Proof: If 2^x+1 is a prime, then obviously $C(2^x+1)=x$. Now, if 2^x+1 is not a prime, $\phi(2^x+1)<2^x$, and since $\phi(2^x+1)$ is even, by Theorem 7 we see that $C[\phi(2^x+1)] \leqq x-2$, and hence $C(2^x+1)=1+C[\phi(2^x+1)] \leqq x-1<x$.

THEOREM 17. *The only Mersenne primes which are in section I of their class are* 2^2-1, 2^3-1, 2^5-1, *and* $2^{17}-1$.

Proof: If a Mersenne prime $M_p=2^p-1$ is in section I of its class we have

$$C(2^p - 1) = p - 1$$

$$\begin{aligned} C(2^p - 1) &= 1 + C(2^{p-1} - 1) \\ &= 1 + C(2^{(p-1/2)} - 1) + C(2^{(p-1)/2} + 1), \quad p \geqq 3. \end{aligned}$$

But

$$C(2^{(p-1)/2} - 1) \leqq (p - 1)/2 - 1$$

$$C(2^{(p-1)/2} + 1) \leqq (p - 1)/2,$$

and thus

$$1 + C(2^{(p-1)/2} - 1) + C(2^{(p-1)/2} + 1) \leqq p - 1. \tag{28}$$

However, the equality sign holds in (28) so that

$$C(2^{(p-1)/2}+1) = (p-1)/2 \tag{29}$$

$$C(2^{(p-1)/2}-1) = (p-1)/2-1. \tag{30}$$

By the lemma, and (29) we see that $2^{(p-1)/2}+1$ is a Fermat prime and hence

$$\frac{p-1}{2} = 2^m \tag{31}$$

$$p = 2^{m+1}+1. \tag{32}$$

Combining (30) and (31) we get

$$C(2^{2^m}-1) = 2^m-1.$$

Then assuming $m>0$, we have

$$C(2^{2^{m-1}}-1)+C(2^{2^{m-1}}+1) = 2^m-1.$$

But

$$C(2^{2^{m-1}}-1) \leqq 2^{m-1}-1$$
$$C(2^{2^{m-1}}+1) \leqq 2^{m-1},$$

so that

$$C(2^{2^{m-1}}-1)+C(2^{2^{m-1}}+1) \leqq 2^m-1. \tag{33}$$

Since the equality sign holds in (33) we have

$$C(2^{2^{m-1}}-1) = 2^{m-1}-1 \tag{34}$$

$$C(2^{2^{m-1}}+1) = 2^{m-1}. \tag{35}$$

Then by the lemma, and (35) we see that $2^{2^{m-1}}+1$ is also a Fermat prime. By continuing this process we can then show that $2^{2^{m-2}}+1$ must also be a Fermat prime, and so on. From this, and the fact that $2^{2^5}+1$ is not a prime, we have $m\leqq 4$. Then from (32), and the fact that we originally assumed $p\geqq 3$, we see that p may be either 2, 3, 5, 17. Then our Mersenne primes may be correspondingly 2^2-1, 2^3-1, 2^5-1, $2^{17}-1$. That these are all primes is known, and that they are all in section I of their respective classes is easily verified.

From our discussion we have been able to get some idea of the constitution of a class. But this discussion is by no means complete, and there exist numerous conjectures which we have been unable to resolve one way or another. Foremost among these conjectures is the proposition, justified by calculation with respect to the first eight classes, that the smallest number in each class is a prime. This may prove to be very difficult to establish. So far we have been unable even to demonstrate that there is at least one prime in each class.

ADJUSTMENTS IN MATHEMATICS TO THE IMPACT OF WAR

G. B. PRICE, University of Kansas

Frank B. Jewett said in a public address [1] early in 1942: "Without insinuating anything as to guilt, the chemists declare that this is a physicist's war. With about equal justice one might say that it is a mathematician's war." This note contains a report on how the mathematicians have participated in the war, the problems that have arisen, the measures that have been taken to meet them, the changes that have been made in the mathematics curriculum, and the effects that war has had on graduate work in mathematics [2]. It is based on information obtained through questionnaires [3] from forty-one of the fifty-four institutions in the United States and Canada that offer the Ph.D. degree in mathematics and from thirteen additional colleges and universities in the United States. No information was obtained from Canada. It is important to observe that the information was gathered between November 17 and December 15, 1942. Since the government has now announced (December 17, 1942) its program for the utilization of the colleges and universities during the war, this report can be considered a record of the first year, and of the first phase, of the participation of mathematics in the war.

The report can be summarized in the following conclusions.

1. There has been an enormous increase in enrollments in mathematics courses; although the most frequently reported increase was thirty per cent (enrollment for the first semester of 1942–43 compared with enrollment for the first semester of 1941–42), many institutions reported increases up to sixty and seventy per cent, and six colleges and universities reported increases ranging from 100 per cent to 300 per cent. These increases took place in spite of decreased university enrollments and enlarged summer schools during the past summer. The number of men studying mathematics is large; for example, in the first semester of 1942–43 at the University of Michigan 939 men were enrolled in undergraduate and first-year graduate courses in mathematics in the arts college and 1820 others were in similar courses in the engineering college.

2. The percentage increase in enrollments in mathematics is much smaller for women than for men. Four women's colleges in the East reported increases ranging from twenty-five to forty per cent. Most coeducational institutions reported either no increase or only a small one; exceptions are one university, 62 per cent; three colleges and universities, 100 per cent; one university, 125 per cent. The total number of women studying mathematics is small. For example, the University of Michigan, after a ten per cent increase over last year, had only 337 women in undergraduate and first-year graduate courses in mathematics in the arts college; approximately two-thirds of this group is enrolled in first-year courses. Michigan apparently has one of the largest groups of women studying mathematics [4]. Of two large universities reporting 100 per cent increases in women in mathematics, one still has fewer than 150 in all undergraduate and graduate courses and the other has fewer than 200.

3. There exists a tremendous and acute shortage of mathematics instructors.

This shortage is attested by such statements as "we could have used four additional instructors this fall" and "twenty per cent of the staff is in the army or navy," but perhaps even more by the emergency steps taken to secure instructors. These include (1) increasing the teaching load of the individual instructor; (2) increasing the size of sections; (3) offering only essential courses (this step has involved dropping many undergraduate and graduate courses); (4) abolishing summer vacations (this step involves continuous operation under a three semester or four quarter plan); (5) using practically all graduate students as instructors and even some senior undergraduates as instructors, assistants, and readers; (6) calling retired professors back to active duty, engaging former mathematics teachers who had entered business or other occupations, and employing retired Army officers; (7) borrowing instructors from other departments (in some cases they are given refresher or preparatory courses [**5**]); borrowing instructors for part-time work from a local air base or from some government agency; (8) "robbing other institutions"; (9) employing faculty wives.

4. Spherical trigonometry, in many cases combined with solid geometry, and the mathematics of navigation have been added to the curriculum almost universally.

5. Ten schools reported that they had added additional elementary courses, other than solid geometry and spherical trigonometry, in plane geometry, algebra, and trigonometry, or some combination of these, to enable students to remove high school deficiencies in mathematics or to obtain training needed for entrance into some branch of the armed forces. Some of this work, especially in algebra and trigonometry, has always been given in many institutions.

6. There has been practically no change in the content or organization of the foundation courses in mathematics (algebra, trigonometry, analytic geometry, and calculus); in continuing to teach the fundamentals in courses which have long been standard the mathematicians have carried out the recommendations of the Army, the Navy, and of their own War Preparedness Committee.

7. Fifteen institutions reported that they had dropped some of their undergraduate courses, especially advanced courses in pure mathematics but also mathematics courses for special groups of students. Among the courses not being given or being given with less frequency are pure geometry, number theory, history of mathematics, fundamental concepts of mathematics, and survey courses in mathematics; forestry mathematics, and various courses for students of business. The reasons for dropping these courses are (1) students are taking standard courses instead of special ones; (2) students are taking courses in applied mathematics instead of in pure and abstract mathematics; (3) the shortage of instructors has made it necessary to drop courses in order to obtain instructors for elementary courses. The shift in the curriculum indicated here and in the next conclusion has already assumed large proportions.

8. There has been a big increase in teaching and research in applied mathematics: fifty-seven *new courses* (other than those named in 4 and 5 above) in approximately twenty different subjects in applied mathematics are being given

in the fifty-four institutions reporting. There are eight courses in mathematics of artillery fire or exterior ballistics; eight courses in cryptography and cryptanalysis; five courses in principles of mechanics; five courses in mathematics for meteorology; and many others with smaller frequencies. In addition, there was inaugurated at Brown University in the summer of 1941 a notable program of Advanced Instruction and Research in Mechanics; Brown will begin publication of *Journal of Applied Mathematics* early in 1943. The University of Wisconsin has established a four-year course leading to the degree of Bachelor in Applied Mathematics and Mechanics.

9. Mathematicians are doing a large amount of teaching in training schools for the Army and Navy, in ESMWT courses, and in refresher courses for those about to enter the army and navy. These schools instruct in various branches of engineering, meteorology, radio and electronics, Diesel motors, navigation and other pre-flight subjects, and train workers for aircraft plants, research workers for aeronautics laboratories, mathematics teachers for the Army and Navy, and so on. The information gathered on this subject is incomplete because the mathematics in these schools is often under the administrative control of the engineering school or some other department of the university. Two examples will illustrate the proportions to which this work has grown. The mathematics department in a state school in the mid-west, far from water of any kind, is giving instruction in mathematics continuously to 1600 students in two training schools for the Navy. The Department of Mathematics of the University of California at Los Angeles is giving approximately forty courses for aircraft workers; during the summer it conducted a series of courses in anti-aircraft mathematics for the men enlisted in the Coast Artillery assigned to anti-aircraft stations in California [6].

10. Five institutions have provided special courses involving mathematics for the training of women; in general, however, there have been no special provisions for women and no concerted efforts to enlist them in the study of mathematics. Purdue is introducing a new sequence of courses to prepare women as statistical workers. Chicago has a special course in mathematics for women students of electronics. The University of Pennsylvania is providing a special course in engineering drafting which requires mathematics. The University of California at Los Angeles has blocked out several curricula designed to prepare women as aircraft workers [6]; these include plane and solid analytic geometry, engineering mechanics, descriptive geometry, and so on. The University of New Mexico is planning a two-year specialized program for women that will involve the College of Engineering and the Departments of Mathematics and Physics. Smith College has plans under way for a summer school next summer that will likely emphasize mathematics and science. All the information indicates a strong demand for women trained in mathematics and an increasing tendency for them to enter the field. Positions on all levels are open to them. The case is cited of a young woman who graduated from a university in the mid-west with an A.B. degree in mathematics (and a Phi Beta Kappa key!) and who obtained an at-

tractive position in the Research Laboratory of the United Aircraft Corporation at East Hartford, Connecticut.

11. Graduate work in mathematics is rapidly approaching the vanishing point. One school reported that such work had been abolished; in many others it is rapidly disappearing as a result of (a) the loss of graduate students, (b) the employment of graduate instructors on war research and for elementary instruction, and (c) the use of graduate students as instructors. One large university reports that it is offering only ten graduate courses this year instead of the usual fifteen, and that only five are planned for next year. Many institutions reported decreases in enrollment in graduate courses in mathematics ranging from fifty to seventy-five per cent. The following are the actual numbers of graduate students in mathematics reported at ten institutions: Brown, 20; Cincinnati, 5; George Washington, 20; Iowa State, 11; University of Iowa, 11; Lehigh, 5; Northwestern, 12; Ohio State, 11; St. Louis, 18; Virginia, 11. But even these figures are misleading; the information indicates that most of these *students* are actually part-time or full-time *instructors* or *research workers*. A careful analysis of the information indicates that only two of these ten institutions are giving essentially a full program of graduate courses to full-time students. M.I.T. reports that the only students attending graduate classes are (a) undergraduates taking advanced work, (b) staff members from the mathematics or other departments, and (c) men employed on war research projects.

12. One school reported that both the mathematics club and the departmental colloquium had been discontinued; two others reported that the departmental colloquium had ceased to function. Twelve other institutions reported either fewer meetings of their club and colloquium or more programs devoted to applied mathematics and subjects related to war. These facts emphasize further the shortage of mathematicians, the extent to which they have turned from pure to applied mathematics, and their complete devotion to winning the war.

References

1. Frank B. Jewett, The Mobilization of Science in National Defense, Science, vol. 95, 1942, pp. 235–241.

2. This paper was prepared for the program of the Mathematical Association of America at a meeting planned for December 31, 1942 in New York City; the meeting was cancelled at the request of the government because of a shortage of transportation.

3. The author takes this opportunity to express his thanks to those who supplied the data for this report. The replies to his request for information were prompt and generous.

4. The University of California at Los Angeles is training many women for the aircraft industries, but the author does not know the exact number; see 6 below.

5. This work is beginning to take organized form. At New York University sixty college professors and high school teachers, formerly instructors in English, history, philosophy, education, and foreign languages, are taking intensive courses in mathematics and physics, sponsored by the U. S. Office of Education, to prepare them as teachers of these subjects; see War Emergency Courses in the University, Science, vol. 96, 1942, pp. 488–489. Several ESMWT courses of a similar nature have been reported also.

6. For more details of the work being done at UCLA see On the Educational Front, A.A.A.S. Bulletin, vol. 1, 1942, no. 9 (November), p. 70.

EQUIAREAL PATTERNS

M. A. SADOWSKY, Illinois Institute of Technology

1. Introduction. In a 3-page paper entitled "*Equiareal pattern of stress trajectories in plane plastic strain*" published in the June, 1941, copy of the Journal of Applied Mechanics, the present writer has reduced the problem of plasticity in two dimensions to pure differential geometry by establishing the following characteristic property of the orthogonal pattern of stress trajectories: using the trajectories as parametric curves in an orthogonal curvilinear system u, v with the first fundamental form

$$ds^2 = Edu^2 + Gdv^2, \tag{1}$$

we will have

$$EG = 1, \tag{2}$$

which expresses the equiareal property of the u, v system. To interpret (2) geometrically, let us compute the area A of a curvilinear rectangle as bounded by four parametric curves $u = u_1$, $u = u_2$ and $v = v_1$, $v = v_2$. The result

$$A = \int_{u_1}^{u_2} \cdot \int_{v_1}^{v_2} \sqrt{EG - F^2}\, dudv = (u_2 - u_1)(v_2 - v_1),$$

shows that the pattern formed by parametric curves as selected by

$$u = 0,\ h_1,\ 2h_1, \cdots nh_1, \cdots,$$
$$v = 0,\ h_2,\ 2h_2, \cdots mh_2, \cdots,$$

consists of curvilinear rectangles all equivalent in area (h_1h_2 each). Figure 1 illustrates this by showing equiareally re-parametrized polar coordinates ($u = \sqrt{2r}$, $v = \theta$). Figure 2 shows an equiareal pattern formed by orthogonal logarithmic spirals.

A complete analytical determination of all equiareal orthogonal patterns is a problem of differential geometry which has not yet been solved. In the present paper, a particular solution involving an arbitrary function of one argument will be derived.

2. Derivation of fundamental equations. In order that (1) be a linear element of the euclidean plane, the Gaussian curvature of the uv-space has to vanish, which is expressed by the equation (known as the Gauss equation)

$$\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right) = 0. \tag{3}$$

We now introduce a new quantity, Ω, as defined by

$$\Omega = \frac{\partial\sqrt{E}}{\partial v} - \frac{\partial\sqrt{G}}{\partial u} \tag{4}$$

into the Gauss equation (3) in the following manner:

$$\frac{\partial}{\partial u}\left[\frac{1}{\sqrt{E}}\left(\frac{\partial\sqrt{E}}{\partial v} - \Omega\right)\right] + \frac{\partial}{\partial v}\left[\frac{1}{\sqrt{G}}\left(\frac{\partial\sqrt{G}}{\partial u} + \Omega\right)\right] = 0. \tag{5}$$

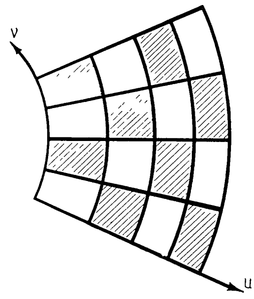
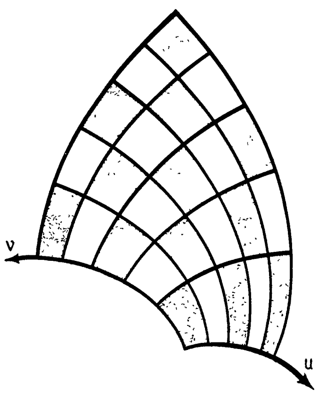


FIG. 1



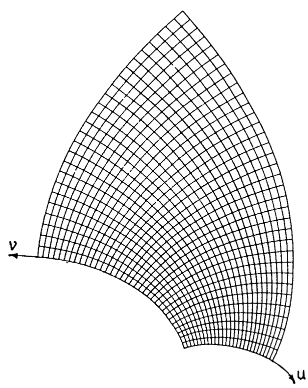


FIG. 2 FIG. 3

Expanding (5) and using the equiareal condition (2) we may reduce the Gauss equation to

$$\frac{\partial}{\partial u}\left(\frac{\Omega}{\sqrt{E}}\right) = \frac{\partial}{\partial v}\left(\frac{\Omega}{\sqrt{G}}\right).$$

At this point we leave the general case and turn to a particular solution of the above equation as given by

$$\Omega = 0.$$

By (4), this means

$$\frac{\partial\sqrt{E}}{\partial v} = \frac{\partial\sqrt{G}}{\partial u}, \tag{6}$$

which may be integrated formally by putting

$$\sqrt{E} = \frac{\partial \phi}{\partial u}, \qquad \sqrt{G} = \frac{\partial \phi}{\partial v}, \tag{7}$$

in which $\phi = \phi(u, v)$ is an arbitrary function of u and v. The equiareal condition (2) expressed in terms of ϕ becomes

$$\frac{\partial \phi}{\partial u} \cdot \frac{\partial \phi}{\partial v} = 1. \tag{8}$$

To solve (8) generally we have at our disposal a complete solution (a solution with two arbitrary constants), easy to guess:

$$\phi = au + \frac{1}{a} v + b.$$

Putting $b = f(a)$, in which $f(a)$ stands for an arbitrary function of a, and eliminating a from

$$\begin{aligned} \phi &= au + \frac{1}{a} v + f(a), \\ 0 &= u - \frac{1}{a^2} v + f'(a), \end{aligned} \tag{9}$$

we obtain ϕ in terms of u and v as expressed by $\phi = \phi(u, v)$. However, as we are not interested in $\phi(u, v)$ itself, but in its partial derivatives (7) only, we will go on as follows: assuming $a = a(u, v)$ to be the solution of the second equation in (9) for a in terms of u and v, we obtain from (7) and (9)

$$\begin{aligned} \sqrt{E} = \frac{\partial \phi(u, v)}{\partial u} &= \frac{\partial \phi(u, v, a)}{\partial u} + \frac{\partial \phi(u, v, a)}{\partial a} \cdot \frac{\partial a(u, v)}{\partial u} \\ &= a + \left(u - \frac{1}{a^2} v + f'(a) \right) \frac{\partial a}{\partial u} = a, \end{aligned}$$

which can be written in a more complete form by (2) as

$$\sqrt{E} = a(u, v), \qquad \sqrt{G} = \frac{1}{a(u, v)} \cdot \tag{10}$$

As for $a(u, v)$, the second equation (9) may be written as

$$au - \frac{1}{a} v = F(a), \tag{11}$$

in which $F(a)$ symbolizes the arbitrary function of a. Now, the function $a(u, v)$ in (10) stands for the solution of (11) for a in terms of u and v.

To imbed the curvilinear system u, v in a Cartesian system of reference x, y we have to integrate by quadratures

$$dx + idy = \sqrt{E}\, e^{i\omega}du + i\sqrt{G}\, e^{i\omega}dv, \tag{12}$$

in which $\omega = \omega(u, v)$ stands for the local, at the point (u, v), directional angle of the parametric u-curve relative to the x-axis. The general differential geometric relation

$$d\omega = -\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}du + \frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}dv,$$

can be shown to lead under (2) and (6) to

$$\omega = \log\sqrt{E}, \tag{13}$$

(an arbitrary constant has been dropped from the above because it would not be essential). This transforms (12) to

$$dx + idy = (\sqrt{E})^{1+i}du + i(\sqrt{G})^{1-i}dv, \tag{14}$$

which is ready for integration by quadratures, provided that $\sqrt{E}$ and $\sqrt{G}$ have been expressed in terms of u and v according to (10) and (11).

3. A geometric derivation of the solution. Eqs. (11), (10) and (14) define the solution analytically. A purely geometrical derivation can be made by using the following property of the solution: the trajectories intersecting the parametric u-curves at $+45°$ (bisecting the first and third quadrants formed by the parametric curves) are straight lines. The arbitrary function $F(a)$ in (11) corresponds to an arbitrary envelope of those lines. Hence, if an arbitrary one-parameter system of straight lines be given, its $\mp 45°$ trajectories will form an orthogonal equiareal system u, v of the variety as defined by eqs. (11), (10) and (14). Figure 2 belongs to this type, but Figure 1 does not, which reminds us of the fact that the solution found is but a particular one, far from being the general solution of the orthogonal equiareal pattern problem.

To establish the above geometric property by proof, let us determine the isoclinics, $\omega(u, v) = \text{constant}$. Let $\theta = \theta(u, v)$ be the angle as formed by the parametric u-curve and the isoclinic at a point (u, v). By (13), $\sqrt{E}$ is constant along an isoclinic, whence

$$\frac{\partial\sqrt{E}}{\partial u}du + \frac{\partial\sqrt{E}}{\partial v}dv = 0,$$

for a displacement from (u, v) to $(u+du, v+dv)$ along an isoclinic. The lengths of displacements in parametric directions are $\sqrt{E}du$ and $\sqrt{G}dv$. Computing θ and using (2) and (6), we have

$$\tan\theta = \frac{\sqrt{G}\,dv}{\sqrt{E}\,du} = -\frac{\sqrt{G}\,\dfrac{\partial\sqrt{E}}{\partial u}}{\sqrt{E}\,\dfrac{\partial\sqrt{E}}{\partial v}} = \frac{\dfrac{\partial(\sqrt{E}\sqrt{G})}{\partial u} - \sqrt{G}\,\dfrac{\partial\sqrt{E}}{\partial u}}{\sqrt{E}\,\dfrac{\partial\sqrt{E}}{\partial v}} = 1,$$

whence

$$\theta = 45^\circ + k\cdot 180^\circ. \tag{15}$$

Now, as ω is constant along an isoclinic (by the very definition of an isoclinic), $\omega+\theta$ will be constant by (15). But $\omega+\theta$ is the direction angle of the isoclinic itself relative to the x-axis. In other words: the direction of an isoclinic is invariably the same at all points of that isoclinic. This proves that the isoclinics are straight lines. Because of (15) they are the $+45^\circ$ trajectories of the parametric u-curves, which completes the proof.

4. The "equimileage" property. Let us mention briefly a peculiar property of the solution—an "equimileage" property—which certainly is in no general relation to the equiareal property: choosing an arbitrary finite selection of parametric curves as a map of one-way streets in a curvilinear city block system (mediaeval European) to drive in the direction of increasing coordinates only as stated by the following driving rule:

$$\begin{aligned} &\text{either } du = 0 \text{ and } dv > 0,\\ &\text{or } dv = 0 \text{ and } du > 0; \end{aligned} \tag{16}$$

we have for the mileage, s, of a ride from $A(u_1, v_1)$ to $B(u_2, v_2)$

$$\begin{aligned} s &= \int ds = \int \sqrt{Edu^2 + Gdv^2} = \int (\sqrt{E}\,du + \sqrt{G}\,dv)^* \\ &= \int \left(\frac{\partial\phi}{\partial u}\,du + \frac{\partial\phi}{\partial v}\,dv\right) = \int d\phi = \phi_2 - \phi_1. \end{aligned}$$

which is independent of the itinerary chosen (independent of the number and locations of the right and left turns made). We now have a geometric interpretation of the function $\phi(u, v)$: it is the "mileage function" of the map (taxi-cab fare function). Reversing all traffic against arrows, we still preserve the equimileage property with $-\phi(u, v)$ as mileage function. A partial reversal of traffic (on u-streets only, or on v-streets only) would destroy that property, making short-cuts possible since $ds = \pm(\sqrt{E}du - \sqrt{G}dv)$ is not integrable. For illustration, see Figure 2.

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* This, I believe, is true because of (16).

5. Examples. I. Assuming $F(a)=0$ in (11), we get

$$a=\sqrt{\frac{v}{u}}, \qquad \sqrt{E}=\sqrt{\frac{v}{u}}, \qquad \sqrt{G}=\sqrt{\frac{u}{v}}\,.$$

An integration of (14) gives the Cartesian coördinates

$$x = \sqrt{2uv}\cos\left(\log\sqrt{\frac{v}{u}}+45^\circ\right),$$

$$y = \sqrt{2uv}\sin\left(\log\sqrt{\frac{v}{u}}+45^\circ\right).$$

From the above, we readily obtain the polar coordinates

$$r=\sqrt{2uv}, \qquad \theta=\log\sqrt{\frac{v}{u}}+45^\circ,$$

whence, by (13),

$$\omega=\log\sqrt{\frac{v}{u}}=\theta-45^\circ.$$

The parametric uv-curves are logarithmic spirals intersecting the isoclinics $\theta=$constant at $\mp 45^\circ$ respectively. The envelope of the isoclinics, the point $r=0$, is improper. A random set of parametric curves is shown in Figure 3, while a selection forming an equiareal pattern is seen in Figure 2.

II. Assuming in (11) $F(a)=2c$ (with c a positive constant), we get

$$a=\sqrt{E}=\frac{c+\sqrt{c^2+uv}}{u}, \qquad \sqrt{G}=\frac{-c+\sqrt{c^2+uv}}{v},$$

$$\omega=\log\frac{c+\sqrt{c^2+uv}}{u}\,.$$

Integration of (14) gives here

$$x=\sqrt{2}\,c\cos(\omega+135^\circ)+\sqrt{2}\sqrt{c^2+uv}\cos(\omega+45^\circ)$$

$$y=\sqrt{2}\,c\sin(\omega+135^\circ)+\sqrt{2}\sqrt{c^2+uv}\sin(\omega+45^\circ)$$

from which we obtain the polar coordinates r and θ

$$r=\sqrt{4c^2+2uv}, \qquad \theta=\omega+45^\circ+\arcsin\frac{c\sqrt{2}}{r}\,.$$

Using the last equation, it can be shown that the envelope of the rectilinear isoclinics is a circle of radius $c\sqrt{2}$.

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The Department of Discussions and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

ON ELLIPTIC INTEGRALS

IVAN NIVEN, Purdue University

It is common knowledge that every elliptic integral is expressible in terms of integrals of three basic types.* It does not seem to have been noted, however, that every elliptic integral is expressible in terms of integrals of the so-called third type. We prove this by showing that the first and second types are dependent upon the third.

If we write

$$P = (at^2 + b)(ct^2 + d), \qquad abcd \neq 0,$$

then the three types of elliptic integrals can be written, respectively,

$$(1) \qquad I_1 = \int P^{-1/2}dt, \qquad I_2 = \int t^2P^{-1/2}dt, \qquad I_3(N) = \int (1 + Nt^2)^{-1}P^{-1/2}dt.$$

We begin with the identity

$$\frac{d}{dt}(t^3P^{-1/2}) = t^2P^{-1/2} + (d/c + b/a)P^{-1/2} - d/c(1 + ct^2/d)^{-1}P^{-1/2} - b/a(1 + at^2/b)^{-1}P^{-1/2},$$

and integrate to get

$$(2) \qquad t^3P^{-1/2} = I_2 + (d/c + b/a)I_1 - d/cI_3(c/d) - b/aI_3(a/b).$$

Similarly we integrate the identity

$$\frac{d}{dt}(tP^{-1/2}) = -P^{-1/2} + (1 + ct^2/d)^{-1}P^{-1/2} + (1 + at^2/b)^{-1}P^{-1/2}$$

to get

$$(3) \qquad I_1 = -tP^{-1/2} + I_3(c/d) + I_3(a/b).$$

Eliminating I_1 between (2) and (3) we obtain

* Cf. W. E. Byerly, Integral Calculus, 2nd edition, p. 217; or G. Mittag-Leffler, An introduction to the theory of elliptic functions, Annals of Mathematics, vol. 24 (1922–23), p. 292; or Whittaker & Watson, Modern Analysis, 4th edition, p. 515.

$$I_2 = t^3P^{-1/2} + (d/c + b/a)tP^{-1/2} - b/aI_3(c/d) - d/cI_3(a/b). \tag{4}$$

The trigonometric forms of these integrals are

$$E_1 = \int Q^{-1/2}d\phi, \qquad E_2 = \int Q^{1/2}d\phi, \qquad E_3(N) = \int (1 + N\sin^2\phi)^{-1}Q^{-1/2}d\phi$$

where

$$Q = 1 - k^2\sin^2\phi, \qquad k^2 = bc/ad.$$

We obtain results corresponding to (3) and (4) by integrating the identities

$$\frac{d}{d\phi}(Q^{-1/2}\tan\phi) = -Q^{-1/2} + Q^{-1/2}(1 - k^2\sin^2\phi)^{-1} + Q^{-1/2}(1 - \sin^2\phi)^{-1},$$

and

$$\frac{d}{d\phi}(k^2Q^{-1/2}\sin\phi\cos\phi) = Q^{1/2} - (1 - k^2)(1 - k^2\sin^2\phi)^{-1}Q^{-1/2}.$$

Thus we obtain the results

$$E_1 = -Q^{-1/2}\tan\phi + E_3(-k^2) + E_3(-1),$$

and

$$E_2 = k^2Q^{-1/2}\sin\phi\cos\phi + (1 - k^2)E_3(-k^2).$$

APPROXIMATIONS TO A CENTRAL ANGLE

J. M. BRUCE, Chicago, Ill.

Problem 1. To compute a central angle in terms of its arc and the height of its arc.*

Let the central angle θ of a circle of radius r intercept an arc of length a and height h. (See figure.) Then

$$h = r(1 - \cos\theta/2), \qquad r = a/\theta,$$

so that

$$\omega = h/a = (1 - \cos\theta/2)/\theta,$$

$$\omega = \theta/8 - \theta^3/384 + \theta^5/46080 + \cdots. \tag{1}$$

We take as an approximation

$$\omega = \theta/8 - \theta^3/384. \tag{2}$$

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* An approximation to a central angle in terms of its chord and the chord of half its arc has been given by the author; this MONTHLY, vol. 49, 1942, p. 184.

To solve this cubic put $\theta=8\sin\phi/3$. Then (2) becomes

$$\omega = \sin\phi/3 - \tfrac{4}{3}\sin^3\phi/3,$$

or

$$3\omega = \sin\phi.$$

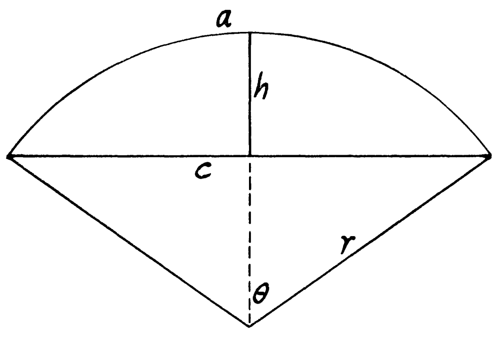
Hence

$$\theta = 8\sin\left(\tfrac{1}{3}\operatorname{arc\,sin} 3\omega\right)$$

is an approximate solution of (1).

The error is about $6'$ when $\theta=90°$.

Problem 2. To compute a central angle in terms of its arc and the chord of its arc.



Letting c be the chord, we have

$$c = 2r\sin\theta/2,$$

which gives, on putting $\theta=4\phi$,

$$\eta = c/a = (\sin 2\phi)/2\phi,$$

$$\eta = 1 - 2\phi^2/3 + 2\phi^4/15 + \cdots.$$

Then

$$-1+\sqrt{1+3\eta} = 1 - \phi^2/2 + 3\phi^4/80 + \cdots.$$

Also

$$\cos\phi = 1 - \phi^2/2 + \phi^4/24 + \cdots.$$

Hence we take for an approximation

$$\cos\phi = -1+\sqrt{1+3\eta}.$$

This gives an error in θ of about $4'$ when $\theta=90°$.

Note by the Editor. The method used by Mr. Bruce can be extended so as to give almost any degree of approximation. The method depends essentially on the selection of a function whose series expansion begins with the same terms as the given series. Thus in Problem 1, we may consider

$$\alpha \sin \beta\theta = \alpha\beta\theta - \frac{\alpha\beta^3}{6}\theta^3 + \frac{\alpha\beta^5}{120}\theta^5 + \cdots .$$

If we put

$$\alpha\beta = \frac{1}{8}, \qquad \frac{\alpha\beta^3}{6} = \frac{1}{384},$$

so that

$$\alpha = \beta = \frac{1}{2\sqrt{2}},$$

we have

$$\frac{1}{2\sqrt{2}} \sin \frac{\theta}{2\sqrt{2}} - \omega = -\frac{\theta^5}{184,320} + \cdots . \tag{3}$$

Hence we have approximately

$$\theta = 2\sqrt{2} \text{ arc sin } 2\sqrt{2}\,\omega. \tag{4}$$

To find the approximate error in θ we note that (3) can be written in the form

$$\sin \frac{\theta}{2\sqrt{2}} = 2\sqrt{2}\,(\omega - \epsilon),$$

where ϵ is about $\theta^5/184,320$. Hence

$$\begin{aligned} \theta &= 2\sqrt{2} \text{ arc sin } 2\sqrt{2}\,(\omega - \epsilon) \\ &= 2\sqrt{2} \text{ arc sin } 2\sqrt{2}\,\omega - 2\sqrt{2}\,\frac{1}{\cos \frac{\theta}{2\sqrt{2}}} \cdot 2\sqrt{2}\,\epsilon + \cdots , \end{aligned}$$

so that the error in (4) is roughly

$$8\epsilon = \frac{\theta^5}{23,040} \cdot$$

This is about 1.5′ when $\theta = 90°$.

A still better approximation can be obtained by comparing the expansions in powers of θ, of $\alpha \sin \beta\theta$ and $\sinh \gamma\omega$. For $\alpha = 1/\sqrt{3}, \beta = 1/\sqrt{23}$, $\gamma = \sqrt{23/192}$, the first three terms of the series agree, and we get the approximation

$$\theta = \sqrt{192/23} \text{ arc sin } (\sqrt{23} \sinh \omega/\sqrt{3}),$$

with an error of about $\theta^7/800{,}000$, or $6''$ when $\theta = 90°$.

In Problem 2 we can use

$$(\cos \alpha\theta)^\beta = \left(1 - \frac{\alpha^2}{2}\theta^2 + \frac{\alpha^4}{24}\theta^4 - \frac{\alpha^6}{720}\theta^6 + \cdots\right)^\beta$$

$$= 1 - \frac{\beta\alpha^2}{2}\theta^2 + \left(\frac{\beta\alpha^4}{24} + \frac{\beta(\beta-1)\alpha^4}{8}\right)\theta^4$$

$$- \left(\frac{\beta\alpha^6}{720} + \frac{\beta(\beta-1)\alpha^6}{48} + \frac{\beta(\beta-1)(\beta-2)\alpha^6}{48}\right)\theta^6 + \cdots.$$

If we put

$$\frac{\beta\alpha^2}{2} = \frac{1}{24}, \qquad \frac{\beta\alpha^4}{24} + \frac{\beta(\beta-1)\alpha^4}{8} = \frac{1}{1920},$$

which gives

$$\beta = \frac{5}{3}, \qquad \alpha = \frac{1}{\sqrt{20}},$$

we have

$$\left(\cos\frac{\theta}{\sqrt{20}}\right)^{5/3} - \eta = \frac{\theta^6}{1134000} + \cdots.$$

Thus, approximately,

$$\theta = \sqrt{20} \text{ arc cos } \eta^{3/5}.$$

As in Problem 1 we find that the error in this approximation is about

$$\sqrt{20}\,\frac{1}{\sin\frac{\theta}{\sqrt{20}}}\,\frac{3}{5}\,\eta^{-2/5}\,\frac{\theta^6}{1134000},$$

which is roughly $\theta^5/94{,}500$ for fairly small values of $\theta/\sqrt{20}$. When $\theta = 90°$, this is about $23''$.

In general the success of this method depends mainly on one's ingenuity in concocting functions with suitable series expansions.

R.J.W.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

Analytic Geometry. By Roscoe Woods. New York, The Macmillan Company, 1939. 14+294 pages. $2.25.

This book is designed as a text for colleges and technical schools; it emphasizes the conservative point of view. With the exception of two chapters, the treatment is confined to rectangular cartesian coordinates. The discussion of the problem of the distance from a line to a point is treated in the usual way, without clinching the significance of the sign, a feature of prime importance in curve tracing. Pencils of lines and of circles, including the radical axis, are discussed, but not of other conics.

The conics are introduced separately, each for itself, but a later chapter treats the general case in a comprehensive way, including poles and polars. A commendable discussion of polar coordinates makes no use of cartesian or other coordinates until the point of view is established independently. Similarly for equations in parametric form.

About fifty pages are devoted to solid analytic geometry. The topics considered include direction numbers, plane and line, and surfaces and curves.

The exercises are numerous, and graded. Answers to the odd-numbered ones are provided, and the others can be obtained from the publisher. The printing, proof-reading, and figures are all well done.

VIRGIL SNYDER

Elements of Spherical Trigonometry. By J. E. Thompson. New York, D. Van Nostrand Co., 1942. 12+144 pages. $1.65.

This book begins by summarizing, with a brief description but no proofs, everything usually studied in a Freshman Course in Plane Trigonometry. It reviews those parts of Solid Geometry which pertain to trihedral angles and spheres. The Law of Cosines for Spherical Triangles is derived from the usual trihedral angle, and all the other formulas are obtained algebraically from this one. A direct geometric proof of the Law of Sines is added as an alternative. Napier's Rules are derived as special cases of the formulas for the general triangle. All triangle solutions are arranged for logarithmic computation, and excellent model forms are given. Checking the solution is stressed in the text, but omitted entirely in the model forms.

There are numerous applications to Solid Geometry, Navigation, and Astronomy. General formulas are derived for the mensuration of a parallelepiped and the regular polyhedrons. A detailed explanation of latitude and longitude

accompanies the problems of great circle sailing. Legendre's Theorem for small triangles is developed and used to obtain the approximate area of a small triangle on the earth, but nothing is said of the accuracy of the approximation. The astronomical applications include finding the time and azimuth of sunrise or sunset, the latitude and local time from a star, and the azimuth and altitude of a star from a given point. For each of these problems, the formula which gives the desired quantity directly in terms of the known one with a minimum of computation, is derived. The right triangle is used very slightly in these applications. A good list of exercises follows each of the last four chapters. The appearance of the book could have been improved by the judicious use of bold face type. There is an excellent index and answers are given for all numerical exercises In spite of its fairly small size, it contains much more Spherical Trigonometry than is actually a prerequisite for Celestial Navigation or Astronomy.

C. E. RHODES

Introduction to the Calculus. Part I. By Samuel Beatty and J. T. Jenkins. Toronto, The University of Toronto Press, 1938. 650 pages, with index.

The aim of this text is clearly shown by two quotations from the authors' preface. "The authors hope to find a place for a type of treatment, which is meant for the student in the first instance and which contains references to the background and applications of the subject, such as are supplied ordinarily by the instructor." Later, "Naturally, this adds to the length of the treatment, without increasing the scope." The reviewer admits frankly that for his own classroom instruction, he prefers to make his own selections from background and applications, choosing them and adapting them for their usefulness and value to the particular class before him at the moment. No doubt the majority of his selections are from material covered in the present book, but a certain pedagogical value accrues in the spontaneous presentation of illustrative material to a class apart from assigned text-book sections. Without much doubt adequate presentation of the material can be found elsewhere in less than six hundred and fifty pages, but the length is due not entirely to the authors' policy of full explanation and a wealth of illustration, but also to the generous spacing of the briefest equations and expressions which adds materially to the clarity of the presentation. The authors and the University of Toronto Press should be complimented for unusually excellent mathematical typography. Misprints are very few and for the most part almost self-correcting. It is unfortunate that figure 40 does not correspond to its discussion in the text, and that orthogonality is not more apparent in figure 89. On page 107 the phrase "acceleration-component" is twice used where "velocity-component" is intended.

If the book be compared with one volume texts in introductory calculus by American authors with which the reviewer is familiar, the adequacy of its scope is apparent. The usual chapters on partial differentiation, multiple integration, and the solution of elementary differential equations are missing, but are to be supplied in the second volume of the authors' work. The reviewer observed the

demonstrated before the student is made acquainted with any formulas of differentiation? Why should a large number of the common applications of integration be treated three separate times, first under the head of "anti-derivation," second, as indefinite integration, and last from the point of view of the fundamental theorem? What purpose is served by intricate operations on series, even on simply convergent series, before an adequate treatment of convergence?

Assuming that the companion volume, Part II, is of similar character, the title used, "An Introduction to the Calculus," is surely modest. They will contain a very complete treatment of the calculus, preparing the student most excellently to pursue the more advanced branches of mathematics which have their foundation in calculus. In the present volume, a persual of the final chapter should convince the reader of this. The excellent proof of the fundamental theorem itself, with extended treatment of its applications, among which especial commendation should be given to the sections on moments, is accompanied by various special results of more than passing interest. Of these one is stated in closing: "Given a closed oval curve A, and let an inner curve B be described by a fixed point on a chord of constant length which moves about A and returns to its original position. The area between A and B is the area of an ellipse whose semiaxes are the segments of the chord."

E. S. HAMMOND

A Study of the Number Concept of Secondary School Mathematics. By H. F. Fehr. (Dissertation, Teachers College. Columbia University, 1940.) Upper Montclair, New Jersey, State Teachers College, 1940. 202 pages. $1.60.

The author begins by defining in a popular way the meaning of the terms Mathematics (Pure and Applied) and shows that by starting with a set of undefined relations—assumed for the sake of argument—we can build a logical structure of theorems. The two important concepts on which all processes depend are those of Number and Function. Function itself can in the ultimate be reduced to Number and the general concept of Number rests in turn on the properties we ascribe to the natural sequence one, two, three, $\cdots$. Thus the concept of number is the basis of all analysis, without which all else must be meaningless.

To most people the word "number" is synonomous with the natural numbers 1, 2, 3, $\cdots$. Even a fraction is not often conceived as a new number, but usually as a relation of two integers, as so many whole parts, when the whole has been divided into equal parts. Zero is the absence of all quantity, a symbol for nothingness. This limited knowledge of number suffices to give man all that is needed to carry on his everyday affairs.

For the person who studies more advanced mathematics, or the theory of its application, the arithmetic of the natural numbers is not sufficient. To explain the length of the diagonal of a rectangle, to describe such phenomena as velocity and force, etc., the natural numbers find used only in a very limited number of cases.

Today the world is using negative numbers as common knowledge. Even such symbols as $\sqrt{3}$, log 2, sin π, are appearing in some of the more popular scientific journals. While the number $\sqrt{-1}$ has not appeared as common knowledge, it can be expected that the history of the negative number can well be repeated in this case also.

If such numbers are to be used, it is essential that the proper concept of these numbers be taught so that the meaning put into them by the layman will lead to real service. This meaning can only come through a fundamental understanding of the operations and extensions of the number system.

Experimental evidence of practical need is one way of introducing negative numbers, but the method of insuring the omnipossibility of the subtraction process is a logical way. This is the concept underlying pure mathematics. If students are to have a thorough knowledge as a basis for future study of mathematics, the logical concepts must find a place in their instruction. It is the purpose of this study to show historically, the practical and logical extension of the number system of elementary secondary algebra, and to create for the teachers of elementary algebra a logical development consistent with modern interpretation.

Many teachers of mathematics have no clear idea of the number concept. The author shows by examples taken from commonly used text-books that the fault is often due to the lack of knowledge of the correct principles which define the number concept.

The following paragraph from the introduction gives clearly the method to be used:

"The logical basis for extending a number system is thru the omnipossibility of certain operations. The operations of high school algebra are addition, subtraction, multiplication, division, involution, and evolution, and these are the only operations here considered. These operations, with the notion of order are sufficient to give a clear understanding of all numbers involved in high school mathematics."

The particular purpose of this study is to create for the teachers of secondary mathematics a practical and a logical concept of the number system of high school mathematics that will enable them to impart their subject with better understanding and more lasting values.

In Chapter II is investigated the historical development of number. Each number system is traced both logically and practically through all the different civilizations. First is investigated the so-called natural numbers; then in turn are treated the fractions, negatives, rationals, real and complex numbers. In Chapter III there is similarly investigated the operations on number, the inverse of these operations and the fundamental laws of number. Particular attention is paid to the law of permanence of mathematical form.

In Chapter IV, there is given a survey of the nature of a logical development, and some recent studies on mathematical thought as applied to the number system. Some of the logical difficulties of the number concept and definition,

especially in regard to the extension of number, are examined but in no manner is a solution of such problems undertaken. The references will indicate to the reader where further study material of this nature is to be found. In Chapter V there is created a complete logical "postulate-definition-theorem-proof" development of the number system of high school algebra. From this study and the writer's own experience, there has resulted a suggested procedure for teaching the various numbers in a meaningful manner. These suggestions and criticisms of current practice are given in Chapter VI.

The above suggestions will indicate the great value of this doctor's thesis to teachers of elementary and high-school mathematics, especially to those who have had little function theory. The author gives a very complete bibliography of the subject and points out the weaknesses in many of the high-school texts.

E. B. Escott

Symmetric Functions in the Theory of Integral Numbers. By Hansraj Gupta. Lucknow University Studies, No. XIV. 1940. 7+105 pages.

The text represents a short study in certain theorems in elementary theory of numbers and their generalizations. These theorems are for the most part in the nature of congruences, such as Fermat's Theorem: $a^{\phi(n)} \equiv 1 \pmod{n}$ for $(a, n) = 1$ and Wilson's Theorem: $(p-1)! \equiv -1 \pmod{p}$ for p prime. The book uses a rather general method for obtaining generalizations of theorems of this type and other interesting congruences. The results are partly known and partly new. Many of the former are contained in Hardy and Wright's Introduction to the Theory of Numbers, in particular, in Chapters VII and VIII of that book.

The author introduces the function $G(n, r)$ which is defined for $0 < r \leqq n$ as the rth elementary symmetric function of the n first positive integers. Thus Wilson's Theorem can be written in the form $G(p-1, p-1) \equiv -1 \pmod{p}$. Similarly, a theorem by Lagrange states that $G(p-1, r) \equiv 0 \pmod{p}$ for $p \geqq 3$ and $1 \leqq r \leqq p-2$ and Wolstenholme's Theorem says that $G(p-1, p-2) \equiv 0 \pmod{p^2}$ when $p \geqq 5$. All these theorems and more general results are obtained in the text by a detailed study of the properties of the function $G(n, r)$.

In Chapter I, the author gives a short introduction into certain aspects of the elementary theory of numbers, which need not be mentioned here in detail. He introduces the notation $\text{pot}_p n$ (p-potency of n) for the highest power of the prime p that divides n and $\text{piq}_p n$ for the highest positive exponent β (if one exists) such that $\phi(p^\beta)$ divides n. Theorems concerning the p-potency of certain binomial coefficients are derived.

Chapter II deals with the theory of primitive roots, with the order of a number a modulo m, if $(a, m) = 1$, and with the k-*ic* residues modulo m. The well-known results concerning these topics are derived. It might be said at this point that some of the proofs in this chapter lack clearness, such as the proofs of Theorems 16 and 17 on pp. 26–28.

Chapter III introduces the function $G(n, r)$, mentioned above; recursion

formulas for this function are derived such as the relation $G(n, r) - G(n-1, r) = nG(n-1, r-1)$, later referred to as the "fundamental relation." The definition of $G(n, r)$ is first extended to all values of n and later to all values of r. Of importance for later applications are a number of theorems, concerning the values of the G-function for negative integral n. At the end of the Chapter, the function $S(n, r)$ (sum of the rth powers of the first n positive integers) and through it the Bernoulli numbers are expressed in terms of the G-function.

The final Chapter IV starts with an extension of the original definition of $G(n, r)$ (see above) and of $S(n, r)$ to $G(A, r)$ and $S(A, r)$, where A is a set of k non-negative integers. In this way more general sets of numbers can be considered than the set of the first n positive integers, for example, the set of the positive integers smaller than and relatively prime to n. If the corresponding functions for the latter set are denoted by $G'(n, r)$ and $S'(n, r)$, respectively, then certain congruences of these functions modulo n^2 are obtained. These congruences represent a generalization of Leudesdorf's Theorem: if $(n, 6) = 1$ then $G'(n, \phi(n) - 1) \equiv 0 \pmod{n^2}$. A very general recursion formula for the G-function leads through specialization to the Theorems of Lagrange, Wilson, Wolstenholme and Fermat which are thus shown to spring from the same source. In the last articles divisibility properties of $S'(n, r)$ and $S(n, r)$ are derived, also the well-known theorem of von Staudt concerning the denominators of the Bernoulli numbers. Finally Wilson's Theorem is generalized to such results as $G(p^u - 1, \phi(p^u)) \equiv -1 \pmod{p}$ and similar ones.

The text contains in the form of a summary a list of symbols, terms, main results, theorems. This seems almost necessary in view of the density of notations, definitions and theorems throughout the book.

The author states at the beginning that the text is written for students who have "gone through a course of numerical arithmetic," that seems to correspond approximately to a course in college algebra. While this may be theoretically true, it can hardly be expected that a student without some experience in theory of numbers and, more generally, in advanced mathematical methods should be able to follow the book without a good deal of trouble.

For anyone interested in the part of the theory of numbers, dealt with in it, Gupta's book certainly constitutes very interesting, though by no means easily digestible, reading.

FRITZ HERZOG

Fundamentals of Radio. By E. C. Jordan, P. H. Nelson, W. C. Osterbrock, F. H. Pumphrey, and L. C. Smeby. W. L. Everitt, Editor. 1 vol., Prentice-Hall, New York, 1942. 400 pages, \$5.00.

The purpose of this volume is to present the basic material of radio for civil or military work. It covers the field very completely and thoroughly, and gives up-to-date information on all matters concerning long, medium and short waves. The ultra short waves or "microrays" are not included. The volume starts with a short summary of mathematics needed in radio. This is very elementary

indeed: a few notions of algebra, trigonometry, vectors and logarithms—no definition of the derivative of a function, no integrals, not even the imaginaries. With such a meager background of mathematics, it is easy to believe that most of the book will be purely descriptive, with a great many statements given without any proof. It is remarkable to see how much of the radio problems the authors are able to explain and discuss under such circumstances. They first start with a short summary of the main laws of electricity, for D.C. and A.C. circuits. A number of drawings and curves are given, to explain fundamental physical laws which should require some higher mathematics for a more precise statement.

Looking at this point of the book, and remembering some others built on the same scheme, one can hardly avoid the conclusion that it would have been much easier for the reader to learn first, and once for all, the meaning of a derivative, than to have to plot curves, draw their tangents, and discover again and again the role played by the "rate of change" of some quantities. In this review of the basic notions of electricity, it is very surprising that nothing is said about electrostatics. How can the reader understand anything on condensers, vacuum tubes, cathodic oscillographs, etc., when the only thing he is told about positive and negative electric charges is (p. 97):

> One of the most fundamental of all electrical concepts is the repelling action of charges of same polarity and the attractive forces between charges of opposite polarity.

This short sentence seems to be the only one in which electrostatic laws are summarized. It is to be hoped that the reader already has some notions of his own; otherwise he might find it hard to understand the properties of the electrons and electronic devices of all sorts.

The laws of electric currents are based on the hydraulic analogy, the volt being called "electric pressure," while a magnetic field is baptized "magnetic pressure"—and this is quite a reasonable way of introducing these notions by comparisons with the laws of liquids. There is a rather loose use of the word "flux," when (p. 61) the authors speak of the "lines of magnetic flux" meaning magnetic induction. The definition of a self-induction coefficient is given (p. 85) without any reference to the magnetic flux in the coil, but this notion appears suddenly on p. 114 for the mutual induction, where it is almost impossible to avoid. This shows how dangerous it is not to suppose a certain minimum of scientific background when discussing technics of such high standards as radio and electronics. A good working knowledge of elementary mathematics and classical physics should really be a necessary prerequisite for any student entering the field of radio; otherwise he will never be able to get any precise and accurate notion of radio itself.

The discussion and explanations of radio problems, as given in this book, are fairly complete, and supported by a great number of curves and diagrams of all sorts, which enable the authors to give a good and clear description of the most important facts and methods. Of course the whole treatment is purely

descriptive, and there are all the time a great many assumptions which the reader has to accept without discussion, as well as statements he should just memorize without any real proof of their validity. In order, however, to help the student, the authors have worked out a number of qualitative explanations which make the results quite plausible. Such methods have their advantages, but also their dangers, and some minor errors can be found, as for instance p. 124, where, speaking of the role played by the space charge in a diode, the authors write "only those electrons emitted with sufficient velocity will have enough energy to penetrate the cloud" and reach the plate. It is well known that electrons in a diode are emitted without velocity, and put into motion by the electric field inside the diode. But for this minor mistake, the chapter on electronic tubes is very good, and contains very clear descriptions of the main properties of triodes, tetrodes, and pentodes of modern designs. The discussion of the different types of amplifiers, with a clear distinction of classes *A. B. C.*, and very numerous diagrams of explanation, is certainly the best part of the book. The description of wave propagation conditions is pretty good, and summarizes in twenty pages a great many valuable informations on the Heaviside layer and its practical importance. A few words on frequency modulation give a fair account of this new development of radio. The book will certainly be found very valuable for young engineers entering the field of radio, with very little scientific preparation, and it will enable them to get a fairly accurate view of the main principles and methods of radio.

L. BRILLOUIN

NEW BOOKS RECEIVED

Laboratory Geometry. By Elizabeth Roudebush. New York, Prentice-Hall, 1942. 5+192 pages. $1.12.

Basic Mathematics. By W. W. Hart. Boston, D. C. Heath and Co., 1942. 6+456 pages. $1.52.

A Review of Arithmetic. By Z. L. Smith. Chicago, Institute of Military Studies, University of Chicago, 1942. 37 pages. $0.25.

Plane Trigonometry. By E. R. Heineman. New York and London, McGraw-Hill Book Co., Inc., 1942. 12+167 pages. $2.00.

Spherical Trigonometry. By R. W. Brink. New York, D. Appleton-Century Co., Inc., 1942. 8+62 pages. $0.75.

The Non-Singular Cubic Surface. A new method of investigating with special reference to questions of reality. By B. Segre. Oxford, University Press and Humphrey Milford, 1942. 11+180 pages. $4.25.

Descriptive Geometry for Engineering and Architectural Draftsmen. By F. A. P. Fischer and J. A. Clear. Milwaukee, The Bruce Publishing Co., 1942. 7+244 pages. $3.00.

Plane and Spherical Trigonometry. By F. A. Rickey and J. P. Cole. New York, The Dryden Press, 1942. 10+209 pages. $2.25.

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Allegheny College, Meadville, Pa.

SOME APPLICATIONS OF THE SLIDE RULE

A number of the club reports this year indicate an interest in the use of the slide rule. Students who have mastered the technique of reading the scales, and who know how to multiply, divide, and extract square roots, are ready to learn more tricks. The tricks shown below are but a few of the many which might serve to illustrate the power of the slide rule as a device for speeding computation.

1. Vector resultants. An important problem in physics is to determine the magnitude and direction of the resultant of two mutually perpendicular vectors. This is equivalent to solving a right triangle in which a and b are given lengths and $\angle C=90°$. Let us assume that b is the longer of the two given sides, so that $a<b$. Then to find A and c, we use the formulas

$$\text{(1)} \qquad \frac{\tan A}{1}=\frac{a}{b}, \qquad 1+\left(\frac{b}{a}\right)^2=\left(\frac{c}{a}\right)^2.$$

Setting a on the **C** scale opposite b on **D**, the angle A is read on the **T** scale opposite the right hand index of the **D** scale. (On some rules this is found under a special hairline on the back side.) Without changing the setting, we read $(b/a)^2$ on the **A** scale opposite the left hand index on **B**. Adding 1 to this number we obtain $(c/a)^2$. Then moving the slide to the right until the index on the **B** scale is opposite $(c/a)^2$ on the **A** scale, the hypotenuse c can be read on the **D** scale opposite the known value a on the **C** scale. Thus the magnitude and direction of the resultant vector are obtained from two settings of the rule.

Example: Given $a=105$, $b=208$. To find A and c. Set 105 on **C** opposite 208 on **D**. Read $\angle A=36°47'$ on **T** opposite the right index on **D**. Also read $(b/a)^2=3.92$ on the **A** scale opposite the left index on **B**. Add 1. Then, setting the left index on **B** opposite 4.92 on **A**, read $c=233$ on **D** opposite 105 on **C**.

2. Graphs of ellipses and hyperbolas. The coordinates of several points on the ellipse $(x^2/a^2)+(y^2/b^2)=1$ may be computed easily by the slide rule, if the equation is written in the form

$$\text{(2)} \qquad \frac{y^2}{a^2-x^2}=\frac{b^2}{a^2}. \qquad \begin{matrix}\textbf{A scale}\\ \textbf{B scale}\end{matrix}$$

The slide is set either by placing a^2 on the **B** scale opposite b^2 on the **A** scale, or

by placing a on the **C** scale opposite b on the **D** scale. For various simple values of x, the values of a^2-x^2 can be computed mentally. (For small integral values of x the values of x^2 are known and for half-integers we note that $(n+\frac{1}{2})^2=n(n+1)+\frac{1}{4}$. Other squares can be read off the slide rule itself, using the **D** and **A** scales). Then with the hairline over a^2-x^2 on the **B** scale we read the corresponding value of y on the **D** scale.

For the hyperbola $(x^2/a^2)-(y^2/b^2)=1$, we use the same set-up except that x^2-a^2 is used in place of a^2-x^2.

3. The parabola in polar coordinates. To compute the polar coordinates of points on the parabola

$$r = 2a/(1 + \cos\theta),$$

it is best to use a trigonometric identity and write

$$r = a + a \tan^2(\theta/2). \tag{3}$$

With the appropriate index of the **T** scale opposite a on the **A** scale we read $a\tan^2(\theta/2)$ on **A** opposite $\theta/2$ on **T**. Adding a, we obtain r.

4. Accurate determination of angles. It is a fact that an acute angle θ can be determined more accurately from $\tan\theta$ or $\cot\theta$ than from $\sin\theta$ or $\cos\theta$, and more accurately from $\log\tan\theta$ or $\log\cot\theta$ than from $\log\sin\theta$ or $\log\cos\theta$. It is also true that angles are determined more accurately from the **T** scale on a slide rule than from the **S** scale. Thus, given $\cos\theta=12/13$, we may read on the **S** scale that $90°-\theta=67°30'$ and hence that $\theta=22°30'$, but we are uncertain of the answer within $10'$. On the other hand, if we write $\tan^2\theta=(13/12)^2-1=25/144$, and set the right index of the **T** scale opposite 14.4 on the **A** scale, we find $\theta=22°37'$ on **T** opposite 2.5 on **A**, and the angle θ is accurate to the nearest minute.

As another example, let $\sin\theta=\sqrt{5/6}$. We have $\sin^2\theta=5/6$, $\cos^2\theta=1/6$, $\tan^2\theta=5$, $\tan^2(90°-\theta)=1/5$. Setting the right hand index on **T** opposite 5 on **A**, we read $90°-\theta=24°6'$ on **T** opposite 1 on **A**. Hence $\theta=65°54'$, accurate to the nearest minute. This is a much more exact result than could have been read from the **S** scale.

If the given angle θ is near 45° it is a good idea to determine θ from $\cos 2\theta$. For example, given $\sin\theta=21/29$, then $\cos 2\theta=1-2\sin^2\theta=-41/841$, and we have $\sin(2\theta-90°)=41/841$. Setting the right index of the **S** scale opposite 84.1 on the appropriate scale (**A** or **D** according to the type of slide rule) we read $2\theta-90°=2°48'$ on the **S** scale opposite 4.1, correct to within $20''$. Hence $\theta=46°24'$ correct to within $10''$.

5. Solution of a cubic equation. An irreducible cubic equation in which the quadratic term is absent may be written in the form

$$ax^3 - bx + 1 = 0, \tag{4}$$

by dividing by the constant term. The roots are the abscissas of the intersections of the two curves

$$y = ax^2 + (1/x), \qquad y = b. \tag{5}$$

A setting of the rule is made with the index of the **B** scale opposite a on the **A** scale. Then when the hairline covers x on the **C** scale it will cover ax^2 on the **A** scale and $(1/x)$ on the **CI** scale. Starting with an approximate value of x on **C**, the hairline is moved until the sum y of the readings on **A** and **CI** has the given value b. Then x is read on the **C** scale.

Example: To find the roots of the equation $3x^3-10x+6=0$, we first note that there are three roots, lying respectively between 1 and 2, between 0 and 1, and between -3 and -2. We then write, as above in (5),

$$y = 0.5x^2 + (1/x), \qquad y = 1.667. \tag{5a}$$

We set the index on **B** opposite 0.5 on **A**, and take as first approximations for the roots the values 1.25, 0.75 and -2.00, respectively. With successive approximations to each root we finally obtain in each case $y=1.667$ as the sum of the readings on **CI** and **A**.

Scale		First root				Second root			Third root		
C	x	1.25	1.40	1.36	**1.368**	0.75	0.70	**0.705**	-2.00	-2.10	$-$**2.073**
CI	$(1/x)$	.80	.71	.735	.731	1.33	1.429	1.418	-0.50	-0.48	-0.483
A	$0.5x^2$	.78	.98	.925	.936	.28	.245	.249	2.00	2.20	2.150
Sum $=y$		1.58	1.69	1.660	1.667	1.61	1.674	1.667	1.50	1.72	1.667

Thus the roots are 1.368, 0.705, and -2.073. More accurate determinations by algebraic methods are 1.367953, 0.705219, -2.073171. As a check we use the fact that the sum of the three roots is 0.

CLUB REPORTS 1941–42

Pi Mu Epsilon, Hunter College

The topic for the year was the *Theory of Matrices*. The papers were: *Vector analysis* by Gertrude Mandel and Charlotte Brown, *Homogeneous linear equations* by Ilse Novak, *Orthogonal complements* by Lucy La Sala, *Non-homogeneous linear equations* by Edris Adams and Helen Trief, *Bilinear forms* by Mae Reiner, *The characteristic equations* by Pearl Schutz, *Determinant of a matrix* by Evelyn Levine and Barbara Samson, *Postulate systems* by Brenda Lansdown, *Matrices with polynomial elements* by Elaine Frankel, *Tschirnhaus transformations* by Blossom Klein and Julia Rubin, *Resultants and discriminants* by Harriet Eisenberg. The officers were: President, Louise Miller; Recording Secretary, Claire Schwartz; Treasurer, Hortense Schindler; Corresponding Secretary, Phyllis Monderer; Faculty Adviser, Professor C. C. MacDuffee.

Mathematics Club, Hunter College

The Mathematics Club held seventeen bi-monthly meetings this year. Faculty and visiting speakers were: Dr. Rosario Candela who spoke on *Cryptography*, Professor Marston Morse on *Equilibrium points*, Professor Jewell Bushey on *Finite geometry*, Mr. Edward Molina of the Bell Telephone Laboratories on *Applications of probability to telephone engineering*, Dr. Raymond Lorch

of Columbia University on *Rings*. Students speakers included Miss Phyllis Monderer, *Proof of the law of gravitation*, and Miss Julia Rubin, *Statistics in Psychology*. The officers: were President, Gertrude Mandel; Vice President, Rosa Weill; Secretary, Evelyn Levine; Treasurer, Julia Rubin; Publicity Manager, Dora Feldman; Assistant Publicity Manager, Harriet Eisenberg; Faculty Adviser, Professor Mina Rees.

Mathematics Club, University of Buffalo

The program for 1941–42 included the following topics: *Mathematical oddities* by Joan Searles and Wallace Barnes, *Curve fitting by the method of least squares*, the paper which won the Sherk Prize for the year 1940–41, by Ruth Euller, *The mathematics of bridge* by Robert Rittenhouse, *Codes and ciphers* by Marvin Messler, *A ladder problem* by Lois Obenauer, *Proving axioms of geometry by algebraic methods* by Jack Castle, *Aesthetic measure of polygons* by Annabel Miller, *An evening at Monte Carlo* by Wallace Barnes, *Airplane designs* by Edward Wallenhorst. Each year the club invites high school students who are interested in mathematics to attend one of the meetings at which a suitable topic is presented. Various informal games and mathematical recreations are planned after the formal meeting. The club officers would welcome an exchange of ideas with other clubs for the planning of the recreation period. Officers for 1941–42 were: President, Joseph Ullman; Vice President, Annabel Miller; Secretary-Treasurer, Ruth Brendel; Faculty Adviser, Professor Harriet Montague.

Pi Mu Epsilon, University of Alabama

The following programs were presented: *Polynomials* by Dr. Paul Hummel, *Klyston, a new ultra high frequency generator*, by Dr. F. H. Mitchell, *Unique factorization* by Dr. H. S. Thurston, *Polynomials* by Dr. R. W. Cowan, *Phases of descriptive geometry* by Dr. W. G. Warnock. The officers for 1941–42 were: Director, Ria Jane Clinkscales; Vice-Director, Louise McClanahan; Secretary, William F. Adams; Treasurer, B. G. Clark; Librarian, Dr. W. P. Ott.

Mathematics Club, Boston University

Professor J. L. Coolidge of Harvard gave a most successful talk on *How the Greeks solved quadratic equations*, and Professor R. M. Frye of the Physics Department presented a paper on *Ballistics*. There was also a joint meeting at Harvard with the Greater Boston Association of College Mathematics Clubs, at which the principal speaker was Professor E. B. Mode, who spoke on *Finite differences*. The officers for 1942–43 are as follows: President, Betty Barry, Vice-President, John Haynes; Secretary, Mary Siteman; Treasurer, Michael Argeros; Executive Committee, Daniel Feer, Earl Maby, Minnie Young; Faculty Adviser, Professor Robert Bruce.

Mathematics Club, St. John's University

The club at its first meeting of the year voted to confine its activities as far as possible to matters concerned with national defense. Accordingly lectures on navigation, surveying, exterior and interior ballistics, bombing, and the slide rule were given by various members of the faculty. One meeting was devoted to an *Information Please* program in which the club members attempted to answer mathematical questions that had been sent in by vaious students and faculty members. Officers were: President, Howard Bryant; Vice-President, Gerard Saur; Secretary-Treasurer, Gerard O'Donnell; Publicity Director, Richard Fantin; Faculty Adviser, Professor J. J. McCarthy.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 551. *Proposed by J. H. Butchart, Grinnell College*

Show that the arcs common to a circular cylinder and a right helicoid whose axis lies on the cylinder are pieces of a circular helix.

E 552. *Proposed by V. Thébault, San Sebastián, Spain*

In certain scales of notation, a number of the form $aabb$ can be the square of a number of the form cc, where c is a multiple of b. Show that (1) a and b are relatively prime; (2) b and $c^2/b-a$ are perfect squares.

E 553. *Proposed by N. A. Court, University of Oklahoma*

If two of the four circles of intersection of two spheres with two planes are cospherical, prove that the remaining two circles are likewise cospherical.

E 554. *Proposed by J. L. Woodbridge, Philadelphia*

Show that n cuts can divide a cheese into as many as $(n+1)(n^2-n+6)/6$ pieces.

E 555. *Proposed by Howard Eves, Syracuse University*

Consider a rectangular parallelepiped of dimensions $a\times b\times c$, made up of abc unit cubes. Imagine the edges of all these unit cubes replaced by material wires, common edges sharing the same wire. Prove that the exterior surface of the resulting wire network is of genus $p=2abc+bc+ca+ab$ (*i.e.*, that it is topologically equivalent to a sphere with p handles).

SOLUTIONS

Fourth Powers of Odd Numbers

E 512 [1942, 195]. *Proposed by V. Thébault, San Sebastián, Spain*

Find the first n odd numbers whose sum divides the sum of their fourth powers.

Solution by F. A. Butter, Jr., University of Southern California

A trivial solution is given by $n=1$, so we suppose that $n>1$. Let

$$T_1(n) = 1 + 3 + \cdots + (2n-1) = n^2,$$
$$T_4(n) = 1^4 + 3^4 + \cdots + (2n-1)^4,$$

and

$$S(N) = 1^4 + 2^4 + \cdots + N^4 = N(N+1)(2N+1)(3N^2+3N-1)/30.$$

Evidently

$$T_4(n) = S(2n) - 2^4 S(n) = n(4n^2-1)(12n^2-7)/15.$$

We must therefore find the least integer $n>1$ such that

$$n(4n^2-1)(12n^2-7) = 15kn^2,$$

where k is an integer. This can be written

$$n(15k - 48n^3 + 40n) = 7,$$

whence $n=7$ necessarily. In fact

$$\begin{aligned} T_4(7)/T_1(7) &= (1^4 + 3^4 + \cdots + 13^4)/(1 + 3 + \cdots + 13) \\ &= 52871/49 = 1079. \end{aligned}$$

Also solved by Adolph Barjansky, D. H. Browne, W. E. Buker, R. E. Crane, Howard Eves, J. F. Heyda, C. A. Murray, E. P. Starke, and the proposer.

Editorial Note. By the Euler-Maclaurin Summation Formula (Boole, *Finite Differences*, p. 90), we have

$$\begin{aligned} \Delta^{-1}(2x)^4 &= \int (2x)^4 dx - \tfrac{1}{2}(2x)^4 + \tfrac{1}{12}D(2x)^4 - \tfrac{1}{720}D^3(2x)^4 \\ &= \tfrac{16}{5}x^5 - 8x^4 + \tfrac{16}{3}x^3 - \tfrac{8}{15}x + C \\ &= \tfrac{8}{15}x(x-1)(2x-1)(3x^2-3x-1) + C \end{aligned}$$

$=f(x)$, say. Therefore

$$\begin{aligned} 1^4 + 3^4 + \cdots + (2n-1)^4 &= f(n+\tfrac{1}{2}) - f(\tfrac{1}{2}) = f(n+\tfrac{1}{2}) - C \\ &= \tfrac{16}{15}n(n^2 - \tfrac{1}{4})(3n^2 - \tfrac{7}{4}) \\ &= \tfrac{1}{15}n(4n^2-1)(12n^2-7), \end{aligned}$$

as before.

Equal Intercepts between Three Planes

E 513 [1942, 195]. *Proposed by N. A. Court, University of Oklahoma*

A line revolves about a fixed point in such a manner that the segment intercepted on it by two intersecting planes has its mid-point in a third given plane. Show that the locus of the variable line is a cone of the second degree.

Solution by Howard Eves, Syracuse University

The cone generated will be of the second degree if a general plane through the vertex cuts the cone in two, and only two, generators. Let P be the vertex and ω

a plane through P. Let ω cut the three given planes in lines a, b, c, line c lying in the third plane. Then, from the conditions of the problem, a and b (in general) intersect. Thus the desired result follows from a known theorem which tells us that in ω there are just two lines through P such that c bisects the segment cut off by a and b. This may be proved as in the *Educational Times*, *Reprints*, vol. 6, 1904, p. 103, Question 15456, or else as follows.

Let P have coordinates (x_1, y_1) referred to the intersecting lines a and b as (oblique) cartesian axes. Then the equation of any line through P has the form

$$y - y_1 = k(x - x_1).$$

We readily find the points where this meets the axes to be $(x_1 - y_1/k,\ 0)$ and $(0,\ y_1 - kx_1)$. The point (x, y) midway between these is given by

$$x = \tfrac{1}{2}(x_1 - y_1/k), \qquad y = \tfrac{1}{2}(y_1 - kx_1).$$

Eliminating k we obtain, as the locus of the midpoint, the conic

$$2xy - xy_1 - yx_1 = 0.$$

Any third line c can cut this conic in at most two points.

Also solved by Adolph Barjansky, L. M. Kelly, E. P. Starke, P. D. Thomas, and the proposer.

Editorial Note. The use of oblique axes suggests the following direct solution. Let the three planes be $x=f$, $y=g$, $z=h$. The line $x/l=y/m=z/n$ meets the first where $z=nf/l$, and the second where $z=ng/m$. The conditions of the problem require the mean of these two values of z to be h, whence

$$\frac{f}{l} + \frac{g}{m} = \frac{2h}{n}\cdot$$

Thus the desired locus is the cone $f/x+g/y=2h/z$, or

$$fyz + gzx - 2hxy = 0.$$

An Exponential Series

E 514 [1942, 195]. *Proposed by J. A. Bullard, University of Vermont*

Find the sum $\sum_{k=0}^{n-1} e^{kx} \sin (y+kz)$.

Solution by B. H. Bissinger, Cornell University

Let $t=x+iz$. Then, assuming x, y, and z to be real, we see that the desired sum is the imaginary part of

$$\sum_{k=0}^{n-1} e^{kt+iy} = \frac{e^{nt}-1}{e^t-1}\, e^{iy} = \frac{(e^{nx+i(y+nz)} - e^{iy})(e^{x-iz}-1)}{(e^{x+iz}-1)(e^{x-iz}-1)},$$

provided $e^t \neq 1$. Thus

$$\sum_{k=0}^{n-1} e^{kx} \sin (y+kz) = \frac{e^{(n+1)x} \sin \{y+(n-1)z\} - e^{nx} \sin (y+nz) - e^x \sin (y-z) + \sin y}{e^{2x} - 2e^x \cos z + 1},$$

unless $x=0$ and z is a multiple of 2π, in which case the sum is simply $n \sin y$.

Also solved by F. A. Butter, Jr., E. P. Starke, and the proposer.

Commensurable Triangles

E 515 [1942, 195]. *Proposed by H. T. R. Aude, Colgate University.*

Find all the triangles with integral sides which have one side equal to 16 units and the cosine of an adjacent angle equal to $-\frac{1}{4}$.

Solution by E. P. Starke, Rutgers University

Let the other sides be a and b, with a opposite the stated angle. The law of cosines gives $a^2=b^2+256+8b$, or

$$(a+b+4)(a-b-4) = 240.$$

If the factors of the left member are equated to the possible pairs of even integral factors of 240, the greater factor being equated to $a+b+4$, the following systems are obtained:

$$a+b+4 = 120,\ 60,\ 40,\ 30,\ 24,\ 20;$$
$$a-b-4 = \ \ 2,\ \ 4,\ \ 6,\ \ 8,\ 10,\ 12.$$

These yield the pairs of values

$$(61, 55),\quad (32, 24),\quad (23, 13),\quad (19, 7),\quad (17, 3),\quad (16, 0)$$

for (a, b). There are thus exactly five non-trivial triangles of the desired kind.

Also solved by Adolph Barjansky, F. A. Butter, Jr., Howard Eves, L. M. Kelly, P. D. Thomas, and the proposer.

A Balanced Incomplete Block Design

E 516 [1942, 257]. *Proposed by W. E. Buker, Pittsburgh Public Schools*

During one week, Mr. X invited just four of his seven friends for dinner each night. The invitations were arranged so that any given pair of guests dined together on just two occasions, and for any two given nights there were two guests who were present both times. Show how this was managed. In how many different ways can it be done?

Solution by W. B. Campbell, Philadelphia

Among the 7 guests there are $\binom{7}{2}=21$ guest-pairs, to be put in one-to-one correspondence with the $\binom{7}{2}$ night-pairs, so one of the stated requirements is redundant. If a given quadruple of guests, such as $ABCD$, appears on one night, each of the 6 pairs contained in it must also appear on just one of the other 6 nights. Any pair from the remaining guests must appear on two nights on which the two pairs from the given quadruple do not overlap, as otherwise there would be more than one repeated guest-pair for that night-pair; *e.g.*, if EF is associated with AB, it is also associated with CD. Since there are 3 available pairs in EFG, and 3 pairs of distinct pairs in $ABCD$, the association can be affected in $3!=6$ ways. Thus each of the $\binom{7}{4}=35$ available quadruples generates 6 different ar-

rangements of 7 compatible quadruples; but as each arrangement could be derived from any one of its 7 constituent quadruples, the actual number of arrangements is $35\cdot 6/7=30$. Considering permutations of the quadruples in a given arrangement over the 7 nights, we find the total number of admissible lists to be

$$30\cdot 7! = 151200$$

(still ignoring the order of seating at the table). A typical arrangement is *ABCD ABEF, CDEF, ACEG, BDEG, ADFG, BCFG.*

Also solved by E. P. Starke.

Editorial Note. This problem was taken from R. D. Carmichael, *Groups of Finite Order*, Boston, 1937, p. 25, Ex. 15, 16. A cyclic solution (in terms of the residues modulo 7) is given by

$$0124,\ 1235,\ 2346,\ 3450,\ 4561,\ 5602,\ 6013$$

and is represented graphically by the white squares of the second "anallagmatic tessellation" drawn on p. 109 of Rouse Ball, *Mathematical Recreations and Essays* London, 1939. The guests whom Mr. X did *not* invite, each night, form the famous triple system of the finite geometry $PG(2, 2)$, whose collineation group is of order 168 (Carmichael, *op. cit.*, p. 22). Therefore the number of such arrangements is $(7!)^2/168=151200$, as above.

Unequal Tetrahedra

E 517 [1942, 257]. *Proposed by V. Thébault, San Sebastián, Spain*

It two tetrahedra have equal areas for corresponding faces, do they necessarily have the same volume?

Solution by E. P. Starke, Rutgers University

A negative answer is indicated by specific cases. The tetrahedra with vertices

$$(0, 0, 0),\quad (2, 0, 0),\quad (0, 2, 0),\quad (\sqrt{3}, -\sqrt{3}, 1);$$
$$(0, 0, 0),\quad (2, 0, 0),\quad (0, 2, 0),\quad (1, 1, \sqrt{3})$$

have equal areas for corresponding faces (viz. 2, 2, 2, $\sqrt{6}$), but the volumes are $2/3$ and $2/\sqrt{3}$. (Note that one pair of corresponding faces are congruent.) There is some interest in the tetrahedron

$$(0, 0, 0),\quad (6, 8, 0),\quad (12, 0, 0),\quad (6, -1, 3\sqrt{7}),$$

whose four faces are congruent isosceles triangles of area 48 each, and whose volume is $48\sqrt{7}=127.0$. A regular tetrahedron of edge $8\cdot 3^{1/4}$ has the same area for each face, but its volume is $2^{15/2}\ 3^{-1/4}=137.5$.

Also solved by Howard Eves and the proposer.

Three Gaussian Integers

E 518 [1942, 257]. *Proposed by J. Rosenbaum, Bloomfield, Conn.*

Find three Gaussian integers x, y, z which satisfy the equation

$$x^p + y^p = z^p$$

for every prime p greater than 3.

Solution by the Proposer

$$x = 1 + i\sqrt{3}, \qquad y = 1 - i\sqrt{3}, \qquad z = 2.$$

Since $x/2$ and $y/2$ are sixth roots of unity, we merely have to verify $x^n+y^n=z^n$ for $n=\pm 1$, and we can deduce the same for $n=6k\pm 1$ where k is any rational integer.

Ballistics

E 519 [1942, 257]. *Proposed by Paul Brock, Brooklyn College*

If a projectile is aimed at a given point from a given origin, find the direction to which the two possible paths are equally inclined initially.

Solution by E. A. Nordhaus, University of Wisconsin in Milwaukee

If the initial slopes of the two paths from the origin to the point (x, y) are $m_1=\tan\alpha_1$ and $m_2=\tan\alpha_2$, then m_1 and m_2 are the roots of the equation (for m)

$$x^2m^2 - kxm + x^2 + ky = 0,$$

where $k=2v_0^2/g$. (See W. D. MacMillan, *Statics and the Dynamics of a Particle*, New York, 1927, p. 254.) On using the expressions for the sum and product of the roots, we find

$$\tan(\alpha_1 + \alpha_2) = -x/y = \cot(-\theta),$$

where $\tan\theta=y/x$. Then $\alpha_1+\alpha_2=\theta+90°$, and the required direction-angle is

$$\tfrac{1}{2}(\alpha_1 + \alpha_2) = \tfrac{1}{2}\theta + 45°.$$

Also solved by Howard Eves, L. M. Kelly, G. A. Yanosik, and the proposer.

Similar Figures

E 521 [1942, 335]. *Proposed by J. R. Musselman, Western Reserve University*

(a) On the sides BC and CA of a triangle ABC, construct externally any two directly similar triangles, CBA_1 and ACB_1. Show that the midpoints of the three segments BC, A_1B_1, CA form a triangle directly similar to the two given triangles.

(b) On BC externally, and on CA internally, construct any two directly similar triangles CBA_1 and CAB_1. Show that the midpoints of AB and A_1B_1 form with C a triangle directly similar to the two given triangles.

Solution by Howard Eves, Syracuse University

The two theorems are very special cases of the fundamental theorem concerning two directly similar figures: If the lines joining corresponding points of two directly similar figures be divided proportionally, the locus of the points of division will be a figure directly similar to the given figures.

$A_1B_1C_1D_1$; A, *etc.* denote the areas of the faces BCD, *etc.* and $2S=A+B+C+D$.

Note. The formula (2) is due to Genty, *Nouvelles Annales de Mathêmatiques*, 1880, p. 528 and 1881, p. 341.

SOLUTIONS

Cyclic Numbers

4012 [1941, 639]. *Proposed by V. Thébault, San Sebastián, Spain*

Find a number of n digits $N_0=a_1a_2 \cdots a_n$, $(a_1\neq 0)$, such that if we transpose the first k digits from left to right, $(k=1, 2, 3, \cdots, n-1)$, the $n-1$ numbers thus obtained $N_1=a_2a_3 \cdots a_na_1$, $N_2=a_3a_4 \cdots a_na_1a_2, \cdots, N_{n-1}=a_na_1a_2 \cdots a_{n-1}$ are each multiples of N_0.

Solution by E. P. Starke, Rutgers University

If we exclude the trivial case where the digits of N_0 are all alike and $N_k=N_0$ for every k, we may show that N_0 is 142857 (the repetend of the decimal expansion of 1/7) or is a number made up of two or more repetitions of these digits.

Notice first that, for every j,

$$a_1 \leqq a_j, \tag{1}$$

for otherwise N_{j-1}, being less than N_0, is not a multiple. If a_1 is not the smallest digit of N_0, let $j+1$ be the least subscript for which $a_{j+1}=a_1$. Then N_j is neccessarily equal to N_0, and $a_{j+2}=a_2$, $a_{j+3}=a_3, \cdots$: N_0 consists of the repetend $N_0' =a_1a_2 \cdots a_j$ written in succession two or more times; and, furthermore, N_0' has the stated property as well as N_0.

Suppose $a_1=1$. Then, because 1 is the final digit of N_1, N_1/N_0 is not an even integer and not 5. Let $N_1/N_0=3$, $(a_1=1)$. Then $a_2a_3 \cdots a_n1/1a_2 \cdots a_{n-1}a_n=3$ implies $a_n=7$; $a_2a_3 \cdots a_{n-1}71/1a_2 \cdots a_{n-1}7=3$ implies $a_{n-1}=5$; similarly $a_{n-2}=8$, $a_{n-3}=2$, $a_{n-4}=4$. If $n=6$, so that $a_{n-4}=a_2=4$ is the leading digit of N_1, the process is complete, giving $N_0=142857$. If $n>6$, the above sequence is repeated: $a_{n-5}=1$, $a_{n-6}=7$, $a_{n-7}=5, \cdots$, giving N_0 made up of two or more repetitions of 142857. There exists no value of a_2 (with $a_1=1$) for which $a_2 \cdots /1a_2 \cdots =N_1/N_0$ can equal 7 or 9.

For $a_1=2$ and $N_1/N_0=2$, $a_2 \cdots a_n2/2a_2 \cdots a_n=2$ implies $a_n=6$ (since by (1) $a_n\neq 1$), but there exists no a_{n-1} such that $a_2 \cdots a_{n-1}62/2a_2 \cdots a_{n-1}6=2$. For $a_1=2$, $N_1/N_0=3$, $a_2 \cdots a_n2/2a_2 \cdots a_n=3$ implies $a_n=4$ and then $a_{n-1}=1$, contrary to (1). Similar elementary considerations eliminate all other values for a_1 and N_1/N_0, and thus establish the initial statement.

Editorial Note. A generalized form of the problem is as follows:

Find the numbers $N=d_1d_2 \cdots d_q$, $d_1>0$, in a scale of radix s, which do not consist of consecutive sets of the same digits in the same order, and such that

$$N_k = d_{k+1}d_{k+2} \cdots d_qd_1 \cdots d_k = r_kN, \qquad k = 1, 2, \cdots, q-1, \ r_q = 1.$$

From the given conditions it is obvious that $q\geqq 2$ and $r_k<s$; and it can be shown by Starke's reasoning that $d_1<d_k$, $k\neq 1$, and that $r_i\neq r_j$, $i\neq j=1, 2, \cdots, q$. Also we have

$$
(1)\qquad \begin{gathered} N = P_k s^{q-k} + Q_k, \qquad N_k = Q_k s^k + P_k, \qquad P_q = N, \qquad Q_q = 0, \\ P_k = d_1 d_2 \cdots d_k, \qquad Q_k = d_{k+1} d_{k+2} \cdots d_q. \end{gathered}
$$

From these relations we have

$$
(2)\qquad N(s^k - r_k) = P_k(s^q - 1), \qquad N(r_k s^{q-k} - 1) = Q_k(s^q - 1).
$$

Let u be the g.c.d. of N and s^q-1 so that $N=nu$, $s^q-1=pu$, where n and p are relatively prime. We then have from (2)

$$
(3)\qquad n(s^k - r_k) = P_k p, \qquad n(r_k s^{q-k} - 1) = Q_k p, \qquad k = 1, 2, \cdots, q,
$$

and from the first equations of (3) considered in turn we must have $n|d_1$, $n|d_2$, $\cdots$, $n|d_q$. After removing this common divisor n of the digits, if $n>1$, we have another number $\overline{N}$ which obviously satisfies all of the requirements; we shall suppose that this has been done and in order to avoid new notations we shall write

$$
(4)\qquad s^k - r_k = P_k p, \qquad r_k s^{q-k} - 1 = Q_k p, \qquad s^q - 1 = Np.
$$

It now follows that p and s are relatively prime, p and r_k are relatively prime, N is the repetend of the repeating decimal for $1/p$, and that $p<s$. The first set of equations in (4) give the ordinary arithmetic process (P) for calculating the decimal $1/p$, and $q \geqq 2$ must be the smallest integer for the given p for which $r_q=1$ in order to avoid the repetition of digit sequences in N. It now follows from the process (P) for developing $1/p$ that $r_k<p<s$ and we again find that no two of the multipliers, or residues, are equal. Thus (P) is also a process for finding q. We have also $3 \leqq p < s-1$, and that p is not a divisor of $s-1$.

In order to find all of the solutions for a given s we consider all the positive integers $p \geqq 3$, prime or composite, less than $s-1$, prime to s and not divisors of $s-1$. If no one of these values of p yields a solution by means of (P), then the problem has no solution. If (P) yields solutions we first obtain all such and then examine each as follows. If for a solution given by p, there exists a greatest positive integer $a>1$ such that $ad_k<s$, $1 \leqq k \leqq q$, for this solution, then to this solution there correspond a solutions with the same q and the same set of multipliers r_k. If a' is an integer such that $2 \leqq a' \leqq a$ and a' is a divisor of p, $p=a'p'$, then this solution is given by (P) for p'. If a' is not a divisor of p, we have a solution which is not given by (P). When this is done for all of the (P) solutions we have all of the solutions of the problem.

The first value of s for which there is a solution is $s=5$, and the following table gives the number n_s of solutions of both types for each s, $5 \leqq s \leqq 19$. The value $s=19$ is the only one of the table which has a solution of the second type, a number of two digits 2(10) which is obtained from the solution of two digits 15 given by (P) for $p=15$, with $r_1=4$, $r_2=1$.

s	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
n_s	1	0	2	2	2	1	6	2	6	4	4	4	11	4	13

Necessary and sufficient conditions that there exists for a given s a solution of the second type are as follows:

There must be a p which gives a solution N of the first type with multipliers r_k, $1 \leqq k \leqq q$, $r_q = 1$, and there must be a positive integer r' less than p and prime to p which is not one of the q multipliers r_k for p. Finally we must have $r'r_k < p$, $1 \leqq k \leqq q$. Then $r'N$ is a solution of the second type with the same multipliers r_k as for N.

This theorem reduces the labor of constructing a table for n_s; its proof follows without difficulty from what is given above but seems long, and this is left to the reader. A large number of special cases of s having numbers of the second type may be obtained as follows: Let r' and a be two positive integers each greater than unity, and set $r_1 = r'a$, $r_k = r_1^k$, $1 \leqq k \leqq q-1$, $r_q = 1$, $q \geqq 2$, $p = r_1^q - 1$, $s = p + r_1$. Then an N of the first type is given by $d_k = r_1^{k-1}$, $1 \leqq k \leqq q-1$, $d_q = r_1^{q-1} + 1$, and $r'N$ is of the second type. If for example $q = 2$, $r' = 2$, $a = 4$, we get $p = 63 = 3 \cdot 21$, $s = 71$. This suggests trying $p = 21$, $s = 29$, $r' = 2$, $r_1 = 8$, and it turns out that this gives also a second type. By trial we find that no s, $19 < s < 29$, yields a second type.

The proposer remarked that the problem and solution result from theorems of his article in the *Gazeta Matematica*, Bucharest, 1935, pp. 385–392. A remark by Walter B. Carver states that the unique solution is the repetend of the decimal for $1/7$. He stated also that, if the requirement $a_1 \neq 0$ is omitted, there are many solutions obtained similarly from primes p for which 10 is a primitive root, for example $1/17$ gives a solution where $a_1 = 0$. He referred to the paper *On cyclic numbers*, by Solomon Guttman, in this MONTHLY [1934, 159]. The paper mentioned by Carver refers to an earlier one by Robert E. Moritz, in this MONTHLY [1927, 33], which begins with a remark that the properties of the number of the problem have been known for some years.

Squares within a Square

4014 [1941, 704]. *Proposed by P. Erdös, University of Pennsylvania*

Show that, if S_1 and S_2 are two squares contained in the unit square so that they have no point in common, the sum of their sides is less than unity.

It is very likely true that, if we have k^2+1 squares contained in the unit square so that no two of them have a point in common, the sum of their sides is less than k.

Solution by A. Seidenberg, Johns Hopkins University

If two squares S_1 and S_2 are contained in a unit square and have no point in common, then the sum of their sides is less than one.

Proof: Since the squares S_1 and S_2 have no point in common, a line l having no points in common with S_1 or S_2 can be drawn dividing the unit square into two parts U_1 and U_2, with U_1 containing S_1 and U_2 containing S_2. Moreover l may be taken so as to form with the two diagonals of the unit square a non-isosceles right triangle. Let P be the vertex of the acute angle of this triangle which is greater than 45 degrees. Then P must lie inside the unit square as other-

wise l would lie entirely outside the open unit square. Let P lie on the diagonal A_1A_2, A_1 in U_1, A_2 in U_2. Since the angle at P is greater than 45 degrees, the square with diagonal PA_1 lies inside U_1 and the square with diagonal PA_2 lies inside U_2. The theorem then follows from the inequalities:

$$\text{side of } S_i < \frac{\sqrt{2}}{2} PA_i, \qquad i = 1, 2.$$

These inequalities follow immediately from the following theorem.

THEOREM. *The square of maximum area contained in a right triangle has a vertex at the right angle of the triangle.*

Proof: Let the right triangle ABC have its vertices at $A=(a, 0)$, $B=(0, b)$, $C=(0, 0)$, $a>0$, $b>0$, in some rectangular cartesian coordinate system. Consider the set Σ of squares $RSTU$ such that R is on $y=0$, S is on $x=0$, and T is on $bx+ay-ab=0$, where RST is a right angle. Clearly, if there is a maximum square contained in ABC then it has a vertex on each side of the triangle and, for a proper lettering of the triangle, it may be supposed to be an element of Σ. By elementary considerations, one sees that S is equidistant from the line AB and the line m through R perpendicular to AB. Hence the equation of m if $S=(0, s)$ is $-ax+by-s(a+b)+ab=0$, so that R is the point $(r, 0)$ where

$$ar = -s(a+b)+ab \tag{1}$$

Let $R_1S_1T_1U_1$ be the element of Σ inside ABC for which $R=(0, 0)$. Let $R_2S_2T_2U_2$ be the element of Σ inside ABC for which T_2U_2 is on the line AB. These elements are unique because, for example, $S=(0, ab/(a+b))$ and $R=(ka, 0)$, $S=(0, kb)$ where $k(a^2+ab+b^2)=ab$.

When $s=ab/(a+b)$ then $r=0$; when $s=kb$, $r=ka$. Hence, because of the minus sign in (1),

$$r < 0 \quad \text{if} \quad s > ab/(a+b), \qquad r > ka \quad \text{if} \quad s < kb.$$

In the first case, when $s>ab/(a+b)$, R is outside the triangle ABC because $r<0$, and in the second case, when $s<kb$, U is outside the triangle because the slope of TU is greater than that of AB. Thus $RSTU$ is inside ABC only if $kb \leqq s \leqq ab/(a+b)$.

By (1), $(RS)^2=r^2+s^2$ is quadratic in s, and since $(RS)^2\to\infty$ as $s\to\infty$, $(RS)^2=d(s)$ attains a minimum (and not a maximum). Hence to obtain the maximum square $RSTU$ lying within ABC it remains only to compare $R_1S_1T_1U_1$ with $R_2S_2T_2U_2$. A simple inequality shows that $R_1S_1T_1U_1$ is the larger. This completes the proof.

Editorial Note. The auxiliary theorem may be proved in another way. If S is a square inside any given triangle ABC, a homothetic transformation with center A carries S into a square S_a with a vertex on BC which is the transform of the vertex of S nearest BC so that S_a lies inside ABC. Similarly using the cen-

ter B, the square S_a goes into S_b with a vertex on BC and another on CA; and finally, using the center C, S_b goes into S_c inside ABC with a vertex on each side, where $S \leqq S_a \leqq S_b \leqq S_c$.

Suppose now that C is a right angle and that the notation is such that $S_c = RSTU$ has the consecutive vertices R, S, T on CA, CB, AB, where angle $SRC = \theta$, $0 < \theta < \pi/2$, and a is the length of a side of S_c. Then the square of side a with the vertices U' on CA, R' at C, S' on BC has the fourth vertex T' actually inside ABC. For, with CA, CB as x, y axes the slopes of TT' and TU are respectively,

$$\frac{1 - \sin\theta - \cos\theta}{1 - \sin\theta}, \qquad -\frac{\sin\theta}{\cos\theta};$$

and the value of the left expression is less than that of the right since $\cos\theta(1 - \sin\theta) < 1 - \sin\theta$. Also T' is nearer CA than T, and hence T' lies actually inside ABC. Thus the maximum square inside ABC has one side along CA, another along CB, and the fourth vertex on AB.

A proof of the conjecture concerning k^2+1 squares, stated by the proposer in the second sentence of this problem 4014, will still be welcomed.

Tetrahedron with Variable Volume

4015 [1941, 704]. *Proposed by N. A. Court, University of Oklahoma*

If the base of a variable tetrahedron is fixed and the opposite vertex varies on a fixed sphere, the volume of the tetrahedron is numerically equal to the power of the variable vertex with respect to another fixed sphere.

Solution by C. E. Springer, University of Oklahoma

Let the vertices A_i of the tetrahedron have coordinates (x_i, y_i, z_i, t_i) $(i = 1, 2, 3, 4)$ with $t_i = 1$, and let A_1 be the variable vertex which lies on the fixed sphere $x^2+y^2+z^2=r^2$, so that $x_1^2+y_1^2+z_1^2=r^2$. If X_1, Y_1, Z_1, T_1, are the cofactors of x_1, y_1, z_1, t_1 in the determinant $|x_1y_2z_3t_4|$, the numerical value of the volume V is given by $6V = |x_1y_2z_3t_4| = x_1X_1 + y_1Y_1 + z_1Z_1 + t_1T_1$, and the latter expression is six times the power of the point (x_1, y_1, z_1, t_1) with respect to the fixed sphere $(x-a)^2+(y-b)^2+(z-c)^2=R^2$, where

$$a = \mp X_1/12, \qquad b = \mp Y_1/12, \qquad c = \mp Z_1/12,$$
$$R^2 = a^2 + b^2 + c^2 + r^2 \mp T_1/6.$$

These two spheres may be both real, or one real and the other an imaginary or null sphere.

Solved also by the proposer.

The proposer's solution is based upon and follows easily from the theorem in his *Modern Pure Solid Geometry*, p. 184, art. 581; and he stated that the two desired spheres are both real if the plane of the base cuts the given sphere. He also stated that the analogous problem in the plane was considered in *Educational Times, Reprints*, vol. 48, 1893, p. 51, Q. 11294.

Squares with Distinct Digits

4020 [1942, 64]. *Proposed by V. Thêbault, San Sebastian, Spain*

With three consecutive odd digits form a number of five digits whose square has ten distinct digits.

Solution by E. P. Starke, Rutgers University

Three digits of N are 1, 3, 5, or 3, 5, 7 or 5, 7, 9. N^2, being composed of the digits $0, 1, 2, \cdots, 9$, is divisible by 9; whence N is a multiple of 3. Thus the digits of N must be 1, 3, 3, 3, 5 or 1, 1, 3, 5, 5, or 3, 3, 3, 5, 7 or 3, 5, 5, 7, 7 or 5, 7, 9, 9, 9 or 5, 5, 7, 7, 9. Permutations (150 in all) of these digits give those values of N formed from three consecutive odd digits whose squares consist of nine or ten digits and are divisible by 9. If, after discarding all of these which are less than 31622 ($<10^{9/2}$), one squares the others, the desired numbers are obtained by inspection. But since a square which presents duplication of digits in its first few and last few places may be eliminated, most of the labor of squaring may be avoided by comparison with a table of squares. Upon carrying out this program, one obtains:

$$35337^2 = 1248703569, \qquad 35757^2 = 1278563049, \qquad 75759^2 = 5739426081.$$

Editorial Note. The proposer gave the same results without indicating his method of derivation.

A Differential Operator

4021 [1942, 127]. *Proposed by Orrin Frink, Jr., Pennsylvania State College*

The differential operator D^2+1 may be factored in many ways; for example, it may be written $(D+\cot x)(D-\cot x)$, or $(\sec x\cdot D)(\cos x\cdot D+\sin x)$, or $(\sin x\cdot D+2\cos x)(D\csc x)$. Show that the most general method of expressing the differential operator $(D+a)^2+b^2$ as the product of two real first order differential operators is given by the formula

$$(D+a)^2+b^2 = r^{-1}[D+a-b\tan(bx+c)-r'/r]\cdot r[D+a+b\tan(bx+c)],$$

where a, b, and c are real numbers, and r is a differentiable function of x.

Solution by R. P. Agnew, Cornell University

The problem is that of characterizing functions p, q, r and s of x such that

$$(1) \qquad [(D+a)^2+b^2]y = (pD+q)(rD+s)y$$

is, for each function $y(x)$ having two derivatives, an identity in x over some interval of values of x. Upon expanding the operators and equating powers of D, we obtain the relations

$$(2) \qquad ps'+qs = a^2+b^2; \qquad pr'+ps+qr = 2a; \qquad pr = 1.$$

The existence of the derivatives r' and s' and the validity of the relations (2) may also be established by setting $y=1$, $y=x$, and $y=x^2$ successively in (1). From the third of equations (2), we see that $r\neq 0$ and $p=r^{-1}$. It turns out that

q and s can also be expressed in terms of r in the following way. Setting $p=r^{-1}$ in the first two of the equations (2) and eliminating q from the resulting equations gives

$$rs' - sr' = s^2 + 2ars + (a^2 + b^2)r^2. \tag{3}$$

Since $r \neq 0$, we can divide (3) by r^2 and set $u=s/r$ in the result to obtain $u'=u^2 - 2au+a^2+b^2$. In case $b\neq 0$, we obtain

$$\frac{d}{dx}\tan^{-1}\frac{u-a}{b} = \frac{bu'}{(u-a)^2+b^2} = b \tag{4}$$

and accordingly $u=a+b\tan(bx+c)$ where c is a constant. Thus

$$s = ru = r[a + b\tan(bx+c)]. \tag{5}$$

Putting the expressions for p and s in the second of equations (2) and solving for q gives

$$q = r^{-1}[a - b\tan(bx+c) - r'/r]. \tag{6}$$

Conversely, if (5) and (6) hold and $p=r^{-1}$, where r is a nonvanishing differentiable function and c is a constant, then the three equations (2) hold when x lies in an interval over which $\tan(bx+c)$ is defined. Thus, when $b\neq 0$, the result is as stated; an interval over which the factoring is valid is an interval over which $bx+c$ is not an odd multiple of $\pi/2$.

When $b=0$, the differential equation involving u becomes $u'=(u-a)^2$. The solution $u=a$ of this equation leads to the formula

$$(D+a)^2 = r^{-1}[D + a - r'/r]\cdot r[D+a]. \tag{7}$$

Over each interval in which u is differentiable and different from a, the equation $u'=(u-a)^2$ implies that

$$\frac{d}{dx}\frac{1}{u-a} = \frac{-1}{(u-a)^2}\frac{du}{dx} = -1 \tag{8}$$

and hence that $(u-a)^{-1}=c-x$ and $u=a+(c-x)^{-1}$ where c is a constant. On computing q and s for this case, we obtain the formula

$$(D+a)^2 = r^{-1}[D + a - (c-x)^{-1} - r'/r]\cdot r[D + a + (c-x)^{-1}] \tag{9}$$

where again r is a differentiable function and c is a constant. The factoring in (9) is valid over the infinite interval $x<c$ and over the infinite interval $x>c$. The factorings in (7) are obtained by setting $b=0$ in the factoring of the statement of the problem; those in (9) are not.

Solved also by E. A. Nordhaus, John Williamson, and the proposer. In these solutions the case of $b=0$ was not considered.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, N. Y.

Applications for Benjamin Peirce Instructorships at Harvard University for the academic year 1943–1944 should be sent to the Chairman of the Department of Mathematics. Candidates should have received the Ph.D. degree or have had equivalent training.

Dr. G. A. Sarton of Harvard University delivered, on October 13, the Averill Lecture at Colby College.

Professor V. G. Grove of Michigan State College has resigned as chairman of the department of mathematics.

Professor E. R. Hedrick, retired vice-president of the University of California, and Professor Emeritus Virgil Snyder of Cornell University have been appointed visiting professors in mathematics at Brown University.

Assistant Professors M. R. Hestenes and W. T. Reid of the University of Chicago have been promoted to associate professorships.

Associate Professor C. G. Killen of Louisiana State Normal College has been promoted to a professorship.

Professor A. C. Lunn of the University of Chicago has retired after forty years of service.

Professor E. B. Mode has been made chairman of the department of mathematics at Boston University.

Professor E. E. Moots of Cornell College, Iowa, has been elected Associate Director of Research on Sound Control at Cruft Laboratory, Harvard University.

Dr. Haim Reingold is now in charge of the Mathematics Division of the Communication Engineering Courses of the Signal Corps at the Illinois Institute of Technology.

Assistant Professor M. F. Smiley of Lehigh University is now a Lieutenant (j.g.) in the U. S. Navy, and is teaching at the U. S. Naval Academy.

Dr. Alfred Tarski, formerly of the University of Warsaw, has joined the faculty of the University of California.

Dr. H. P. Thielman of the College of St. Thomas has been appointed an assistant professor at Iowa State College.

Professor A. D. Campbell, chairman of the department of mathematics of Syracuse University, was drowned in Green Lake, New York, on September 23, 1942. He was a charter member of the Mathematical Association.

Professor I. M. DeLong, since 1925 emeritus professor of mathematics at the University of Colorado, died September 2, 1942. He was a charter member of the Mathematical Association.

Dr. C. N. Haskins, Chandler Professor of Mathematics at Dartmouth College, died November 13, 1942, at the age of sixty-eight.

Professor J. F. Reilly of the State University of Iowa died August 18, 1942, at the age of sixty-six. He was a charter member of the Mathematical Association and served as secretary of the Iowa Section from 1921 to 1933.

FROM THE EDITOR

The Editor-in-Chief wishes to express his indebtedness to many persons during the first year of his editorship. At the beginning certain artistic and minor typographical improvements in the cover of the MONTHLY and in its internal arrangement were made. In this he was assisted by Miss Mary D. Alexander of the University of Chicago Press as well as by experts of the Banta Publishing Company.

For assistance in reading and appraising papers presented to the MONTHLY he wishes to express his appreciation of the work of the following:

A. A. Albert, R. C. Archibald, I. A. Barnett, L. M. Blumenthal, Louis Brand, H. E. Bray, H. E. Buchanan, Herbert Busemann, W. H. Bussey, W. B. Carver, R. V. Churchill, N. A. Court, A. B. Coble, H. S. M. Coxeter, C. J. Coe, L. W. Cohen, N. B. Conkwright, W. M. Davis, J. L. Doob, Otto Dunkel, Philip Franklin, Orrin Frink, Jr., M. R. Hestenes, R. V. Heath, Dunham Jackson, R. A. Johnson, L. C. Karpinski, A. J. Kempner, J. E. LaFon, R. E. Langer, N. H. McCoy, E. J. McShane, Karl Menger, W. E. Milne, J. R. Musselman, E. J. Moulton, Ivan Niven, Rufus Oldenburger, I. E. Perlin, H. B. Phillips, Tibor Radó, W. T. Reid, M. A. Sadowsky, Otto Schilling, W. S. Snyder, Norman Steenrod, M. H. Stone, R. J. Walker, H. S. Wall, J. H. Weaver, Marie J. Weiss, L. R. Wilcox and R. L. Wilder.

In an early issue of the MONTHLY a Department of War Information will be instituted. This will be under the editorship of Professor C. V. Newsom, University of New Mexico, Albuquerque, New Mexico, to whom all pertinent items should be sent.

SUGGESTIONS FROM THE OFFICE OF SCIENTIFIC PERSONNEL OF THE NATIONAL RESEARCH COUNCIL

As soon as the Army and Navy training programs are in full operation there will be an unprecedented demand for teachers of physics and of mathematics. The situation will be particularly critical in the field of physics where the teaching ranks of colleges and universities have already been seriously depleted.

It is the business of the Office of Scientific Personnel to assist in the placing of the scientific specialist where he can serve the war effort. Because the present supply of readily available physicists approximates zero and the supply of mathematicians is running low, perhaps this office can assist best by suggesting two sources close at hand to the institutions which will participate in the Army and Navy training programs.

The first source of supply is the nearby institutions which will not have Army or Navy training programs. Although these institutions should continue to teach physics to as many students as possible, some departments of physics will find it impossible to continue in operation and their staff members should be added to departments in need of their services.

The reason why all departments of physics which can possibly do so should continue in operation is that it is probable that only the needs of the Army and the Navy will be satisfied through the training programs. The further needs of war research, war industry, and teaching must be satisfied, for the most part, by women and others ineligible for military service. In this connection it should be mentioned that there seems to be no possibility of meeting the need for competent teachers of physics in the secondary schools.

Accordingly, it is important that colleges which are not fortunate enough to secure training contracts should continue to teach physics to even larger numbers than before. So great is the need for men and women trained in physics that every effort should be made to recruit into departments of physics all students with the necessary aptitude. There must be many students with aptitude for physics who do not normally find their way into this field.

These considerations make it clear that expanding departments should not take teachers from other institutions unless it is necessary for them to find employment elsewhere. The other source of supply, and the one that should be utilized wherever possible, is within the institution itself.

In any college or university there are teachers of other subjects, including such fields as botany, geology, physiology, psychology, and zoology (as well as the obvious fields of chemistry and engineering from which a few may be recruited), who have sufficient knowledge of physics so that, with a little brushing up and some observation of good physics teachers at work, they can become proficient teachers of beginning physics. These men and women should be found at once and encouraged to prepare themselves for the teaching service which they will almost certainly be called upon to perform either at their own institutions or elsewhere.

Still another consideration should be kept in mind by physics departments in making their adjustments to these new demands—all teaching and research in

physics must be directed to war ends. Advanced undergraduate and graduate work will soon make only light demands upon staff time except in the small number of institutions where the Army and Navy training programs will call for advanced work. Therefore, in many institutions there will be physicists who can serve the war effort better by going into war research than by teaching beginning physics. Institutions must release such men freely and replace them from the two sources already mentioned which promise to furnish an adequate supply. There seems to be no hope of being able to satisfy the present and future needs for war research and it is urged that all physicists who are as good research men as they are teachers of beginning physics should register with this office at once indicating when they will be available.

Everything that has been said with regard to physics applies, although to a lesser degree, in the field of mathematics.

It is the hope of this office that most of the required readjustments in the staffs of physics and mathematics departments may be worked out by drawing in new staff members from nearby institutions or by adjustments within the institutions themselves. This office will facilitate such adjustments in every way possible if they cannot be worked out locally and will be glad to receive information regarding institutional needs and available personnel. This will be done, however, only with the understanding that we will attempt to prevent bidding of institution against institution in mad competition for personnel. It is to the interest of all concerned that a sufficient supply of teachers of physics and mathematics should be developed and that the necessary readjustments be made easily and efficiently.

HOMER L. DODGE
Director, Office of Scientific Personnel
National Research Council

2101 Constitution Avenue
Washington, D. C.
January 11, 1943

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

The following is a list of the Sections of the Associations, with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Notre Dame, Ind., April 9–10, 1943
INDIANA, Notre Dame, April 9–10, 1943
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, Brooklyn, N. Y., May 8, 1943
MICHIGAN, Notre Dame, Ind., April 9–10, 1943
MINNESOTA
MISSOURI, Kansas City
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, Jan. 30, 1943
OHIO, Columbus, April 1, 1943
OKLAHOMA
PHILADELPHIA, Philadelphia, Nov. 27, 1943
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 13, 1943
SOUTHWESTERN
TEXAS, Lubbock, April, 1943
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May 7, 1943

MATHEMATICAL ASSOCIATION OF AMERICA

The office of the Secretary-Treasurer of the Mathematical Association, which has been at Oberlin, Ohio, since the organization in 1915, has been moved to Ithaca, New York, as of January 1, 1943, with the beginning of Professor Walter B. Carver's term as Secretary-Treasurer. All business correspondence, all subscriptions to the

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THE APPLICATIONS OF MATHEMATICS IN METEOROLOGY*

B. HAURWITZ, Massachusetts Institute of Technology

1. Introduction. The aim of this paper is to enumerate the mathematical prerequisites which are needed to take up the study of modern meteorology. The discussion will be confined to the more elementary topics. This limitation appears especially justified at the present time because the five institutions which offer complete training courses in meteorology have shortened and simplified their curricula, owing to the large and urgent demand for meteorologists in the armed services. At the Massachusetts Institute of Technology, for instance, meteorology was taught in a two-year course to graduate students. After a successful completion of this course, the students received the degree of Master of Science. At present the course is reduced to a period of nine months at all the institutions which offer these meteorological courses. The student body is made up mainly of Aviation Cadets, and further of Naval Ensigns and some civilian students. The Aviation Cadets are, after successful completion of the course, commissioned as Second Lieutenants in the Army Air Corps. At present, only a small fraction of the civilian students remain for two years in order to qualify for the Master's Degree.

The most important practical function of the meteorologist in wartime, and, for that matter, in peacetime, is weather forecasting. Consequently, the meteorological courses, especially now, stress those problems which are closely related to the analysis of the present weather and to the prognosis of the future weather. Since the advances of the forecasting technique are largely due to the introduction of physical principles, in particular due to the application of the appropriate laws of thermodynamics and hydrodynamics, a knowledge of these subjects and the mathematical tools necessary for their understanding is indispensable for the modern meteorologist. The branch of meteorology which studies the application of the laws of thermodynamics and hydrodynamics to meteorological problems is called "dynamic meteorology," and it is mainly dynamic meteorology where a knowledge of some mathematics is necessary.

It may be mentioned here that statistical methods are also widely used in meteorology. The statistical approach permits one to overcome to a certain extent the handicap which arises out of the fact that it is impossible to conduct controlled experiments in meteorology. But the use of mathematical statistics in meteorology will not be discussed here since statistical methods are not required in daily forecasting, even though the results of statistical investigations may be of great assistance to the forecaster.

2. The hydrodynamic equations in meteorology. A study of dynamic meteorology would logically begin with the hydrodynamic equations. In the introductory courses in dynamic meteorology it is, however, preferable to begin in a

* Presented at the meeting of the Mathematical Association of America at Vassar College, September 1942.

more elementary fashion by considering simple cases of balanced motions. But for the sake of completeness we shall here start out with the hydrodynamic equations.

Let $\mathbf{v}$ denote the velocity vector of a particle of air, $\mathbf{\Omega}$ the vector of the earth's rotation, ρ the density, p the pressure, g the acceleration of gravity, $\boldsymbol{\kappa}$ the unit vector directed vertically upward. Then the equations of motion and of continuity are:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} + 2\mathbf{\Omega} \times \mathbf{v} = -\frac{1}{\rho}\nabla p - \boldsymbol{\kappa} g, \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\mathbf{v}) = 0. \tag{2}$$

Counting the vector equation (1) as three equations, *four* equations are available to determine the five unknown variables, *viz.*, the three velocity components u, v, w, the pressure p, and the density ρ. The number of equations is equal to the number of unknown variables only if the fluid is incompressible because the density ρ can, under this condition, be regarded as a known parameter. But in a study of atmospheric motions the compressibility has in many cases to be taken into account. A fifth equation is then required to complete the system of equations. Such an equation can be obtained from physical considerations. Often it may be assumed that the relation between pressure and density changes follows the adiabatic law of compression and expansion, so that the adiabatic relation would complete the system of equations. After the introduction of appropriate boundary conditions for a particular problem, the motion is fully determined by the equations, the boundary conditions and the initial state of the fluid. The assumption of adiabatic motion has, for instance, been made mostly in the study of atmospheric wave motions.

In elementary instruction in dynamic meteorology such advanced problems are not taken up. The topics which are discussed comprise rather simple matters which are nevertheless of great practical importance. Some of these topics will here be reviewed briefly in order to show the approach of dynamic meteorology to its problems. A more detailed discussion can be found in the text books on dynamic meteorology.*

3. The geostrophic wind. One of the variables which is entered on the weather maps is the atmospheric pressure reduced to sea level. Its horizontal distribution is represented by the "isobars," that is, by lines connecting points of equal pressure. A study of the weather maps shows that the wind velocity is approximately proportional to the horizontal pressure gradient and that, in the northern hemisphere, the wind direction is such that, facing downwind, one has the lower pressure to the left, the higher pressure to the right. These empirical results can easily be explained with the aid of the equation of motion (1). The

* D. Brunt, Physical and Dynamical Meteorology, Second Edition, Cambridge, 1939.
B. Haurwitz, Dynamic Meteorology, New York, 1941.

vertical component of the motion may be neglected, and it may be assumed that the horizontal wind components u and v are independent of space and time. Then, from (1)

$$
\begin{aligned}
-2\Omega_z v &= -\frac{1}{\partial}\frac{\partial p}{\partial x}, \\
2\Omega_z u &= -\frac{1}{\rho}\frac{\partial p}{\partial y},
\end{aligned}
\tag{3}
$$

where $\Omega_z = \omega \sin \phi$, the vertical component of the vector of the earth's rotation at the place under consideration, ω is the angular velocity of the earth's rotation, and ϕ is the latitude. If, for the sake of convenience, the coordinate system is rotated around the z-axis until the y-axis coincides with the wind direction, it follows that

$$
\begin{aligned}
v &= \frac{1}{\rho}\frac{1}{2\omega \sin \phi}\frac{\partial p}{\partial x}, \\
u &= 0.
\end{aligned}
\tag{4}
$$

Hence the velocity of the wind is proportional to the pressure gradient, and the wind direction is perpendicular to the pressure gradient. If the pressure increases in the positive x-direction, v is positive, that is, the wind blows in the direction of the positive y-axis. It follows that an observer facing downwind (in the direction of the positive y-axis) has the higher pressure to his right (in the direction of the positive x-axis), the lower pressure to his left. This result holds for the northern hemisphere. In the southern hemisphere ϕ is negative, since the latitude has to be counted negative. Consequently, the low pressure is to the left of the wind in the southern hemisphere.

In the layers next to the ground, up to about 3000 feet, the geostrophic wind relation is not as well satisfied as at higher levels, owing to the effect of the friction at the earth's surface. The wind here does not blow perpendicular to the pressure gradient but makes a smaller angle with it.

With the aid of the geostrophic wind relation one can frequently plot the pressure distribution more accurately than would otherwise be possible, especially over the oceans where, even in peacetime, observations are fewer and more widely apart than over land. The influence of friction is much smaller over the water than over land and much more uniform, so that a correction for friction can easily be applied to the geostrophic wind relation. The intensity and direction of the wind indicate according to the geostrophic relation the direction of the isobars and their distance in the environment of the point of observation. If pressure and wind are given at places which are far apart, a more accurate representation of the pressure distribution may thus be obtained by noting not only the pressure at each station, but also the wind velocity which indicates the distance between the isobars in the neighborhood of each station, this distance being equivalent to the pressure gradient.

The geostrophic relation permits one, furthermore, to obtain a fairly reliable estimate of the wind velocity above the layer of frictional influence, that is, at levels of 3000 feet, approximately. Such estimates are particularly desirable when direct observations of the winds at higher levels are impossible, due to overcast skies. At greater heights wind estimates based on the distribution of the surface pressure become, however, less satisfactory, since the pressure gradient changes as a rule with the altitude. But it is possible to take the effect of this vertical variation of the pressure gradient on the wind into account by means of the so-called thermal wind.

4. The thermal wind. To determine this variation of the geostrophic wind with the height the equations (3) have to be differentiated with respect to the height coordinate z. Since not the density but the pressure and temperature of the air are measured in meteorology, it is appropriate to eliminate the density from the two equations (3) with the aid of the equation of state for ideal gases:

$$\rho = \frac{p}{RT}$$

where R is the gas constant for air, T the absolute temperature. The use of this equation is justified, since the air may be treated as an ideal gas at the temperatures and pressures which occur in the atmosphere. Hence

$$\begin{aligned} \frac{\partial}{\partial z}\left(\frac{v}{T}\right) &= \frac{R}{2\Omega_z}\frac{\partial}{\partial z}\left(\frac{\partial\, ln\, p}{\partial x}\right), \\ \frac{\partial}{\partial z}\left(\frac{u}{T}\right) &= -\frac{R}{2\Omega_z}\frac{\partial}{\partial z}\left(\frac{\partial\, ln\, p}{\partial y}\right). \end{aligned} \tag{4}$$

It is convenient to write the equations in this form, since T is in general a function of z. Since hydrostatic equilibrium can be assumed in vertical direction

$$g = -\frac{1}{\rho}\frac{\partial p}{\partial z},$$

or substituting from the equation of state

$$\frac{g}{T} = -R\frac{\partial\, ln\, p}{\partial z}. \tag{5}$$

On the right side of the two equations (4) the order of the partial differentiations may be interchanged. It follows with the aid of (5) that

$$\begin{aligned} \frac{\partial}{\partial z}\left(\frac{v}{T}\right) &= \frac{g}{2\Omega_z}\frac{1}{T^2}\frac{\partial T}{\partial x}, \\ \frac{\partial}{\partial z}\left(\frac{u}{T}\right) &= -\frac{g}{2\Omega_z}\frac{1}{T^2}\frac{\partial T}{\partial y}. \end{aligned} \tag{6}$$

These two relations show that the variation of the geostrophic wind in vertical direction depends on the horizontal temperature gradient, a quantity which can easily be obtained from the observations. In the lower part of the atmosphere, the troposphere, the temperature decreases at a linear rate with the altitude

$$T = T_0 - mz.$$

If it is assumed that the rate m of the vertical variation of the temperature is independent of the horizontal coordinates, equations (6) can be integrated. Integrating between the surface, indicated by the subscript "0," and the level z

$$(7)\qquad \begin{aligned} v &= v_0 \frac{T}{T_0} + \frac{gz}{2\Omega_z} \frac{1}{T_0} \frac{\partial T_0}{\partial x}, \\ u &= u_0 \frac{T}{T_0} - \frac{gz}{2\Omega_z} \frac{1}{T_0} \frac{\partial T_0}{\partial y}. \end{aligned}$$

These relations give a fairly reliable estimate of the upper winds from observations of the surface wind and of the temperature field if direct observations of the upper winds are not available, for instance, due to low clouds. Under such weather conditions, reliable estimates of the upper winds are particularly important for air navigation.

If, on the other hand, the vertical wind distribution is known but the temperature distribution unknown, equations (7) can be used to obtain some information about the horizontal temperature distribution. As an example, suppose that the wind is from the south at the surface and from the west at 10,000 feet, while its intensity is the same at both heights. Choosing the x-axis positive eastwards, the y-axis positive northwards, the assumptions imply that

$$\begin{aligned} u_0 &= 0, \qquad & v_0 &> 0, \\ u &> 0, & v &= 0, \end{aligned}$$

and hence $\partial T_0/\partial y < 0$, $\partial T_0/\partial x < 0$. The numerical values of the two components of the horizontal temperature gradient are almost equal under the above conditions, since the ratio T/T_0 is not much different from unity. It follows that the temperature gradient is directed to the southwest. Since the wind turns from a southerly direction at the ground to a westerly direction aloft, it can be expected that the temperature at the place of observation will rise, due to the advection of warmer air.

Such forecasts based on observations made at a single station are of special importance in wartime when a meteorologist, on a warship for instance, may not be able to receive weather reports from other localities. He will then have to base his forecasts entirely on observations made at his own post. This procedure is called "spot forecasting." Under such circumstances when as much information as possible has to be obtained from very few observations, conclusions based on dynamical reasoning are of special importance.

5. The variation of the wind in the frictional layer. It has been mentioned previously that in the layer from the surface up to a few thousand feet the geostrophic wind relation is much less satisfactory than at higher levels, owing to the influence of surface friction. In order to study the dynamical effects determining the vertical distribution of the wind in this layer, the frictional terms have to be included in the hydrodynamic equation (1). The term containing the divergence of the velocity may be omitted because it is of less importance. If the coefficient of viscosity μ is regarded as constant, the effect of viscosity is expressed by $\mu/\rho\, \nabla^2\mathbf{v}$. Since the variations in the vertical direction are much larger than in the horizontal direction, it is generally sufficient to take only the term $\mu/\rho\, \partial^2\mathbf{v}/\partial z^2$ into consideration.

For the sake of simplicity, it may be assumed that the wind field is steady and that the wind is the same everywhere in the horizontal direction; furthermore that the pressure gradient is directed in the positive x-direction and that $1/\rho\, \partial p/\partial x$ is independent of the altitude. If, finally, the pressure gradient is replaced by the geostrophic wind velocity v_g according to equations (4), the two equations determining the vertical wind distribution in the frictional layer are

$$
(8) \qquad \begin{aligned} -\, 2\Omega_z(v - v_g) &= \frac{\mu}{\rho}\,\frac{d^2u}{dz^2}, \\ 2\Omega_z u &= \frac{\mu}{\rho}\,\frac{d^2(v - v_z)}{dz^2}\,. \end{aligned}
$$

Upon multiplying the second equation by the imaginary unit i, adding it to the first equation and introducing

$$V = u + i(v - v_z),$$

it follows that

$$\frac{d^2V}{dz^2} = i\,\frac{2\Omega_z\rho}{\mu}\,V.$$

The following boundary conditions may be assumed. With increasing altitude the function V, which represents the deviation from the geostrophic wind, should tend to zero, since the frictional effects are greatest next to the ground and decrease with the altitude and since the observations show that the wind approaches the geostrophic wind with increasing elevation. At the surface of the earth the wind velocity may be zero. This condition is not entirely satisfactory because the wind distribution in the lowest few meters next to the ground is subjected to other forces than those considered in equations (8). But even with this simple boundary condition the essential features of the wind distribution in the frictional layer, except at the very lowest levels, are well brought out. In symbols the two boundary conditions require that

$$V \to 0 \quad \text{when} \quad z \to \infty$$

$$V = -\, iv_g \quad \text{when} \quad z = 0.$$

Hence

$$V = - iv_g e^{-az}(\cos az - i \sin az),$$

where

$$a = \sqrt{\frac{\Omega_z \rho}{\mu}} \cdot$$

Separating the real and imaginary parts, it follows that

$$(9) \qquad \begin{aligned} u &= - v_g e^{-az} \sin az \\ v &= v_g(1 - e^{-az} \cos az). \end{aligned}$$

The end points of the wind vector, when plotted in the u, v plane, lie on a spiral around the end point of the geostrophic wind vector. The actual wind direction coincides with the direction of the geostrophic wind, which has here been chosen as the y-direction, at the level where $u=0$ while v is finite. The lowest level H above the surface where u vanishes is according to the first equation (9) given by the formula

$$(10) \qquad H = \frac{\pi}{a} = \pi\sqrt{\frac{\mu}{\Omega_z \rho}} \cdot$$

This height is called the "geostrophic wind level" or, less accurately, "gradient wind level." Since $e^{-\pi}=0.043$, the frictional deviations from the geostrophic wind above the geostrophic wind level are very small and below the limits set by the errors of observation. Since $\Omega_z=\omega \sin \phi$ where the angular velocity of the earth's rotation $\omega=7.29 \cdot 10^{-5}$ sec^{-1}, $\rho=1.26 \cdot 10^{-3}$ gm cm^{-3}, $\mu=1.7 \cdot 10^{-4}$ gm cm^{-1} sec^{-1}, it would follow that at latitude 45° the geostrophic wind level is only about 1.6 m. In reality, the average value of H is about 1500 m, although it may be considerably smaller or larger. With this value for H one finds that $\mu=150$ gm cm^{-1} sec^{-1}. This figure can of course only be regarded as a rough approximation, owing to the simplifying assumptions and to the varying atmospheric conditions. The order of magnitude may be smaller by as much as a factor 1/10. But even making allowance for these effects, the coefficient of viscosity for atmospheric motions is very much larger than the one determined in the laboratory for laminar flow. Obviously, the value of μ determined by means of the height of the geostrophic wind level is not the coefficient of molecular viscosity but the coefficient of eddy viscosity caused by the turbulence of atmospheric motions. The turbulent eddy motion brings about a transport of momentum at right angles to the direction of mean motion. The effect of this momentum transport is similar to that due to the molecular viscosity, but of far greater intensity.

6. Turbulent mass exchange. In a manner similar to the transfer of momentum atmospheric turbulence gives rise also to a transport of heat, water vapor, dust and other characteristics of the air. The transfer of heat is analogous to the

molecular conduction of heat, the transfer of water vapor analogous to the process of molecular diffusion. The mechanism responsible for this transfer is called turbulent mass exchange. Since most of the atmospheric characteristics change much more rapidly in vertical than in horizontal direction, the distribution of a property s due to turbulent mass exchange can be described by the differential equation

$$\frac{\partial s}{\partial t} = \frac{\partial}{\partial z}\left(k \frac{\partial s}{\partial z}\right), \tag{11}$$

which is similar to the equation for the conduction of heat. But while in the process of molecular transfer the coefficient k can, as a rule, be regarded as constant, such is not the case when turbulent mass exchange is considered, and the assumption of a constant k (or μ as in the preceding section) represents only a rough approximation to reality. Since equations of the type (11) describe problems involving atmospheric turbulence, all the mathematical methods used in the theory of the conduction of heat may find application in dynamic meteorology also.

One of the standard examples concerns the daily temperature period. The air temperature at the earth's surface has a period of 24 hours, due to the daily period of the insolation, with a maximum about two hours after noon and a minimum at sunrise. According to the observations, the amplitude of the daily temperature wave decreases and the maximum occurs later with increasing altitude. The solution of the differential equation (11) for turbulent mass exchange shows that both these phenomena are due to the role of turbulent mass exchange in propagating the daily temperature wave upwards.

Another problem involving turbulent mass exchange arises in connection with the heating of cold continental air masses from the ground upwards as they move over increasingly warmer land. Such heating takes place when air from northern Canada and Alaska travels southward. Eq. (11) has then to be solved under a certain initial condition, depending on the original vertical temperature distribution in the air and under a boundary condition given by the temperature at the earth's surface with which the air is in contact. The solution is an expression which contains the error function.

7. Conclusion. In the foregoing discussion, nothing has been said about more advanced topics, such as the perturbation theory of atmospheric motions, because under the present emergency training program these subjects cannot be included in the curriculum. But the few examples presented here will show that in meteorology, as well as in physics, at least a knowledge of differential and integral calculus and of the more elementary parts of the differential equations of mathematical physics is highly desirable, even for the meteorology student who intends to specialize in weather forecasting. These subjects are rather minimum requirements, and there is no scarcity of more difficult problems in meteorology whose solutions require a considerablv more advanced and complete mastery of mathematics.

DIAGNOSTIC TESTING PROGRAM IN PURDUE UNIVERSITY

3. A Report of the Results of the Program

M. W. KELLER, D. R. SHREVE and H. H. REMMERS, Purdue University

In general, universities attempting to remedy the deficiencies of beginning students in mathematics have separated the poorly prepared and given them extra instruction. This sectioning of the students has usually been determined by the number of units of high school algebra the student has had, by his score on some prognostic test, by a "weeding out" on the basis of teachers' judgments of the obviously weak during the first few weeks in the regular college course, or by some combination of these factors. In most cases this extra instruction has consisted in starting at the first part of an algebra textbook—rather than with some later chapter—and devoting more time to the study of algebra.

In those colleges and universities where the students study trigonometry first and algebra second the need for remedial instruction in the prerequisite algebra is most apparent. It is difficult, however, to select those topics and concepts in algebra which can be reviewed quickly with most benefit to the students. Some institutions have made an *a priori* determination of the material to be included in the review which has been satisfactory for them. Very few have attempted to determine objectively by an analysis of preliminary diagnostic tests exactly what topics and concepts in elementary algebra are actually causing the most trouble and the characteristic kinds of mistakes which are made so that a definite effort could be made to correct these specific difficulties in the review instruction.

Prior to June, 1942 the freshmen engineering students enrolled in mathematics were not sectioned according to abilities. The curriculum provided that the first two-thirds of the first semester be devoted to the study of trigonometry and the remainder of the semester to certain more advanced topics in algebra. With this course of study, the student who could not perform the fundamental prerequisite algebraic manipulations and numerical computations with a reasonable degree of mastery was definitely lost before he started the trigonometry.

One of the objects of the diagnostic testing program at Purdue University was to attempt to correct this situation:

(1) By an analysis of the problems in the textbook to find out specifically those mathematical skills and concepts that are needed by the student.

(2) By giving diagnostic tests to determine the abilities of entering students to perform these necessary operations before remedial work was undertaken.

(3) By remedial instruction aimed at correcting the specific weaknesses revealed by the diagnostic tests both as to the kind and type of error.

The work of the first year in our research program was experimental exploration in a field where relatively little is known. Preliminary diagnostic and achievement tests were administered. Error analyses were made and the problems were validated and revised where necessary.*

* Reports of the error analyses of the diagnostic tests were published in the papers listed at the end of this article.

The students who were tested during the second year were used as the control group and those taught in the third year as the experimental group. Evidence for the comparability of the experimental and control groups has been presented below. Remedial instruction with the control group was given incidentally as would ordinarily be done by many instructors in trigonometry. However, the specific knowledge of students' difficulties acquired by detailed analysis of the diagnostic and achievement tests may have influenced the remedial instruction as well as the presentation of new material. If so, this would only serve to minimize the observable differences between the experimental and control groups resulting from the remedial instruction given the students the third year. Otherwise the course followed the regular prescribed outline.

The achievement of the experimental group was measured by use of the same objective tests as those used for measuring the achievement of the control group. The tests were scored both years on the same basis. To insure uniformity in the scoring, the test items were graded on a right or wrong basis. There was no partial credit given where only trivial errors were made except on two tests. These two tests were almost entirely tests of computational ability for which the partial scoring was definitely made uniform for both groups.

The same diagnostic tests, psychological examination, and mathematics training test were given both groups at the beginning of the semester. All measurements of the initial abilities of the two groups indicate that the control group was on the average somewhat superior. The data of initial abilities is given in Table 1.

Immediately after the control group had been given the diagnostic tests they started the study of trigonometry. The experimental group, however, was given seven periods of remedial instruction in algebra first. The topics reviewed were (1) factoring, (2) simplification, multiplication, and division of simple fractions, (3) addition and subtraction of fractions, (4) positive integers, negative integers, and zero used as an exponent, (5) fractional exponents and radicals, (6) linear equations and systems of linear equations, and (7) quadratic equations. The material to be included in this review was determined in the manner previously outlined. A definite effort was made to correct the specific types of errors which the analysis of the diagnostic tests revealed were most common. After this review the students were given tests equivalent in difficulty to the preliminary tests and covering the same fundamental operations. The mean gain on the tests given after the review was approximately forty per cent using the initial mean as the base.

There was no attempt made in this experiment to have the instructors use the same teaching methods. In teaching the experimental group it was agreed to emphasize certain concepts and to attempt to correct certain types of mistakes. Each instructor, however, presented the material in the manner which he believed was most effective for him.

After this short review of algebra the experimental group studied exactly the same topics and in the same order as the control group. As each new topic

TABLE I

Summary of Data on Initial Comparability of the Experimental and Control Groups

		N	Mean	$S.D.$	$S.E._M$	$S.E._\sigma$
Psychological	1940	131	59.40	25.00	2.18	1.54
Examination	1941	139	53.75	26.70	2.26	1.60
Diff.			5.65±3.14	1.70±2.22		
C.R.			1.80	.76		
Significant at*			5% level	not significant		
Iowa Mathematics	1940	128	59.25	23.50	2.08	1.47
Training Test	1941	134	58.40	26.55	2.29	1.62
Diff.			.85±3.09	3.05±2.19		
C.R.			.27	1.39		
Significant at			not significant	not significant		
K-S-R	1940	130	17.78	6.50	.57	.40
Number Technique Test	1941	138	15.44	6.62	.56	.40
Diff.			2.34± .81	.12± .56		
C.R.			2.89	.21		
Significant at			1% level	not significant		
Purdue Mathematics	1940	130	16.72	6.70	.59	.41
Training Test	1941	136	15.10	6.60	.57	.40
Diff.			1.62± .82	.10± .57		
C.R.			1.97	.17		
Significant at			5% level	not significant		

* As is customary, the fiducial limits of significance have been here set at 95 per cent as an acceptable fiducial probability. See Peters, C. C. and Van Voorhis, Statistical Procedures and Their Mathematical Bases, McGraw Hill, 1940, pp. 137 ff.

was presented to the experimental group a definite effort was made to correct those mistakes and wrong concepts which an error analysis of the control group's achievement tests had shown to be common. Thus the emphasis was placed on those errors which the students actually make and not on what the instructors might, on other grounds, have thought needed emphasis.

The findings and grade distributions for each of the eight achievement tests in trigonometry for both groups are given in Table 2. The content of each test is briefly indicated:

Test 1. 44 Problems. Definitions of the trigonometric functions; trigonometric functions of angles of any magnitude in standard position; complementary angles; trigonometric functions of angles of 0°, 30°, 45°, 60°, 90°, 120°, 135°, etc.

Test 2. 18 Problems. Approximate numbers; 3-place tables of trigonometric functions; solutions of right triangles.

Test 3. 40 Problems. Reduction formulas; trigonometric functions of angles of any magnitude from 3-place tables.

Test 4. 6 Problems. Numerical computations using Law of Sines and Law of Cosines, without logarithms.

Test 5. 12 Problems. Logarithms; Law of Sines by use of logarithms.

Test 6. 4 Problems. Solutions by Law of Tangents and half-angle formulas, using logarithms.

Test 7. 26 Problems. Radian measure; trigonometric equations; simple identities.

Test 8. 18 Problems. Addition formulas; multiple-angle formulas; inverse functions.

TABLE II

Data for the Eight Achievement Tests in Trigonometry

Test	Group	N	Mean	$\%A$	$\%B$	$\%C$	$\%D$	Pass	Instruction time in days	$\frac{M_1-M_2}{\sigma_{M_1-M_2}}$	In favor of	Significance level
1	Con.	138	26.6	13	32	24	31	69.3	6	2.9	Exp.	1%
	Exp.	142	30.2	24	32	23	21	78.8	6			
2	Con.	137	12.4	27	29	20	24	75.9	5	2.4	Exp.	1%
	Exp.	141	13.5	33	36	15	16	83.7	4			
3	Con.	138	34.1	63	14	12	11	89.1	3	3.9	Exp.	1%
	Exp.	143	37.6	83	10	1	6	94.4	2			
4	Con.	136	4.2	29	18	24	28	72.1	4	0.0	Neither	not sig.
	Exp.	142	4.2	25	28	20	27	72.6	3			
5	Con.	134	6.8	20	24	21	35	64.9	8	0.0	Neither	not sig.
	Exp.	140	6.8	20	33	20	27	72.8	5			
6	Con.	133	2.0	14	29	17	40	60.2	4	0.0	Neither	not sig.
	Exp.	141	2.0	18	21	21	40	60.2	4			
7	Con.	131	12.8	16	28	21	35	64.9	7	2.2	Exp.	1%
	Exp.	134	14.8	28	27	20	25	75.4	7			
8	Con.	131	9.5	14	10	30	47	53.4	8	1.8	Exp.	5%
	Exp.	135	10.7	19	20	21	40	60.0	8			

Explanation of Data: (by columns of entries)

Mean: The arithmetic mean score, based on number of problems marked *correct* on "right or wrong" scoring.
$S.E._M$ Standard error of the arithmetic mean.
$\%A$: The per cent of grades of "A," the highest passing letter mark.
$\%B$: The per cent of grades of "B," the second passing mark.
$\%C$: The per cent of grades of "C," the lowest passing mark.
$\%D$: The per cent of grades of "D," the failing mark.
Pass: The per cent of passing grades, to the nearest 0.1%.

Days: The number of days (50 minutes per day) of instruction devoted to the subject matter for each test.

$\frac{M_1 - M_2}{\sigma_{M_1 - M_2}}$: Ratio of difference of means to its standard error.

The observable effect of the remedial instruction in the distribution of the grade averages for the whole course in trigonometry is perhaps of greater interest than any observations of effects in single tests. The distribution of grades for trigonometry is now indicated:

TABLE III

Distribution of Semester Grades in Trigonometry

	%A	%B	%C	% Pass	%D
Control	14.6	34.6	25.4	74.6	25.4
Experimental	21.9	35.7	31.4	89.0	11.0

The critical ratio of the mean difference is 2.3, which is significant at the 1% level. All tests were converted to letter grade equivalents and summed, thus weighting them equally for the determination of the semester grades. This table shows a net gain by the experimental group of fifty per cent in "A" grades, of seventeen per cent in "A" and "B" grades combined, and of nineteen per cent in all passing grades, with a reduction of more than fifty-six per cent in failures.

Since the teaching personnel changed during the second and third years, this paper gives only the results for those classes taught by Dr. Keller and Dr. Shreve each of whom taught three classes each year. The inclusion of the classes of the other instructors would show results even more favorable for the experiment.

Through all these tests, there are two rather obvious trends: (1) In those tests where manipulations of algebraic symbols occur frequently, the increase in mean score, per cent of passing grades, per cent of "A's," and the decrease in per cent of failures are significantly high. (2) In those tests which are largely computational, the differences in results are not statistically significant. The reduction in time for teaching some of these topics, however, ranges up to 37 per cent below the time given to the same topics with the control group.

The results are equally significant if the work in algebra is included, even though there were eleven less days the third year to teach the trigonometry and algebra. This difference in time was due to the seven days used for review and a four day shorter semester.

The data warrant the conclusion that remedial instruction is of positive value when:

(1) Specific weaknesses of students are first determined by diagnostic tests,
(2) Remedial instruction in regular lesson assignments is based specifically upon the diagnostic findings, and

(3) In the teaching of each new topic those concepts and operations are emphasized which the analysis of acihevement tests shows to be common errors.

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ON AN INTEGRAL TRANSFORMATION OF GENERAL CIRCUIT THEORY

BERNHARD GROSS, National Institute of Technology, Rio de Janeiro, Brazil

It has been stated by Carson that the behavior of a network is completely determined if either the real or the imaginary component of the complex steady state admittance is specified over the entire frequency range [1]. The same can be said of the impedance. The direct relations existing between the two components have been derived by M. Bayard [2], who has studied the impedance function of a Kirchhoff dipole consisting of resistors, capacitors and inductors. His results have been quoted by Van der Pol [3]. The same relations appear again in the theory of anomalous dispersion and absorption, where they have been obtained by R. de L. Kronig [4] and H. A. Kramers [5]. Some applications have been given afterwards by C. J. Gorter and R. de L. Kronig [6]. Anomalous dispersion and absorption is but a particular case of anomalous behavior of dielectrics. A derivation from a very general point of view covering both networks and anomalous dielectrics, of the relations mentioned here, has been given recently by B. Gross, [7] and, independently and nearly at the same time, by R. H. Cole [8]. A numerical application in the field of anomalous dielectrics has already been given by Silva and Gross [9]. Independently of their physical significance, the relations referred to above, or quite similar ones, have been considered incidentally in mathematical papers [10]. Their closer study seems justified as much from a physical as from a mathematical point of view. This study has been made by B. Levi, L. Gama and the present author [11]. Considering the numerous applications of the formula referred to above and the general technical and scientific interest in all research concerning dielectric behavior, it seems worth while to report briefly here our main results. For details we must refer to the original papers.

I

The relations in question are integrals of the type:

$$(1)\qquad A(\beta) = \frac{2}{\pi}\int_0^\infty B(\alpha)\,\frac{\beta}{\beta^2-\alpha^2}\,d\alpha \quad \text{where} \quad -\infty < \beta < +\infty, \quad \beta \neq 0.$$

An integral of this form exists as a *principal value* for a given β, if the function $B(\alpha)$ is such that the 4 derivate numbers of B for $\alpha=\beta$ are finite, provided furthermore that the integrals $\int_0^{\beta-\delta}$ and $\int_{\beta+\delta}^{\infty}$ exist, where δ is a positive number sufficiently small, but $\neq 0$. (L. I. Gama [11]).

The kernels of (1), given by

$$K(\alpha,\beta)=\frac{\beta}{\beta^2-\alpha^2} \tag{2}$$

are *orthogonal* in the interval 0 to ∞, *i.e.*,

$$\int_0^{\infty}K(\alpha,\beta)K(\alpha,\sigma)d\alpha=0,\quad\text{for}\quad\beta\neq\sigma. \tag{3}$$

The *eigenfunctions* and *eigenvalues* of the integral (1) are given, respectively, by

$$\phi_n(\beta)=c(n)\beta^n,\qquad -1<n<+1 \tag{4}$$

and

$$\lambda_n=\frac{1}{\dfrac{2}{\pi}\displaystyle\int_0^{\infty}\frac{x^n}{1-x^2}dx}\qquad -1<n<+1. \tag{5}$$

For $0>n>-1$, this expression reduces to $\lambda_n=-\tan(\pi/2n)$. All these relations are easily verified by direct substitution.

The integral (1) can still be transformed for periodic functions. If B is an even, bounded function of period l, average value (averaged over one period) B_0, and $0<\beta<\beta_0<1$, then we can write (1) in the form

$$A(\beta)=\frac{1}{l}\int_0^{l}[B(\alpha)-B_0]\cot\frac{\pi}{l}(\beta-\alpha)d\alpha. \tag{6}$$

This equation has the form of Hilbert's transformation [12]. It is obtained by expanding the integral (1) in a series of integrals each of which extends over one period of the function B.

The *derivative* of (1) is given by

$$A'(\beta)=-\frac{2}{\pi}\int_0^{\infty}B'(\alpha)\frac{\alpha}{\alpha^2-\beta^2}d\alpha. \tag{7}$$

The integral (1) becomes particularly simple if $B(\alpha)$ is $\cos\sigma\alpha$ or $\sin\sigma\alpha$. Then

$$\frac{2}{\pi}\int_0^{\infty}\cos\sigma\alpha\frac{\beta}{\beta^2-\alpha^2}d\alpha=\sin\sigma\beta \tag{7a}$$

$$\frac{2}{\pi}\int_0^{\infty}\sin\sigma\alpha\frac{\alpha}{\alpha^2-\beta^2}d\alpha=\cos\sigma\beta. \tag{7b}$$

These formulas can be interpreted as representations of the functions $\sin \sigma\beta$ and $\cos \sigma\beta$ by Fourier integrals.

II

The integral (1) is closely related to the theory of conjugate Fourier series and integrals.

Consider the expressions:

$$P^*(\beta) = \sum_1^\infty k_n \sin n\beta \tag{8a}$$

$$Q^*(\beta) = \sum_1^\infty k_n \cos n\beta \tag{8b}$$

and

$$P^{**}(\beta) = \int_0^\infty \phi(\sigma) \sin \sigma\beta \, d\sigma \tag{9a}$$

$$Q^{**}(\beta) = \int_0^\infty \phi(\sigma) \cos \sigma\beta \, d\sigma. \tag{9b}$$

The functions P and Q are not independent of each other. Suppose that

$$\sum_1^\infty | k_n | \quad \text{and} \quad \int_0^\infty | \phi(\sigma) | \, d\sigma$$

exist, the series (8) and integrals (9) being in consequence absolutely and uniformly convergent.

Under these conditions the relations between P and Q are given by the formulas.

$$P(\beta) = \frac{2}{\pi} \int_0^\infty Q(\alpha) \frac{\beta}{\beta^2 - \alpha^2} \, d\alpha \tag{10a}$$

$$Q(\beta) = \frac{2}{\pi} \int_0^\infty P(\alpha) \frac{\alpha}{\alpha^2 - \beta^2} \, d\alpha. \tag{10b}$$

These relations are easily verified by direct substitution with the aid of the integrals (7).

III

The relations (10) can be considered as a pair of integral equations. There arises therefore the question as to the general conditions under which the integral (1) can be inverted.

The answer is given by the following theorem: (B. Levi [13]).

The inversion of the integral

$$A(\beta) = \frac{2}{\pi}\int_0^\infty B(\alpha)\,\frac{\beta}{\beta^2-\alpha^2}\,d\alpha \tag{11}$$

is given by

$$B(\beta) = \frac{2}{\pi}\int_0^\infty A(\alpha)\,\frac{\alpha}{\alpha^2-\beta^2}\,d\alpha \tag{12}$$

for any function $A(\beta)$, defined continuous and bounded in the interval 0 to ∞, infinitesimal to at least the first order at $\beta=0$, and such that the integrals

$$\int_0^\infty \frac{A(\beta)}{\beta}\,d\beta \quad \text{and} \quad \int_p^\infty \frac{|A(\beta)|}{\beta^2}\,d\beta, \quad \text{where} \quad p > 0$$

exist.

IV

Arbitrary functions may be represented by integrals of the type (1).

Any function $F(\beta)$ which can be represented by absolutely and uniformly convergent Fourier series or integrals, may also be represented by

$$F(\beta) = \frac{2}{\pi}\int_0^\infty P(\alpha)\,\frac{\beta}{\beta^2-\alpha^2}\,d\alpha + \frac{2}{\pi}\int_0^\infty Q(\alpha)\,\frac{\alpha}{\alpha^2-\beta^2}\,d\alpha \tag{13}$$

where

$$P(\alpha) = \frac{1}{\pi}\int_{-\infty}^\infty F(\beta)\,\frac{\beta}{\beta^2-\alpha^2}\,d\beta \tag{14a}$$

$$Q(\alpha) = \frac{1}{\pi}\int_{-\infty}^\infty F(\beta)\,\frac{\alpha}{\alpha^2-\beta^2}\,d\beta \tag{14b}$$

provided, furthermore, that $\int_{-\infty}^{\infty} F(\beta)d\beta$ exists if the function is non-periodic, or that $\int_0^l F(\beta)d\beta=0$, if the function is periodic, of period l.

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CONVERGENT SEQUENCES OF PROBABILITY DISTRIBUTIONS

J. H. CURTISS, Cornell University

1. Introduction. It is known that many of the more important probability distributions encountered in practical statistics tend to simple limiting forms as certain parameters become infinite. However, because of the lack of an adequate textbook on statistical theory at the intermediate and advanced levels, the teacher interested in the derivation of such convergence theorems is forced to refer to a literature which is widely scattered and often mathematically unsatisfactory. It is the main purpose of this paper to present a collection of some of the more important limit theorems of practical statistics, derived by a simple and uniform method of broad applicability.

The method is based upon the properties of the moment generating function. The use of this function in place of the more usual characteristic function permits us to avoid complex variables without losing certain of the advantages of the characteristic function type of proof. In an earlier expository paper [6, p. 75], the author has set forth reasons why the uniqueness and limit theorems for the moment generating function* might well be assumed without proof in certain courses in statistics. The reader who adopts this point of view will find that the present paper essentially presupposes no mathematical equipment beyond that supplied by the usual first course in calculus.

2. The m.g.f. and the s.g.f. of a distribution. Let X be a variate in one dimensional space. We shall denote the probability associated with an interval such as $a \leqq X \leqq b$ by $P(a \leqq X \leqq b)$. The function $F(x) = P(-\infty < X \leqq x)$ will be called the distribution function of X, a term which we shall generally abbreviate to d.f. The mean value, or expected value, of any function $f(X)$ will be denoted, if it exists, by the usual symbol $E[f(X)]$. We write $\mu = E(X)$, $\sigma^2 = E[(X-\mu)^2]$, and call these quantities the mean and variance of X respectively.

The moment generating function of X, henceforth to be abbreviated to m.g.f., is the function $G(\alpha) = E(e^{\alpha X})$, where α is real. Unless statement is made to the contrary, we shall always impose the restriction upon the distribution of X that this mean value is to exist for all values of α in some neighborhood of the origin. The function $H(\alpha) = \log_e G(\alpha)$ is called the semi-invariant generating function or cumulant generating function of X, to be abbreviated to s.g.f. In the earlier paper referred to above, the author has given a fairly detailed exposition of the purely formal properties of the m.g.f. and the s.g.f. It suffices for the present purposes merely to recall that $\mu = H'(0)$, and $\sigma^2 = H''(0)$. In addition, we shall need the following limit theorem:

THEOREM 2.1. *Let $F_n(x)$ and $G_n(\alpha)$ be respectively the d.f. and m.g.f. of a variate X_n and let $F(x)$ and $G(\alpha)$ be the d.f. and m.f.g. of a variate X. If $\lim_{n\to\infty} G_n(\alpha) = G(\alpha)$ for α in some neighborhood of $\alpha = 0$, then $\lim_{n\to\infty} F_n(x) = F(x)$ uniformly in each finite or infinite interval of continuity of $F(x)$.*

* Theorem 2.1 below is of this type.

A formal proof of this theorem has been supplied elsewhere by the author,* and will be omitted here. We repeat our earlier suggestion that in certain courses in statistical theory, this theorem might not inappropriately be assumed without proof.

3. A convergence test for distributions. If $G(\alpha)$ exists for $|\alpha| < \alpha_1 \neq 0$, then it is easily seen that $H(\alpha)$, $H'(\alpha)$, $H''(\alpha)$, and $H'''(\alpha)$ all exist for $|\alpha| < \alpha_1$. Accordingly we can expand $H(\alpha)$ in a Maclaurin series with remainder in the following two ways:

$$H(\alpha) = H(0) + \alpha\cdot\mu + \frac{\alpha^2}{2}\cdot H''(\theta\alpha), \qquad 0 < \theta < 1, \ |\alpha| < \alpha_1,$$

$$H(\alpha) = H(0) + \alpha\cdot\mu + \frac{\alpha^2}{2}\cdot\sigma^2 + \frac{\alpha^3}{6} H'''(\theta'\alpha), \qquad 0 < \theta' < 1, \ |\alpha| < \alpha_1.$$

The first term in each expansion vanishes, since $H(0) = \log E(e^0)$.

Suppose now that we are dealing with a variate for which $\mu = 0$ and $\sigma = 1$. Such a variate is often called a standardized or reduced variate. In this case our expansions become

$$H(\alpha) = \frac{\alpha^2}{2} H''(\theta\alpha), \qquad 0 < \theta < 1, \ |\alpha| < \alpha_1, \tag{3.1a}$$

$$H(\alpha) = \frac{\alpha^2}{2} + \frac{\alpha^3}{6} H'''(\theta'\alpha), \qquad 0 < \theta' < 1, \ |\alpha| < \alpha_1. \tag{3.1b}$$

If now X is any variate with a m.g.f., then the variate $T = (X-\mu)/\sigma$ is clearly a reduced variate. If we denote the m.g.f. and s.g.f. of T by ${}_TG(\alpha)$, ${}_TH(\alpha)$, and the s.g.f. of X by $H(\alpha)$, then

$$\begin{aligned} {}_TH(\alpha) &= \log\,({}_TG(\alpha)) \\ &= \log\, E\left[\exp\, \alpha\left(\frac{X-\mu}{\sigma}\right)\right] = -\,\mu\,\frac{\alpha}{\sigma} + \log\, E\left[\exp\,\frac{\alpha X}{\sigma}\right] \\ &= -\,\mu\frac{\alpha}{\sigma} + H\left(\frac{\alpha}{\sigma}\right). \end{aligned}$$

Therefore ${}_TH''(\alpha) = H''(\alpha/\sigma)/\sigma^2$, ${}_TH'''(\alpha) = H'''(\alpha/\sigma)/\sigma^3$. Substituting into (3.1a) and (3.1b), we obtain

$${}_TH(\alpha) = \frac{\alpha^2}{2}\,\frac{H''(\theta\alpha/\sigma)}{\sigma^2}, \qquad 0 < \theta < 1, \ |\alpha| < \alpha_1,$$

$${}_TH(\alpha) = \frac{\alpha^2}{2} + \frac{\alpha^3}{6}\,\frac{H'''(\theta'\alpha/\sigma)}{\sigma^3}, \qquad 0 < \theta' < 1, \ |\alpha| < \alpha_1,$$

or

* See [7], where a slightly more general form of the result is stated and proved.

$$(3.2a) \qquad {}_TG(\alpha) = \exp\left\{\frac{\alpha^2}{2}\,\frac{H''(\theta\alpha/\sigma)}{\sigma^2}\right\}, \qquad 0 < \theta < 1,\ |\alpha| < \alpha_1,$$

$$(3.2b) \qquad {}_TG(\alpha) = \exp\left(\frac{\alpha^2}{2}\right)\exp\left\{\frac{\alpha^3}{6}\,\frac{H'''(\theta'\alpha/\sigma)}{\sigma^3}\right\}, \qquad 0 < \theta' < 1,\ |\alpha| < \alpha_1.$$

Now consider the reduced Gaussian or normal distribution, whose d.f. is

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt.$$

(In the sequel, the symbol $N(x)$ will always refer to this d.f.) A simple integration shows that the m.g.f. of the distribution is exp $(\alpha^2/2)$. Thus (3.2a) and (3.2b) suggest that in many cases the m.g.f. of a reduced variate will be approximately equal to that of a normal distribution, at least for values of α sufficiently near the origin. These equations, when taken with Theorem 2.1, yield the following:

CONVERGENCE TEST FOR DISTRIBUTIONS. *Let $H_n(\alpha)$ be the s.g.f., μ_n the mean and σ_n^2 the variance of a variate X_n dependent on a parameter n, and let $F_n(t)$ be the d.f. of the reduced variate $T_n = (X_n - \mu_n)/\sigma_n$. If either of the equations*

$$(3.3a) \qquad \lim_{n\to\infty} \frac{H_n''(\alpha/\sigma_n)}{\sigma_n^2} = 1$$

$$(3.3b) \qquad \lim_{n\to\infty} \frac{H_n'''(\alpha/\sigma_n)}{\sigma_n^3} = 0$$

holds uniformly in some neighborhood of $\alpha = 0$, then $\lim_{n\to\infty} F_n(t) = N(t)$ *uniformly for all t.*

It is possible to show that (3.3a) and (3.3b) are equivalent conditions, but we shall not go into the matter here. We shall now apply our convergence test to a number of important distributions of variable form.

4. Generalized binomial distributions. The variate X_n is said to have the binomial distribution of Poisson with parameters $p_1, p_2, \cdots, p_n$, if X_n can be expressed as the sum of n independent variates $z_1, z_2, \cdots, z_n$ where z_j has the values 0 and 1 assumed with respective probabilities q_j and p_j $(q_j + p_j = 1)$. The variate X_n may be physically interpreted as the number of successes in n independent trials with probability p_j of success in the jth one. The reader will have no trouble in checking the following computation, in which $G_n(\alpha)$, $H_n(\alpha)$, μ_n, and σ_n^2 denote respectively the m.g.f., s.g.f., mean, and variance of X_n:

$$G_n(\alpha) = E\left(\exp \sum_1^n \alpha z_j\right) = \prod_1^n (q_j + p_j e^{\alpha}),$$

$$H_n(\alpha) = \sum_1^n \log (q_j + p_j e^{\alpha}),$$

$$\mu_n = \sum_1^n p_i = n\bar{p}, \qquad \bar{p} = \frac{\sum_1^n p_i}{n},$$

$$\sigma_n^2 = \sum_1^n q_i p_i = n\bar{p}\bar{q} - n\sigma_p^2, \qquad \bar{q} = 1 - \bar{p},\ \sigma_p^2 = \frac{\sum_1^n (p_i - \bar{p})^2}{n},$$

$$H'''(\alpha) = e^\alpha \sum_1^n q_i p_i \frac{q_i - p_i e^\alpha}{(q_i + p_i e^\alpha)^3}.$$

So

$$(4.1) \qquad |H_n'''(\alpha)| \leqq e^\alpha \sum_1^n \frac{q_i p_i}{(q_i + p_i e^\alpha)^2} \leqq e^{|\alpha|} \sigma_n^2.$$

Consider now a sequence $X_1, X_2, X_3, \cdots$, of variates with Poisson binomial distributions. Let the parameters of X_n be $p_1^{(n)}, p_2^{(n)}, \cdots, p_n^{(n)}$, the variance be σ_n^2, and the s.g.f., $H_n(\alpha)$. It is apparent from (4.1) that if $\lim_{n\to\infty}\sum_1^n p_j^{(n)} q_j^{(n)} = \lim_{n\to\infty}\sigma_n^2 = \infty$, where $q_j^{(n)} = 1 - p_j^{(n)}$, then $\lim_{n\to\infty} H_n'''(\alpha/\sigma_n)/\sigma_n^3 = 0$ uniformly in any finite interval. Referring to the Convergence Test of §3, we obtain the following result:

THEOREM 4.1. *Let X_n have a Poisson binomial distribution with parameters $p_1, p_2, \cdots, p_n$, which may depend upon n, and let $F_n(t)$ be the d.f. of the variate $T_n = (X_n - \sum_1^n p_j)/\sum_1^n p_j q_j$, where $q_j = 1 - p_j$. If $\lim_{n\to\infty}\sum_1^n p_j q_j = \infty$, then $\lim_{n\to\infty} F_n(t) = N(t)$ uniformly.**

If $p_j = p$, $j = 1, \cdots, n$, the distribution of X_n is a simple binomial or Bernoulli distribution. If in this case p depends upon n, the condition $\lim_{n\to\infty}\sum_1^n p_j q_j = \infty$ becomes $\lim_{n\to\infty} npq = \infty$. This condition is automatically satisfied if p is constant for all n, $0 < p < 1$, and the theorem then reduces to the familiar DeMoivre-Laplace Theorem.

The variate X_n in the theorem assumes only integral values, and so the values of T_n are equally spaced at intervals of $1/\sigma_n$. If the histogram for T_n and the graph of the reduced normal density function $N'(t)$ are sketched upon the same coordinate axes, the picture will suggest that for a given value of n, the normal distribution will yield a better approximation to $P(k \leqq X_n \leqq l)$, where k and l are integers, if we add on an interval of length $1/(2\sigma_n)$ at each end of the corresponding T_n interval. Numerical checks indicate that the approximation is indeed improved by this device.

5. The Poisson exponential distribution. A variate X is said to have a Poisson exponential distribution with parameter m if its values are $0, 1, 2, \cdots$, and if $P(X = k) = m^k e^{-m}/k!$, $k = 0, 1, 2, \cdots$. It is easily seen [6, p. 379] that $H(\alpha) = m(e^\alpha - 1)$.

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* The m.g.f. of T_n is $\pi_1^n[q_j \exp(-\alpha p_j/\sqrt{\sum p_j q_j}) + p_j \exp(\alpha q_j/\sqrt{\sum p_j q_j})]$, and we have proved that this expression approaches $\exp(\alpha^2/2)$ as $n \to \infty$, provided that $\sum_1^n p_j q_j \to \infty$. This aspect of the theorem is possibly of some interest for itself alone, in that it furnishes a generalized solution to an elementary problem proposed by J. F. Kenney in this MONTHLY, vol. 49, 1942, p. 61, problem E504.

The Poisson exponential distribution is the limit of a generalized binomial distribution under certain conditions on the parameters p_j. For by expanding $\log (q_j+p_ju)$ in a Taylor series with remainder about $u=1$, replacing u by e^α and summing from 1 to n, we obtain

$$H_n(\alpha) = \sum_1^n \log (q_j + p_j e^\alpha) = (e^\alpha - 1)\cdot \sum_1^n p_j - \frac{(e^\alpha - 1)^2}{2}\cdot \sum_1^n \frac{p_j^2}{(q_j + p_j\xi_j)^2},$$

where ξ_j lies between 1 and e^α. If $\alpha > \log (1/2)$, then $\xi_j > 1/2$, $q_j+p_j\xi_j > q_j+p_j/2 = 1-p_j/2 \geqq 1/2$, and $\sum_1^n [p_j/(q_j+p_j\xi_j)]^2 \leqq 4\sum_1^n p_j^2$. Thus if $\lim_{n\to\infty}\sum_j^n p_j = m$ while $\lim_{n\to\infty}\sum_1^n p_j^2 = 0$, then $\lim_{n\to\infty} H_n(\alpha) = H(\alpha)$ in some neighborhood of the origin, where $H(\alpha)$ is the s.g.f. in the first paragraph of this section. From Theorem 2.1 we obtain the following result:

THEOREM 5.1. *Let X_n have a Poisson binomial distribution with parameters $p_1^{(n)}, p_2^{(n)}, \cdots, p_n^{(n)}$ which depend upon n, and let $F_n(x)$ be the d.f. of X_n. If $\lim_{n\to\infty}\sum_1^n p_j^{(n)} = m$ while $\lim_{n\to\infty}\sum_1^n (p_j^{(n)})^2 = 0$, then $\lim_{n\to\infty} F_n(x) = M(x)$, where $M(x)$ is the d.f. of a Poisson exponential distribution with parameter m.*

The degree of approximation is discussed by Uspensky [13, pp. 135–137] in the case in which $p_j^{(n)} = p^{(n)}$, $j = 1, \cdots, n$.

From the formula $H(\alpha) = m(e^\alpha - 1)$, we find that for a Poisson exponential distribution $\mu = m$, $\sigma^2 = m$, $H''(\alpha) = me^\alpha$, $H''(\alpha/\sigma)/\sigma^2 = \exp (\alpha/\sqrt{m})$. Since the last expression tends uniformly to one in any finite interval as $m \to \infty$, our Convergence Test applies to this distribution.

THEOREM 5.2. *If X has a Poisson exponential distribution with parameter m, and if $F_m(t)$ is the d.f. of the variable $(X-m)/\sqrt{m}$, then $\lim_{m\to\infty} F_m(t) = N(t)$ uniformly.*

The remarks made at the end of §4 concerning the correction to the range of the variable T_n of Theorem 4.1 apply equally well in the present case to T_m. The correction here has the value $1/(2\sqrt{m})$.

6. The Γ-distribution. If the variate X has the d.f.

$$F(x) = \begin{cases} \displaystyle\int_0^x \frac{h^b}{\Gamma(b)} t^{b-1}e^{-ht}dt, & x \geqq 0, \\ 0, & x < 0, \end{cases}$$

with $b > 0$, $h > 0$, we shall say that X has a Γ-distribution with parameters b and h. Here

$$G(\alpha) = \int_0^\infty \frac{h^b}{\Gamma(b)} t^{b-1}e^{(\alpha-h)t}dt = \left(1 - \frac{\alpha}{h}\right)^{-b}, \qquad |\alpha| < h,$$

$$H(\alpha) = b \log h - b \log (h - \alpha)$$

$$\mu = b/h, \qquad \sigma^2 = b/h^2,$$

$$H''(\alpha/\sigma)/\sigma^2 = (1 - \alpha/\sqrt{b})^{-2}.$$

The last expression obviously approaches unity uniformly in some neighborhood of $\alpha = 0$, as $b \to \infty$. We have:

THEOREM 6.1. *If X has a Γ-distribution with parameters b and h, and if $F_b(t)$ is the d.f. of the variable $T_b = h(X - b/h)/\sqrt{b}$, then $\lim_{b\to\infty} F_b(t) = N(t)$ uniformly.*

The important case is that in which $h = 1/2$, $b = n/2$, where n is a positive integer. The distribution is then usually termed a χ^2 distribution with n degrees of freedom. In this case we shall denote the variate X by χ_n^2, and the theorem states that for large values of n, the distribution of $(\chi_n^2 - n)/\sqrt{2n}$ is approximately normal.

Various tables of the distribution of χ_n^2 for low values of n are available. Perhaps the most convenient one for practical purposes is due to R. A. Fisher [8, pp. 118–119], [9]; this table gives for values of n from 1 to 30 the various points $a_n(p)$ such that $P[\chi_n^2 > a_n(p)] = p$, where p is one of several chosen probability levels, as for instance, .05. The table has recently been extended by Thompson [12].

Fisher has pointed out that $\sqrt{2\chi_n^2} - \sqrt{2n-1}$ has approximately a reduced normal distribution for $n > 30$. Wilson and Hilferty [15] showed that rather accurate results could be obtained by taking $(\chi_n^2/n)^{1/3}$ to be normally distributed with mean $1 - 2/(9n)$ and standard deviation $\sqrt{2/(9n)}$. From the fact that $(\chi_n^2 - n)/\sqrt{2n}$ is approximately normally distributed for large values of n, we obtain the approximate formula

$$a_n(p) \doteq n + a(p)\sqrt{2n}, \tag{6.1}$$

where $a(p)$ is the point of the reduced normal distribution corresponding to the probability level p. It is possible to show that actually

$$a_n(p) = n + a(p)\sqrt{2n} + \tfrac{2}{3}\{[a(p)]^2 - 1\} + 0(n^{-1/2}).\text{*} \tag{6.2}$$

so we should not expect to obtain entirely accurate results from (6.1) even for large values of n. Fisher's suggestion yields the approximation

$$a_n(p) \doteq \tfrac{1}{2}[a(p) + \sqrt{2n-1}]^2 = n + a(p)\sqrt{2n} + \frac{[a(p)]^2 - 1}{2} + 0(n^{-1/2}). \tag{6.3}$$

From Wilson and Hilferty's approximation, we obtain

$$\begin{aligned} a_n(p) &\doteq n\left[1 - \frac{2}{9n} + a(p)\sqrt{\frac{2}{9n}}\right]^3 \\ &= n + a(p)\sqrt{2n} + \tfrac{2}{3}\{[a(p)]^2 - 1\} + 0(n^{-1/2}). \end{aligned} \tag{6.4}$$

* A. M. Peiser is preparing a paper dealing from this point of view with approximate formulas for percentage points of various distributions, including the χ^2 distribution.

In the light of (6.2), we note that (6.4) should be the best of the three approximations for large values of n. Numerical comparisons of the results given by (6.3) and (6.4) with the corresponding true points $a_n(p)$ were made by Wilson and Hilferty [15] for values of n from 1 to 30 and by Merrington [11] for values of n from 30 to 100. These comparisons consistently affirm the superiority of (6.4); Merrington found a maximum absolute error of less than .04 between the values given by (6.4) and the true values, for the values of n and p which she considered. The accuracy of (6.4) is especially remarkable for $p=.05$.*

7. The Fisher z-distribution. The variate $Z=[\log(\chi_m^2/m)-\log(\chi_n^2/n)]/2$, where χ_n^2 and χ_n^2 have independent χ^2 distributions with respectively m and n degrees of freedom, is said to have the Fisher z-distribution. The d.f. of Z is

$$\int_{-\infty}^{z} \frac{2m^{m/2}n^{n/2}}{B\left(\frac{m}{2}, \frac{n}{2}\right)} \frac{e^{mt}}{(me^{2t}+n)^{(n+m)/2}}\, dt,$$

where $B(x, y)$ is the Beta function. The m.g.f. $G(\alpha)$ is perhaps most easily obtained by referring the integration back to the two χ^2 distributions:

$$\begin{aligned} G(\alpha) &= E(e^{\alpha Z}) \\ &= E\left\{\exp\left[\frac{\alpha}{2}\left(\log\frac{\chi_m^2}{m}-\log\frac{\chi_n^2}{n}\right)\right]\right\} \\ &= \left(\frac{n}{m}\right)^{\alpha/2} E[(\chi_m^2)^{\alpha/2}]E[(\chi_n^2)^{-\alpha/2}] \\ &= \left(\frac{n}{m}\right)^{\alpha/2} \frac{\Gamma\left(\frac{m+\alpha}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}, \qquad -m<\alpha<n. \end{aligned}$$

Then if we let $\psi(u)=\log\Gamma(u)$, we have for the s.g.f.:

$$\begin{aligned} H(\alpha) &= \frac{\alpha}{2}[\log n-\log m]+\psi\left(\frac{m+\alpha}{2}\right)+\psi\left(\frac{n-\alpha}{2}\right)-\psi\left(\frac{m}{2}\right)-\psi\left(\frac{n}{2}\right), \\ H'(\alpha) &= \frac{1}{2}\left[\log n-\log m+\psi'\left(\frac{m+\alpha}{2}\right)-\psi'\left(\frac{n-\alpha}{2}\right)\right], \\ \mu &= \frac{1}{2}\left[\log n-\log m+\psi'\left(\frac{m}{2}\right)-\psi'\left(\frac{n}{2}\right)\right], \end{aligned}$$

* Aitken [1, p. 104] states that the 5 per cent point of the distribution of χ_n^2 (that is, $a_n(.05)$ in our notation) "is given with good approximation" by $n(2-.025n)$. The assertion is doubtless meant to apply only for small values of n, say $n<20$; but for such values of n, the formula seems to yield a better approximation to $a_{n-1}(.05)$ than to $a_n(.05)$. By the time $n=30$, both interpretations of the formula are about 12 per cent in error.

$$H''(\alpha) = \frac{1}{4}\left[\psi''\left(\frac{m+\alpha}{2}\right) + \psi''\left(\frac{n-\alpha}{2}\right)\right],$$

$$\sigma^2 = \frac{1}{4}\left[\psi''\left(\frac{m}{2}\right) + \psi''\left(\frac{n}{2}\right)\right],$$

where $-m<\alpha<n$.

From Binet's second formula for $\psi(u)$ [14, p. 251], we find that if $u>0$, then

$$\psi'(u) = \log u - \frac{1}{2u} + R_1(u), \qquad |R_1(u)| < \frac{1}{12u^2},$$

$$\psi''(u) = \frac{1}{u} + R_2(u), \qquad |R_2(u)| < \frac{2}{3u^2}\cdot$$

Substituting into the earlier set of equations, we have

$$\mu = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{m}\right) + r_1(m, n), \tag{7.1}$$

$$H''(\alpha) = \frac{1}{2}\left(\frac{1}{m+\alpha} + \frac{1}{n-\alpha}\right) + r_2(m+\alpha, n-\alpha), \quad -m<\alpha<n. \tag{7.2}$$

$$\sigma^2 = \frac{1}{2}\left(\frac{1}{m} + \frac{1}{n}\right) + r_2(m, n), \tag{7.3}$$

where $|r_1(n, m)| < 1/(6m^2)+1/(6n^2)$, $|r_2(m, n)| < 2/(3m^2)+2/(3n^2)$. From (7.2) without using (7.3), we obtain

$$\frac{H''(\alpha/\sigma)}{\sigma^2} = \frac{1}{2}\left[\frac{m+n}{mn\sigma^2 + \alpha\sigma(n-m) - \alpha^2}\right] + \frac{r_2\left(\dfrac{\sigma m+\alpha}{\sigma}, \dfrac{\sigma n-\alpha}{\sigma}\right)}{\sigma^2}. \tag{7.4}$$

According to (7.3), $\sigma m+\alpha$ and $\sigma n-\alpha$ tend to infinity when n and m do, so the last term in (7.4) tends to zero. If the first term on the right in (7.4) be inverted and (7.3) be again applied, it will be seen at once that this term approaches unity uniformly for α in some neighborhood of the origin. Our Convergence Test, trivially extended to the case of a double sequence of m.g.f.'s, now yields the following theorem:*

THEOREM 7. 1. *Let $F_{m,n}(t)$ be the d.f. of the reduced z-distribution. Then* $\lim_{m\to\infty,n\to\infty}F_{m,n}(t) = \lim_{m\to\infty}\lim_{n\to\infty}F_{m,n}(t) = \lim_{n\to\infty}\lim_{m\to\infty}F_{m,n}(t) = N(t)$ *uniformly.*

The uniformity of the limit and the continuity of $N(t)$ imply that if $\{t_{m,n}\}$ is a sequence of points on the t-axis tending to the limit t as m and n tend to infinity in any manner, then $F_{m,n}(t_{m,n})$ tends to $N(t)$. This remark enables us to put the theorem in a more convenient form:

* A proof without the use of the m.g.f. has recently been published by Aroian 2, pp. 439–441.

THEOREM 7.2. *Let* $\mu_1=(1/n-1/m)/2$ *and* $\sigma_1=(1/m+1/n)/2$, *and let* $\mathcal{J}_{m,n}(t)$ *be the d.f. of the variate* $T=(Z-\mu_1)/\sigma_1$. *Then* $\lim_{m\to\infty,n\to\infty}\mathcal{J}_{m,n}(t)=\lim_{m\to\infty}\lim_{n\to\infty}\mathcal{J}_{m,n}(t)=\lim_{n\to\infty}\lim_{m\to\infty}\mathcal{J}_{m,n}(t)=N(t)$.

For we note that

$$P\left\{\frac{Z-\mu_1}{\sigma_1}\leqq t\right\}=P\left\{\frac{Z-\mu}{\sigma}\leqq t_{m,n}\right\},$$

where $t_{m,n}=(\sigma_1/\sigma)t+(\mu_1-\mu)/\sigma$; and from (7.1) and (7.3) we see that $t_{m,n}\to t$ as $m\to\infty$, $n\to\infty$.

A table of points of the z-distribution due to Fisher and Deming is readily available [8, pp. 250–255], [9]. The derivation of the approximate formulas for the 5 per cent and 1 per cent points recommended in connection with these tables [8, p. 239] has recently been published by Cochran [4]. It does not seem to be emphasized sufficiently in most of the texts on statistical methodology which refer to these tables that the instructions which accompany the tables have the effect of doubling the probability listed at the head of the table.*

8. The distribution of a linear form. Suppose that the n independent variates X_j each have the same s.g.f. $H(\alpha)$, $|\alpha|<\alpha_1$. Then the s.g.f. $H_n(\alpha)$ of the linear form $L_n=\sum_1^n c_jX_j$, where the numbers c_j may depend upon n, can be computed as follows:

$$H_n(\alpha)=\log E[\exp(\alpha\textstyle\sum_1^n c_jX_j)]=\log\prod_1^n E[\exp(\alpha c_jX_j)]=\sum_1^n H(\alpha c_j),$$

and $H_n(\alpha)$ exists in some neighborhood of the origin. Letting μ and σ^2 be the common mean and variance of the variates X_j, and μ_n and σ_n^2 the mean and variance of L_n, we have:

$$H_n'(\alpha)=\sum_1^n c_jH'(\alpha c_j),\qquad \mu_n=\mu\sum_1^n c_j$$

$$H_n''(\alpha)=\sum_1^n c_j^2H''(\alpha c_j),\qquad \sigma_n^2=\sigma^2\sum_1^n c_j^2,$$

$$\frac{H_n''(\alpha/\sigma_n)}{\sigma_n^2}=\frac{1}{\sigma^2}\sum_1^n d_j^2H''\left(\frac{\alpha}{\sigma}d_j\right),$$

where $d_j^2=c_j^2/\sum_1^n c_k^2$. (Notice that $\sum_1^n d_j^2=1$). Then

$$\left|\frac{H_n''(\alpha/\sigma_n)}{\sigma_n^2}-1\right|=\frac{1}{\sigma^2}\left|\sum_1^n d_j^2\left[H''\left(\frac{\alpha}{\sigma}d_j\right)-H''(0)\right]\right|$$
$$\leqq\frac{1}{\sigma^2}\left\{\max\left|H''\left(\frac{\alpha}{\sigma}d_j\right)-H''(0)\right|,\ j=1,\cdots,n\right\}.$$

It is apparent that $H_n(\alpha)$ will satisfy the Convergence Test of §3 if the quantities d_j tend to zero. Thus we have proved the following theorem:

* For further explanation of this remark, see [10, p. 144].

THEOREM 8.1. *Let* $L_n=\sum_1^n c_jX_j$ *be a linear form in the* n *identically distributed independent variates* $X_1, \cdots, X_n$, *whose m.g.f. exists in some neighborhood of the origin. Let* D_n *be an upper bound of the quantities* $(c_j)^2/\sum_1^n c_k^2$, $j=1, \cdots, n$. *Let* $F_n(t)$ *be the d.f. of the variate* $T_n=(L_n-\mu\sum_1^n c_j)/\sigma\sqrt{\sum_1^n c_j^2}$, *where* μ *and* σ *are the mean and variance of the variate* X_j. *Then if* $\lim_{n\to\infty}D_n=0$, *it follows that* $\lim_{n\to\infty} F_n(t)=N(t)$ *uniformly.*

The theorem is a special case of the Central Limit Theorem of probability theory [5, pp. 56 ff.]. The important cases are those in which the c_j's depend upon n. If $c_j=1/n$, $j=1, \cdots, n$, the function L_n becomes the "arithmetic mean" $\overline{X}$ of the variates X_j, and the theorem states that the distribution of $\sqrt{n}(\overline{X}-\mu)/\sigma$ tends to normality, provided that the m.g.f. of the variates X_j exists. A slight extension of the theorem serves to prove that the reduced distribution of the difference of two arithmetic means tends to normality. The two means become two proportions if the variate X_j assumes merely the values 0 and 1. The theorem may also be used to derive conditions under which the reduced distributions of the regression coefficients tend to normality.

Recently Berry [3] has obtained the following estimate of the error term in Theorem 8.1 in the more general case in which only a finite third absolute moment of the distribution of X_j is assumed:

$$\sup_{-\infty<t<\infty} |F_n(t) - N(t)| \leqq 1.88\sqrt{D_n}\,\nu_3/\sigma^3, \qquad \nu_3 = E(|X_j - \mu|^3).$$

9. The distributions of positional means. We shall indicate in this final section how Theorem 4.1 can be used to derive limit theorems for certain positional means of a sample. Specifically, we shall consider the median, but our methods are applicable at once to the other quartiles, deciles, and percentiles.

Let $X_1, X_2, \cdots, X_N$ be independent identically distributed variates.* A determination of these N variates will be called a sample. For each point $(x_1, x_2, \cdots, x_N)$ in the space of the joint distribution of the variates X_j, we define a function $w_r=f_r(x_1, x_2, \cdots, x_N)$ as follows: The numbers $x_1, x_2, \cdots, x_N$ are to be arranged in a row in order of magnitude, and w_r is the value of the rth number in this row. The corresponding variate $W_r=f_r(X_1, X_2 \cdots, X_N)$ is the *a priori* rth position in the rank list of the sample. For example, if $N=2n+1$, $r=n+1$, W_r is the *a priori* median of the sample. Henceforth we drop the phrase *a priori*, as is customary in sampling theory.

Let $H_r(w)$ denote the d.f. of W_r and $F(x)$ that of the variate X_j. A moment's reflection will show that $H_r(w)$ is equal to the probability that at least r of the variates X_j fall in the interval $-\infty<x\leqq w$. Since the probability that any one does so is $F(w)$, the law of repeated trials at once gives

$$H_r(w) = \sum_{k=r}^{N} {}_NC_k[F(w)]^k[1 - F(w)]^{N-k}.$$

* No assumption is made in this section as to the existence of a m.g.f. for the X_j's.

That is, if we were to introduce a variate Y with a simple binomial distribution with $N+1$ values and parameter $p=F(w)$, then $H_r(w)=P(r\leqq Y\leqq N)=P(r\leqq Y<\infty)$.

Now let W_r be the median of a sample of $N=2n+1$ observations; in this case, $r=n+1$ and we replace W_r by the symbol $\tilde{X}$. Let us impose upon $F(x)$ the condition that the equation $F(x)=1/2$ should have a unique solution $x=\tilde{\mu}$, which we shall call the median of the distribution of the variates X_j. In §8 we saw that the distribution of $\sqrt{2n+1}(\overline{X}-\mu)/\sigma$ becomes normal as $n\to\infty$, where μ and σ^2 are the mean and variance (assumed in that section to exist) of the variates X_j. This suggests that if we were to choose the constant K correctly, the distribution of $V_n=\sqrt{2n+1}(\tilde{X}-\tilde{\mu})/K$ might behave similarly—a conjecture which we shall now investigate.

Let $J_n(v)$ be the d.f. of V_n. Then $J_n(v)=H_{n+1}(\tilde{\mu}+Kv/\sqrt{2n+1})=P(n+1\leqq Y_n<\infty)$, where Y_n is a variate with a simple binomial distribution with $2n+2$ values and parameter $p(n)=F(\tilde{\mu}+Kv/\sqrt{2n+1})$. We now further assume that $F(x)$ is continuous at $x=\tilde{\mu}$. Then $(2n+1)p(n)[1-p(n)]\to\infty$ as $n\to\infty$; so if

$$T_n = \frac{Y_n - (2n+1)p(n)}{\sqrt{(2n+1)p(n)[1-p(n)]}},$$

then according to Theorem 4.1,

$$\lim_{n\to\infty} P(t \leqq T_n < \infty) = \int_t^\infty \frac{1}{\sqrt{2\pi}} e^{-z^2/2}dz,$$

uniformly in t. Furthermore, by the remark following Theorem 7.1, if $\{t_n\}$ is any sequence tending to t as $n\to\infty$, then $\lim_{n\to\infty}P(t_n\leqq T_n<\infty)=\lim_{n\to\infty}P(t\leqq T_n<\infty)$.

Now $J_n(v)=P(n+1\leqq Y_n<\infty)=P(t_n\leqq T_n<\infty)$, where

$$(9.1)\quad t_n = \frac{(n+1)-(2n+1)F(\tilde{\mu}+Kv/\sqrt{2n+1})}{\sqrt{(2n+1)F(\tilde{\mu}+Kv/\sqrt{2n+1})[1-F(\tilde{\mu}+Kv/\sqrt{2n+1})]}}.$$

At this stage we impose on $F(x)$ the additional (and final) restriction that $F'(\tilde{\mu})$ exists and is positive. Then

$$F(\tilde{\mu}+Kv/\sqrt{2n+1}) = F(\tilde{\mu}) + \frac{Kv}{\sqrt{2n+1}}[F'(\tilde{\mu})+0(1)].$$

Substituting in (9.1) and noting that $F(\tilde{\mu})=\frac{1}{2}$, we find that $\lim_{n\to\infty}t_n=-2KvF'(\tilde{\mu})$, so

$$\lim_{n\to\infty} J_n(v) = \int_{-2KvF'(\tilde{\mu})}^\infty \frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz = \int_{-\infty}^{2KvF'(\tilde{\mu})} \frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz.$$

Obviously the proper choice for K is $K=1/[2F'(\tilde{\mu})]$. Our result may be stated formally as follows:

THEOREM 9.1. *Let $\tilde{X}$ be the median of a sample of $(2n+1)$ independent observations, each with the d.f. $F(x)$. Let $F'(\tilde{\mu})$ exist and be positive, where $\tilde{\mu}$ is the median (assumed to exist) of the distribution specified by $F(x)$. Let $J_n(v)$ be the d.f. of the variate $V_n = 2F'(\tilde{\mu})\sqrt{2n+1}(\tilde{X}-\tilde{\mu})$. Then $\lim_{n\to\infty} J_n(v) = N(v)$.*

More briefly, the median in large samples drawn by simple random sampling from a continuous parent distribution is approximately normally distributed with mean $\tilde{\mu}$ and variance $1/\{4[F'(\tilde{\mu})]^2(2n+1)]\}$.

If applied to the first and third quartiles of a simple sample of $4n+1$ observations, our method yields the results that these statistics in large samples from a continuous parent distribution are approximately normally distributed with respective means q_k and variances $3/\{(64n+16)[F'(q_k)]^2\}$, $k=1, 3$, where q_1 and q_3 are the first and third quartiles of the X_j distribution.

In conclusion, it is worth pointing out that if Z has the Fisher z-distribution of §7, and if m and n are even integers, then an integration by parts will, show that $P(a \leqq e^{2Z} < \infty)$ is equal to the sum of the first $m/2$ terms, and, $P(-\infty < e^{2Z} \leqq a)$ is equal to the sum of the last $n/2$ terms, of the binomial expansion of $(q+p)^{(m+n-2)/2}$, with $p = ma/(n+ma)$, $q = 1-p$. The distribution of e^{2Z} has been tabulated by Snedecor under the name of the F distribution [9], and this provides a means of determining the points of the distribution of W_r in terms of those of $F(w)$. For example, in the case of the median of a sample of $N=23$, we find that if $H_{12}(w) = .95$, then $F(w) = .66$; that is, the 5 per cent point of the exact distribution of the median is the 34 per cent point of the distribution of the variates X_j.

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BOOLEAN ALGEBRA AS AN INTRODUCTION TO POSTULATIONAL METHODS*

E. R. STABLER, Hofstra College

The purpose of this paper is to suggest how a simple system of postulates and theorems for Boolean algebras might be used as an introduction to postulational methods in an elementary course in foundations, or fundamental concepts, of mathematics.

Postulational theory often is introduced in such a course by the study of significant sets of postulates closely related to elementary algebra or geometry. This procedure undoubtedly is valuable. The approach suggested here, however, begins with postulates for a mathematical system quite different from the ordinary systems of elementary mathematics. At the same time, the postulates are so selected that only one of them would seem unnatural or queer to undergraduates. It is believed that there may be some psychological advantages in an approach of this kind.

The postulates. The proposed set of postulates is listed below as Set R. Postulates 1–8 are identical with a set of postulates for Boolean rings given as Set III in a previous paper.† Stone has proved‡ that any Boolean ring with unit element is equivalent to a Boolean algebra (assuming suitable connecting definitions for the two systems). Thus the inclusion of Postulate 9 actually makes Set R a set of postulates for a Boolean algebra.

We presuppose a class K of elements $a, b, c, \cdots$ and two undefined operations $+$, $\times$, which may be referred to as ring addition and Boolean multiplication respectively. Hypotheses concerning elements belonging to K are not completely stated. The postulates are referred to briefly by name in parentheses.

Set R

1. If a, b are in K then i) $a+b$, and ii) ab are in K. (Closure)
2. $(a+b)+c=a+(b+c)$ (Associative addition)
3. If $a+b_1=a+b_2$ then $b_1=b_2$ (Left-hand cancellation)
4. If $a_1+b=a_2+b$ then $a_1=a_2$ (Right-hand cancellation)
5. $(ab)c=a(bc)$ (Associative multiplication)
6. $a(b+c)=ab+ac$ (Left-hand distributive)
7. $(a+b)c=ac+bc$ (Right-hand distributive)
8. $aa=a$ (Idempotent law)
9. There exists in K an element e such that for all a in K $ae=a$. (Unit element)

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* Presented to the Michigan Section of the Mathematical Association of America, Ann Arbor, Michigan, March 14, 1942.

† E. R. Stabler, Sets of postulates for Boolean rings, this MONTHLY, vol. 48, 1941, pp. 20–28. (To be referred to later as [S]).

‡ M. H. Stone, The theory of representations for Boolean algebras, Transactions of the American Mathematical Society, vol. 40, 1936, pp. 37–111.

Odd and even integers. Of course, the one postulate in Set R which is not of standard type is the idempotent law, asserting that for every element a (in K) $aa=a$. Some persons may object that this postulate is impossible or contrary to the ordinary laws of algebra. This reaction immediately calls for a concrete interpretation of the abstract system which will show that all of the postulates can be simultaneously valid. Such an example will establish the logical *consistency* of the postulates. Probably the most convincing example from the standpoint of the beginner would be one in which the operations $+$ and $\times$ are closely related to ordinary addition and multiplication. Accordingly, we can present the example of addition and multiplication of odd and even integers. In particular, let us refer to two elements called "odd" and "even" and formulate the obvious tables as indicated under Interpretation I. It is easy to see that this odd-even system satisfies all of the postulates.

Interpretation I

$+$	even	odd
even	even	odd
odd	odd	even

$\times$	even	odd
even	even	even
odd	even	odd

The theorems. With the consistency of the postulates established (and the postulates thus related to some ideas of a concrete nature) the next step would probably be the abstract deduction of some consequences of the postulates. From the standpoint of abstract algebra the immediate theorems which follow are most concisely stated in the terminology of groups and rings, but it is unnecessary to use this terminology. The results can merely be worked out as a new type of algebra. The proofs are all straightforward and the rôles played by the various postulates can be clearly identified. The need of proceeding in a formal manner should be apparent in view of the novel situation caused by the idempotent law. Some of the theorems will seem equally novel. Thus it will be found, as in [S], that every element is its own inverse with respect to addition ($a+a=0$), so that in solving equations of the type $x+a=b$ it is permissible to transpose terms without changing signs. Or again, as proved by Stone, if there are more than two elements in the system then there always exist "divisors of zero," *i.e.* elements a, b, with $a\neq 0$, $b\neq 0$, but with $ab=0$. The strength of the idempotent law in accounting for these results can be readily appreciated, and further comparisons can be made with standard properties of elementary algebra.

There is a certain analogy here with the classical geometric example for illustrating apparently conflicting mathematical theorems, namely the existence of both euclidean and non-euclidean geometries. In that example, it is true, we find a more clear-cut variation of certain theorems with changes in a crucial postulate. However, difficulties are sometimes encountered in an elementary discussion due to the confusion in the minds of the students between the mathematical-logical and the physical-intuitive aspects of geometry. Furthermore, it is not easy actually to deduce significant non-euclidean theorems in a brief, logical

development. Our algebraic example seems to avoid difficulties of this kind, and may thus prepare for a later postulational study of non-euclidean geometries.

Certain theorems following from Set R will be of a familiar rather than novel nature and will look as fundamental as the postulates; for example, the commutative laws of ring addition and multiplication. Someone may suggest that these principles might have been included with the postulates. In that case they would have been redundant. Thus the question arises: are any of the nine original postulates superfluous, or are these postulates logically independent of each other? It can be understood immediately that Postulate 8, the idempotent law, is independent of the other postulates. The simplest formal example to show this would probably be the following. Let K be the class of all positive integers, with $+$ and $\times$ taken to mean ordinary addition and multiplication; then the idempotent law fails while all of the other postulates are satisfied. In the same way, examples can be found to establish the independence of most of the other postulates.*

Interpretation from logic. Next, we might inquire whether the postulates of Set R can refer to any more substantial type of subject matter than odd and even integers. At this point it would be significant to give an interpretation from logic which satisfies all of the postulates. Consider a collection of propositions $(p, q, r, \cdots)$ of such a nature that they can be classified as "true" or "false." New propositions can be formed from these in various ways. For example, the *conjunction* of p and q is the proposition saying "both p and q are true"; the *disjunction* of p and q is the proposition saying "p is true or q is true, and perhaps both"; while the *complete disjunction* of p and q is the proposition saying "*either* p is true *or* q is true, but not both." Now let us interpret $p+q$ to mean the complete disjunction of p and q, and $p\times q$ to mean the conjunction of p and q. It is evident that the truth or falsity of $p+q$ and $p\times q$ is uniquely determined by the truth or falsity of the propositions p, q, themselves. Thus we can formulate truth value tables as indicated under Interpretation II, and we can think of this interpretation simply as a system of two elements "true" and "false."

Interpretation II

$+$	false	true
false	false	true
true	true	false

$\times$	false	true
false	false	false
true	false	true

On comparing the new tables with the original tables for the odd-even system, it is apparent that the new tables are simply a translation of the old ones in which the word "even" is translated as "false," and "odd" is translated as

* In [S] examples were given which proved the independence of the first eight postulates, the Boolean ring postulates. A number of the examples can be used again for Set R, but independence proofs remains to be found for Postulates 3 and 7 of Set R.

"true." Since all possible cases of the nine postulates are satisfied by the odd-even system, therefore they must be satisfied by the true-false system. Thus we have a simple illustration of two isomorphic interpretations of the same abstract set of postulates. This leads to the concept of categoricalness of postulates.

Categoricalness. A set of postulates is said to be categorical whenever every pair of possible interpretations of the postulates are isomorphic, that is, in 1–1 correspondence throughout their tables of operations. A third simple interpretation of the postulates of Set R will show that these postulates surely are not categorical. Consider the four *combinations* or *sets* of letters formed from the letters α, β, taken two, one, and zero at a time. Let $+$ designate the operation of symmetric difference, performed on these sets, and let $\times$ designate the operation of intersection.* This gives us an example of a Boolean algebra of four elements (the elements of the system in this case being themselves classes or sets). In a similar manner by starting with n letters, examples of a Boolean algebra could be constructed containing as elements the 2^n possible combinations or sets of letters taken $n, n-1, \cdots, 1, 0$ at a time.

In spite of the non-categoricalness of the postulates of Set R, it is easy and instructive to modify Set R so as to obtain a categorical set for a two-element Boolean algebra, or for a four-element Boolean algebra. All we have to do is to add a postulate requiring exactly two elements, or, alternatively, exactly four elements in the system. It is then possible to show (with the aid of our theorems) that the addition and multiplication tables are completely determined. It should be noted, however, that in each case the new postulate makes certain of the original postulates redundant. Thus, in the two-element case, Postulates 5 and 9 become redundant and should be deleted. In the four-element case Postulate 9 is again redundant. In fact, Stone has proved that if the number of elements in a Boolean ring is finite, then a unit element exists; so that Postulate 9 is redundant whenever we require a finite number of elements in the system.

Finally, it might be desirable to outline a proof of the equivalence or inter-deducibility of Set R and a standard set of postulates for Boolean algebras, say Huntington's first set† (to be called Set H). This set is expressed in terms of multiplication and Boolean addition (rather than our ring addition) and is well known for the complete duality between the two operations. In order to deduce Set H from Set R, we first can give definitions in the R system of Boolean addition ($\vee$) and of the complement, a', of an element, a (namely, $a \vee b = (a+b)+ab$ and $a'=a+e$). Then we can continue the development of theorems without any difficulty and arrive at all of the postulates of Set H (not already contained as

* The symmetric difference of two sets is the set consisting of all elements belonging to one or the other of two sets, but not both. The intersection of two sets is the set consisting of all elements common to both.

† E. V. Huntington, Sets of independent postulates for the algebra of logic, Transactions of the American Mathematical Society, vol. 5, 1904, pp. 288–309.

postulates in Set R). Conversely, starting with Set H, and referring to theorems proved by Huntington, it is possible to deduce the postulates of Set R, after first defining ring addition in terms of Boolean addition ($a+b=ab'\vee a'b$).

Summary. The postulational approach to Boolean algebra suggested here seems to have certain possibilities for introducing postulational methods and concepts to persons with little previous knowledge of foundations of mathematics or abstract mathematical systems. Thus, it furnishes a simple approach to the study of consistency, independence, and categoricalness of postulates; and it gives practice in easy formal deduction under novel conditions. At the same time, it serves to emphasize the relative nature of the truth of the theorems in any mathematical system. Finally, by examining two equivalent but quite dissimilar sets of postulates, it suggests that there is considerable freedom of choice as to the undefined concepts and fundamental propositions of such a system.

THE TWENTY-SIXTH ANNUAL MEETING OF THE KENTUCKY SECTION

The twenty-sixth annual meeting of the Kentucky Section of the Mathematical Association of America was held at the University of Kentucky on Saturday April 11, 1942, in conjunction with the annual meeting of the Kentucky Academy of Science. Professor L. A. Fair, chairman of the Section, presided.

There were twenty-eight in attendance including the following twenty-one members of the Association: N. B. Allison, P. P. Boyd, M. C. Brown, L. W. Cohen, H. H. Downing, L. A. Fair, Clarence Ford, Tryphena Howard, Fritz John, C. G. Latimer, F. Elizabeth Le Stourgeon, W. L. Moore, Sallie E. Pence, D. W. Pugsley, J. K. Reckzeh, G. G. Roberts, W. F. Smith, D. E. South, Guy Stevenson, Mary E. Williams, H. A. Wright.

A luncheon meeting was held in the Student Union building at which the following officers were elected for next year: Chairman, Charles Hatfield, Georgetown College; Secretary, M. C. Brown, University of Kentucky.

The following papers were presented:

1. "Asymptotic formulae" by Professor Fritz John, University of Kentucky.
2. "On conservative transformations of functions of two variables" by J. D. Rommel, Jr., University of Kentucky, introduced by Professor Cohen.
3. "Mathematical camouflage" by Professor H. H. Downing, University of Kentucky.
4. "Fractional integration" by Mrs. A. S. Howard, University of Kentucky, introduced by Professor John.
5. "The use of the Bernoulli coefficients in proving the binomial theorem" by Professor G. G. Roberts, Berea College.
6. "The net of Möbius" by Dean P. P. Boyd, University of Kentucky.
7. "A mathematical method of testing concave aspherical mirrors" by Pro-

fessor W. L. Moore, University of Louisville.

8. "An almost universal form" by Professor C. G. Latimer, University of Kentucky.

9. "Hermite polynomials" by J. C. Eaves, University of Kentucky, introduced by Professor Cohen.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Professor John emphasized the importance of asymptotic expansions for numerical calculations. The semi-convergent series for the probability integral, the γ function, and others were discussed as examples.

2. Mr. Rommel discussed double integral transformations of real-valued functions of two variables, $x(\sigma, \tau)$ into real-valued functions of two variables, $y(s, t)$. The formal expressions of the transformation used is given by

$$y(s, t) = \int_0^\infty \int_0^\infty k(s, t, \sigma, \tau) x(\sigma, \tau) d\sigma d\tau; \qquad s, t > 0.$$

Necessary and sufficient conditions that this transformation be conservative were stated by the author.

3. Professor Downing pointed out that criticisms of mathematical difficulties and the severe methods used in presenting the subject matter in mathematics texts, together with the demand for combined courses, unified and correlated courses, survey courses, and popular expositions of mathematics have led to some confusion. Titles of books do not always give an indication of the nature of the content, the same title used by different authors is applied to books carrying different subjects, unlike titles are used with books carrying the same matter. And the chapter headings and section headings are very often imaginative and fantastic.

4. Using the identity

$$\int_0^x \int_0^{\xi_n} \cdots \int_0^{\xi_1} f(\xi_1) d\xi_1 \cdots d\xi_n = \frac{1}{(n-1)!} \int_0^x (x - \xi)^{n-1} f(\xi) d\xi$$

Mrs. Howard defined $\phi(x)$ to be the n-th integral of $f(x)$ by

$$\phi(x) = \frac{1}{(n-1)!} \int_0^x (x - \xi)^{n-1} f(\xi) d\xi.$$

She extended the relationship between this and $f(x) = d^n\phi(x)$ by means of the Gamma function and proved some theorems concerning $\phi(x)$.

5. Professor Roberts defined the Bernoulli numbers, proved some special properties and used them in deriving a proof of the binomial theorem.

6. Dean Boyd presented the pioneer work of August Ferdinand Möbius, 1790–1868, mentioning his homogeneous coordinates, directed line segments, cross ratios, spherical trigonometry, twisted curves. His construction of corresponding elements in two collinear or correlative planes by means of the net

was described in detail and theorems on rational and irrational points were demonstrated.

7. In testing a twelve-inch telescope mirror, Professor Moore used the familiar Ronchii test. Trying to get the shape of the shadows of the Ronchiigram, he set up the equations of the shadows as if the source of light were a point. The wire in the grating would cut the cone of rays from the mirror surface and would appear as shadows on the surface. To get the theoretical curves the wire was considered as a line each point of which lay on the normals to the surface. The traces of these normals on the xz plane turned out to be

$$px^2(x - h)^3 + pz^2(x - h)^3 - d^2x^2(dx - 2px + 2ph) = 0$$

where h is the distance of the line from the optical axis, and d is the distance of the line from the surface of the mirror. When the Ronchii line is on the optical axis, $H=0$ and the trace of the normals on the xz plane consists of a straight line and a circle whose center is at the center of the mirror and whose radius is equal to the projection of the radius of the zone tested on the xz plane.

8. The paper of Professor Latimer was concerned with the integers represented by $f=x^2+y^2+7z^2+7w^2$. By employing known results on the quadratic sub-fields of a quaternion algebra and on the classes of integral sets in such an algebra, he showed that a positive integer is represented by f if and only if it is not in the form $3\cdot 7^n$ or $6\cdot 7^n$.

9. Mr. Eaves showed how a general expression for the Hermite Polynomials may be obtained by using the generating function $\rho - y^2+2xy$ and certain known properties of the polynomials. Other relationships and properties were deduced from these.

D. E. SOUTH, *Secretary*

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The department of Discussions and Notes is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

CLASSIFICATION OF THE CONICS

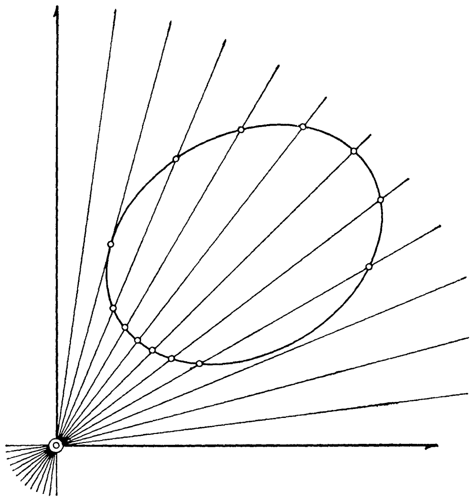
R. C. YATES, U. S. Military Academy

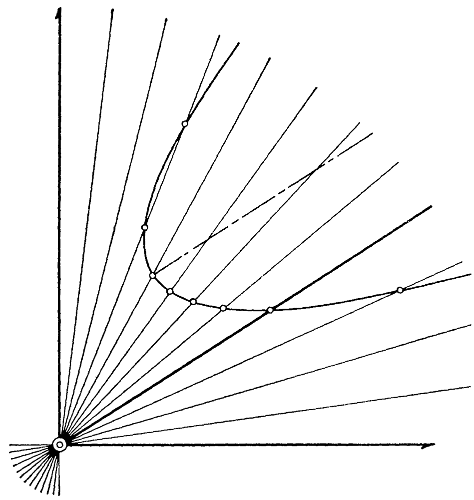
The universal method of obtaining the discriminant relation for the conics by a transformation of axes is tedious, involved, and often omitted in freshman courses. As a first introduction to geometrical invariants, its value is unquestionable. But if the primary object is the identification of the conics, the following approach has several desirable features that warrant its notice.

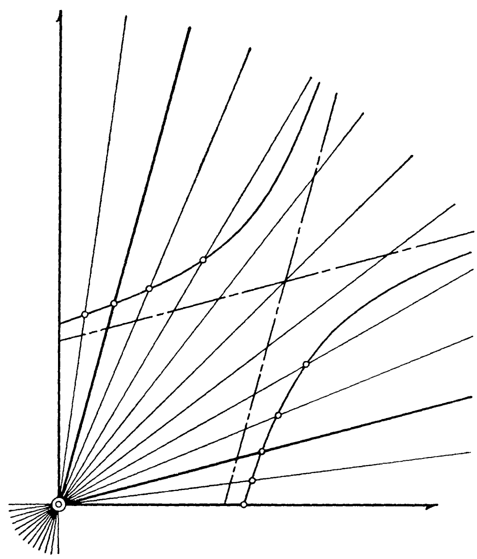
The three types of conics represented by the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \qquad (D, E \neq 0) \tag{1}$$

are characterized* as follows.







The *parabola* is such that there is *one and only one* line of the pencil $y = mx$ that cuts the curve in only one point (*i.e.*, the line parallel to the axis of the curve).† The *hyperbola* is such that there are *two and only two* distinct lines of the pencil

* If thought desirable, these characterizations may be justified by means of the "standard" forms with which the student is already familiar.

† We consider here "algebraic" points and real lines $y = mx$. A point of tangency, for instance, is counted twice.

$y=mx$ that cut the curve each in only one point (*i.e.*, the lines parallel to the asymptotes of the curve). The *ellipse* is such that there is *no* line of the pencil $y=mx$ that cuts the curve in only one point.

The pencil $y=mx$ meets (1) in points whose abscissas are given by

$$(A + Bm + Cm^2)x^2 + (D + Em)x + F = 0. \tag{2}$$

If some lines of the pencil are to yield but one point of intersection, then

$$A + Bm + Cm^2 = 0,$$

or,*

$$\boxed{m = \frac{-B \pm \sqrt{(B^2 - 4AC)}}{2C}}$$

For the parabola, only one such line exists. Thus†

$$\boxed{B^2 - 4AC = 0}\ .$$

For the hyperbola, there are two such lines. Thus‡

$$\boxed{B^2 - 4AC > 0}\ .$$

For the ellipse, there is no such line. Thus

$$\boxed{B^2 - 4AC < 0}\ .$$

The characterization fails when $D=E=0$. However, the abscissas of the points of the curve cut out by the pencil are given by

$$(A + Bm + Cm^2)x^2 + F = 0.$$

Since there are always two values of x, real or imaginary, equal and opposite in sign, it is evident that such conics are central.

* If $C=0$, $A\neq 0$, we establish, instead of (2), the corresponding equation for the ordinates of points of intersection of the pencil $x=ny$ and the conic.

† Here, obviously, $m=-B/2C$ is the slope of the axis of the curve. Rotation of axes through the angle arc tan $(-B/2C)$, accordingly, removes the xy term.

‡ The bisector of an angle formed by the two lines here is parallel to an axis of the curve. The angle of rotation to remove the xy term is thus easily obtained.

If $B^2-4AC>0$, the quantity $A+Bm+Cm^2$ changes sign as m ranges through all real values. Thus some lines of $y=mx$ cut the curve and some do not. The conic is thus a *hyperbola.*

If $B^2-4AC<0$, the quantity $A+Bm+Cm^2$ has the same sign for all real values of m. If this sign is opposite that of F, *all* lines $y=mx$ cut the curve. The conic is therefore an *ellipse.* If the sign is the same as that of F, the curve is imaginary.

AN EXISTENCE PROOF FOR LOGARITHMS

F. E. Hohn, University of Arizona

The purpose of this article is to give a simple proof of the following theorem, customarily assumed in college algebra courses:

THEOREM. *Given a real number $b>1$ and any positive number N, then there always exists a real number q, called the logarithm to the base b of N, such that $b^q=N$.*

The method of proof is to *construct the number q*, and to this end we prove first the following preliminary result:

LEMMA. *Given a number $b>1$ and any positive number N, we can construct a sequence of decimal fractions of the form $q_0=p_0$, $q_1=p_0+p_1/10$, $q_2=p_0+p_1/10+p_2/100$, $\cdots$, such that b^{q_n} approaches N as n increases.*

Proof: Since $N>0$ there is a unique integer p_0, not necessarily positive, such that

$$b^{p_0} \leqq N < b^{p_0+1}.$$

Then

$$1 \leqq N/b^{p_0} < b,$$

and hence

$$1 \leqq N^{10}/b^{10p_0} < b^{10}.$$

There is then a unique integer p_1, $0 \leqq p_1 \leqq 9$, such that

$$b^{p_1} \leqq N^{10}/b^{10p_0} < b^{p_1+1}.$$

Then

$$1 \leqq N^{10}/b^{10p_0+p_1} = N^{10}/b^{10q_1} < b,$$

so that

$$1 \leqq N^{100}/b^{100q_1} < b^{10}.$$

There is then a unique integer p_2, $0 \leqq p_2 \leqq 9$, such that

$$b^{p_2} \leqq N^{100}/b^{100q_1} < b^{p_2+1}.$$

Then

$$1 \leqq N^{100}/b^{100q_1+p_2} = N^{100}/b^{100q_2} < b.$$

This process can evidently be continued to obtain

$$1 \leqq N^{10^n}/b^{10^n q_n} < b,$$

which gives

$$1 \leqq N/b^{q_n} < b^{10^{-n}},$$

$$0 \leqq (N/b^{q_n}) - 1 < b^{10^{-n}} - 1$$

$$0 \leqq N - b^{q_n} < b^{q_n}(b^{10^{-n}} - 1) \leqq N(b^{10^{-n}} - 1). \tag{1}$$

The lemma follows at once if we assume that $b^{10^{-n}}$ approaches 1 as the integer n increases. This can be considered a consequence of the fact that $\lim_{x\to 0} b^x = b^0 = 1$, which most students will be willing to grant.

If we now define q and b^q by

$$q = \lim_{n\to\infty} q_n, \qquad b^q = \lim_{n\to\infty} b^{q_n},$$

then $N = b^q$, or $q = \log_b N$.

It is worth noting that as far as most applications of logarithms are concerned we never need to use anything beyond relation (1). For instance, a table accurate to 5 places can be constructed from (1) with $n = 6$.

Although this method of computing logarithms is not of much practical value, it can be carried out without too much labor. With the help of a table of 10th powers, the value of log 3 can be computed to 5 places in about ten minutes.

Note by the Editor. The proof that $\lim_{n\to\infty} b^{10^{-n}} = 1$, though probably not suitable for an elementary class, is quite simple. In the first place, $b^{10^{-n}} > 1$ for all positive n, for if $b^{10^{-n}} \leqq 1$, then $b = (b^{10^{-n}})^{10^n} \leqq 1$ contrary to assumption. Hence if $\lim_{n\to\infty} b^{10^{-n}} \neq 1$, we must have $b^{10^{-n}} > 1 + \epsilon$ for some $\epsilon > 0$ and arbitrarily large n. Then $b = (b^{10^{-n}})^{10^n} > (1+\epsilon)^{10^n} > 1 + 10^n\epsilon$ by the binomial theorem, and this cannot be true for arbitrarily large n. R.J.W.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Differential- und Integralrechnung. By Otto Haupt and Georg Aumann. I Band: *Einführung in die reelle Analysis*, pages 1–196, II Band: *Differentialrechnung*, pages 1–168, III Band: *Integralrechnung*, pages 1–183. Berlin, Walter de Gruyter and Company, 1938. RM. 30.00.

Differential- und Integralrechnung is designed for an analysis course intermediate between the traditional advanced calculus and the modern super-real variables, which begins with boolean algebras, lattices, topology and continues with Banach Spaces and integrals with values therein. Emphasis is placed on intuitive development rather than on ultra-formal argument. Nevertheless, some novel features are included.

In the first half of the first volume, *Introduction to Real Analysis*, the continuum is introduced rapidly (long before the reader gets tired) and the usual limit properties of number sequences and sets are defined and explored. In the second half real functions of one and more real variables are considered. At the end of this half the author has a section on inequalities and convex functions, a subject not generally found in a book of this type.

The second volume, *Differential Calculus*, again, is divided into two parts. In the first part the notion of a derivative is introduced and discussed. After the rules of differentiation are developed, the mean value theorem, Taylor series, what some texts call evaluation of *indeterminates*, the indefinite integral, and circular and hyperbolic functions are considered. These are then illustrated mainly with geometric examples and then the properties of monotone functions are given further attention. In the second half functions of many real variables are considered, with emphasis, at the end, on systems of equations and Jacobians.

The last volume, *Integration*, is the most abstract of the three, which is of course pedagogically justified when one considers the additional sophistication the reader has supposedly acquired on his way to the end. *Measure* and *content*, for the most part in general fields of sets (*Mengenkörper*) are dealt with in the first half. Here we find more emphasis on the notion of *content* than is customary. In the second half there is considered first, integration associated with *content* and second, integration associated with *measure*. The final section on examples and applications begins with properties of the Lebesgue integral and ends with a consideration of bounded k-dimensional *surfaces* in E_n and generalizations to E_n of the theorems of Gauss and Stokes.

SEYMOUR SHERMAN

An Outline of College Algebra. By Gerald E. Moore. N. Y. Barnes and Noble, Inc., 1942. 224 pages. $1.00.

The author states in his preface that this book is intended primarily for students who have had at least one year of high school algebra. After a brief summary (five chapters) of the elementary rules and operations of algebra the author considers those topics generally found in a freshman course in college algebra: solution of equations of various kinds, the binomial theorem, theory of equations, probability, determinants, etc. His method of presenting each of these topics is to state briefly the theory involved and then to give numerous examples illustrating this and methods of solving problems. These varied and well chosen illustrations are one of the best features of the book, the chapters on curve

plotting and quadratics being especially well done. The author is more careful than most about giving definitions, explaining new terminology, and in stating clearly the assumptions used in each problem. He wisely avoids giving proofs of such basic theorems as $(-a)(-b)=ab$. Proofs of these given in texts of this level are generally muddled; here the author frankly states that they are to be accepted by the student as rules which are best proved by the modern methods of abstract algebra.

As a review for someone who has already been over the material this book seems well suited, but the reviewer is dubious of the author's claim that it can be used as a text by students beginning the subject even if they have had a year of elementary algebra. Though complete and well arranged it seems too brief for this, and there are no problems scattered through the book for the student to cut his teeth on. There is a series of eight general exams in the back of the book, but these are not enough to give the student sufficient practice and a real mastery of the subject. An unusual feature is a bibliography of nineteen contemporary college algebra texts giving for each topic studied the pages where it may be found in each of the texts listed.

W. H. DURFEE

Lezioni di Analisi Matematica. Parte Seconda. Quarta Edizione. By Francesco Tricomi. Cedam—Casa Editrici Dott. A. Milani—Padova. 1939—XVIII. 8+355 pages. Lire 60.

This is the second of two volumes. The first volume ended with the indefinite integral. The second begins with the definition of the definite integral in the Riemann sense. Applications are made to ordinary algebraic and transcendental functions. Other topics treated are development in series, Taylor's formula, Simpson's rule, and the introduction of Fourier series.

Differentiation of functions of more than one variable is discussed with application to mean value, maximum and minimum, and Taylor's series. Topics in differential geometry include: tangent, length, osculating plane, curvature, and torsion for twisted curves; singularities and envelopes for plane curves; analytical representation and the Dupin indicatrix for surfaces; correspondence between points in two planes.

Multiple integrals are treated with application to areas and volumes.

In the chapter on ordinary differential equations types discussed are first order, linear and linear with constant coefficients, development in series, and some mention of equations of higher order.

Partial differential equations are discussed with their use in Laplace's equation and harmonic functions, Green's theorem, the problem of the vibrating cord, and the fundamental problem in the calculus of variation.

This volume is written in the same clear style as the first volume and, like it, has no additional exercises included although they are available.

A. H. BLACK

NEW BOOKS RECEIVED

Rapid Review of Elementary Algebra. (Penny Press Series.) By J. T. May. New York, Emerson Books, Inc., 1942. 32 pages. $0.18.

Technical Handbook for Solving Problems in Shop or Factory. By E. H. Lang. New York, Prentice-Hall, Inc., 1942. 4+100 pages. $1.00.

Plane and Spherical Trigonometry. By F. A. Rickey and J. P. Cole. New York, The Dryden Press, 1942. 10+209 pages. $2.25.

Military and Naval Maps and Grids, Their Use and Construction. By. W. W. Flexner and G. L. Walker. New York, Dryden Press, 1942. 96 pages. $1.00.

On Growth and Form. By D'Arcy W. Thompson. Revised Edition. Cambridge, University Press; New York, Macmillan Co., 1942. 1116 pages. $12.50.

Principles of College Algebra. By M. S. Knebelman and T. Y. Thomas. New York, Prentice-Hall, Inc., 1942. 10+380 pages. $2.50.

Differential Equations. By R. P. Agnew. New York, McGraw-Hill Book Co., 1942. 7+341 pages. $3.00.

Metric Methods in Finsler Spaces and in the Foundations of Geometry. By H. Busemann. (Annals of Mathematics Studies, No. 8.) Princeton, Princeton University Press; London, Humphrey Milford and Oxford University Press, 1942. 247 pages. $3.00.

Topics in Topology. By S. Lefschetz. (Annals of Mathematics Studies, No. 10.) Princeton, Princeton University Press; London, Humphrey Milford and Oxford University Press, 1942. 139 pages. $2.00.

Finite Dimentional Vector Spaces. By P. R. Halmos. (Annals of Mathematics Studies, No. 7.) Princeton, Princeton University Press; London, Humphrey Milford and Oxford University Press, 1942. 5+196 pages. $2.35.

PROBLEMS AND SOLUTIONS

Edited by Otto Dunkel, Orrin Frink, Jr., and H. S. M. Coxeter

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 556. *Proposed by C. W. Bruce, Wesleyan College, Macon, Georgia*

The graph of a quadratic function passes through three given points (x_i, y_i). Show that the abscissa of its maximum or minimum is

$$\begin{vmatrix} x_1^2 & y_1 & 1 \\ x_2^2 & y_2 & 1 \\ x_3^2 & y_3 & 1 \end{vmatrix} \div 2 \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

E 557. *Proposed by V. Thébault, San Sebastián, Spain*

A sphere (S) of constant radius rolls on a fixed sphere (O) in such a way as to pass through a fixed point A. Determine the loci of the centers of similitude of the spheres (S) and (O).

E 558. *Proposed by V. V. Nákladem, Philadelphia*

Let P be any point in the plane of a triangle ABC. Show that the sum of the squares of the areas of the three triangles PBC, PCA, PAB cannot exceed

$$(PA^2 + PB^2 + PC^2)^2/16.$$

E 559. *Proposed by M. A. Sadowsky, Illinois Institute of Technology*

Let a, b, c be non-negative numbers satisfying $a^2+b^2+c^2<1$. Prove that the three planes $x=a$, $y=b$, $z=c$ will cut out from the first octant of the unit sphere $x^2+y^2+z^2=1$ a curvilinear triangle of area

$$\begin{aligned} \tfrac{1}{2}\pi(1 - a - b - c) &+ a(\text{arc sin } b_a + \text{arc sin } c_a) \\ &+ b(\text{arc sin } c_b + \text{arc sin } a_b) \\ &+ c(\text{arc sin } a_c + \text{arc sin } b_c) \\ - \text{arc sin } b_c c_b - \text{arc sin } c_a a_c &- \text{arc sin } a_b b_a), \end{aligned}$$

where a_b stands for $a/\sqrt{1-b^2}$, and so on.

E 560. *Proposed by S. H. Gould, University of Toronto*

In the fifth book of his *Laws*, the philosopher Plato, discussing the distribution of land in a colony, seeks a number divisible by every integer from 1 through 10 and chooses 5040. Show in general that if m and n are positive integers with $n<p$, where p is the smallest prime greater than m, then $m!$ is divisible by n except when $m=3$.

SOLUTIONS

A Perfect Cube

E 522 [1942, 335]. *Proposed by V. Thébault, San Sebastián, Spain*

Find the smallest prime radix for which there exists a perfect cube of the form *abcabc*.

Solution by N. G. Gunderson, Cornell University

Let the perfect cube be

$$n^3 = ap^5 + bp^4 + cp^3 + ap^2 + bp + c = (p^3 + 1)(ap^2 + bp + c).$$

If $p^3+1=\prod q_i^{\alpha_i}$, where the q_i are distinct primes, then n must have as a factor $\prod q_i^{\beta_i}$, where β_i is given by

$$\alpha_i = 3(\beta_i - 1) + \gamma_i, \qquad 0 < \gamma_i \leqq 3.$$

Kraïtchik (*Recherches sur la Théorie des Nombres*, vol. II, 1929, pp. 152–156) has tabulated the factors of $(p^3+1)/(p+1)$ for $p<1000$. Also we have $n<p^2$ and $a\neq 0$. These conditions serve to exclude all values of p less than 1000 except 19, 23, 31, 293. For $p=19$ there are four suitable cubes:

1 2 1 1 2 1, 3 14 1 3 14 1, 8 16 8 8 16 8, 17 5 18 17 5 18.

Also solved by D. H. Browne, W. E. Buker, Daniel Finkel, and E. P. Starke.

Orthocentric Tetrahedra

E 523 [1942, 335]. *Proposed by N. A. Court, University of Oklahoma*

With the vertices of a given orthocentric tetrahedron $ABCD$ as centers, spheres are drawn orthogonal to a given sphere (H) concentric with the polar sphere of $ABCD$. Show that the radical planes of (H) with the four spheres considered form a tetrahedron which is orthocentric, and that its orthocenter coincides with that of $ABCD$.

Solution by L. M. Kelly, U. S. Coast Guard Academy

Let us refer to the four spheres as (A), (B), (C), (D). Since (H) is orthogonal to them, the orthocenter H must lie on the radical axis of (A), (B), (C). Also the intersection of the radical planes of (H) and (A), (H), and (B), (H), and (C), is the radical center of (A), (B), (C), (H), and so lies on the radical axis of (A), (B), (C). Hence, if we refer to this point as D', the line HD' is perpendicular to the plane of centers ABC, and so coincides with HD. But this line is also perpendicular to the radical plane of (H) and (D). Defining analogous points A', B', C', we conclude that H is the orthocenter of the tetrahedron $A'B'C'D'$.

Also solved by Howard Eves and the proposer. The proposer remarks that the two tetrahedra are reciprocal with respect to (H).

A Variant of the Eight Queens Problem

E 524 [1942, 336]. *Proposed by R. V. Heath, Wall St., New York*

Write the numbers 9, 10, 11, 12, 13, 14, 15, 16 in one line. Underneath, place the numbers 1, 2, 3, 4, 5, 6, 7, 8 in such an order that the eight sums and eight differences are sixteen different numbers. In how many ways can this be done?

Solution by H. D. Larsen, University of New Mexico

Consider the grid in the XY plane formed by the abscissas $1, 2, \cdots, 8$ and the ordinates $9, 10, \cdots, 16$. Each possible pair of values in the desired arrangement are the coordinates of a lattice point occurring in this grid. All points on the same "positive" diagonal represent pairs of numbers having the same sum, and all points on the same "negative" diagonal represent pairs having the same difference. Clearly, the conditions of the problem require the selection of eight lattice points such that no two of them lie on the same vertical line, the same horizontal line, or the same diagonal. This is precisely the classical Eight Queens Problem. (W. W. R. Ball, *Mathematical Recreations and Essays*, eleventh edition, pp. 165–171.) There are 92 different ways of selecting the eight points in the

manner stated. However, we must further exclude those cases in which the sum of one pair of numbers is equal to the difference of another pair, *i.e.*, in which diagonals of opposite kinds through two of the points intersect on the Y axis. Since this property is maintained by reflection, *i.e.*, by reversing the order of the digits, we need only test 46 of the 92 possible orders. Five of the 46 possibilities are found to pass this test; so there are just ten satisfactory arrangements, *viz.*

37286415,	42736815,	27581463,	36824175,	25741863,
51468273,	51863724,	36418572.	57142863,	36814752.

Also solved by Howard and Ruth Eves, and E. T. Frankel. Eight of the ten arrangements were found by the proposer. (See *Judge*, Apr. 1935, p. 23, and May 1935, p. 27.) Jekuthiel Ginsburg (*Scripta Mathematica*, vol. 5 (1938), p. 63) mistook this for the arithmetical form of the Eight Queens Problem itself, as he failed to notice the extra restriction.

A Commensurable Parallelepiped

E 525 [1942, 336]. *Proposed by Maurice Kraïtchik, New School for Social Research, New York City*

Find parallelepipeds with commensurable edges and diagonals.

Solution by E. P. Starke, Rutgers University

If the parallelepiped is to be rectangular, solutions are easily found. Let a, b, c be three mutually perpendicular edges. The diagonals are equal and have length $x=\sqrt{a^2+b^2+c^2}$, whence

$$x^2 - c^2 = a^2 + b^2.$$

For a solution, take a and b to be any two integers not both odd, and put $x = \frac{1}{2}(m+n)$, $c=\frac{1}{2}(m-n)$, where $mn=a^2+b^2$, m and n are of like parity, and $m>n$. For example:

$$a = 1, \qquad b = 2, \qquad c = 2, \qquad x = 3.$$

Any parallelepiped is determined completely when a trihedral angle is known and the lengths of the edges which form that angle. Let the three edges have lengths a, b, c; and let the face angles formed by b and c, c and a, a and b, be respectively α, β, γ. Let d and d' be the diagonals of the face containing angle α and sides b and c. Then

$$\begin{aligned} d^2 = b^2 + c^2 - 2bc\cos\alpha, \qquad d'^2 = b^2 + c^2 + 2bc\cos\alpha,\\ d^2 + d'^2 = 2(b^2 + c^2). \end{aligned} \tag{1}$$

The bisecting parallelogram whose sides are a and d has as its diagonals y and z, two diagonals of the parallelepiped. Thus

$$y^2 + z^2 = 2(a^2 + d^2).$$

Corresponding to a and d', we have similarly for the other two diagonals of the

parallelepiped,

$$w^2 + x^2 = 2(a^2 + d'^2)$$

and

$$w^2 + x^2 + y^2 + z^2 = 4a^2 + 2(d^2 + d'^2) = 4(a^2 + b^2 + c^2). \tag{2}$$

From (1) and analogous results for the other faces, we have

$$\begin{cases} \cos \alpha = (w^2 + x^2 - y^2 - z^2)/8bc, \\ \cos \beta = (w^2 - x^2 + y^2 - z^2)/8ca, \\ \cos \gamma = (w^2 - x^2 - y^2 + z^2)/8ab. \end{cases} \tag{3}$$

A tentative method for finding parallelepipeds with integral sides and diagonals is suggested by (2) and (3). Take any integers $a \leqq b \leqq c$. Take any of the representations (2) of $4(a^2+b^2+c^2)$ as a sum of four squares with $w \geqq x \geqq y \geqq z \geqq 0$. These numbers will correspond to a solution provided each fraction in (3) is less than 1 in absolute value and each angle in (3) is less than the sum of the other two. Examples follow:

(A) $a = b = 2$, $c = 3$, $w = x = 5$, $y = z = 3$, $\alpha = \text{arc cos } \frac{2}{3}$, $\beta = \gamma = 90°$.

(B) $a = b = c = 3$, $w = 7$, $x = y = 5$, $z = 3$, $\alpha = \beta = \text{arc cos } 5/9$, $\gamma = \text{arc cos } 1/9$.

(C) $a = b = 4$, $c = 1$. Here $4(a^2+b^2+c^2) = 132$, which is the sum of four squares in several ways, *viz.*

$$(w, x, y, z) = (11, 3, 1, 1), \quad (9, 5, 5, 1), \quad (9, 7, 1, 1), \quad (8, 6, 4, 4), \quad (7, 7, 5, 3).$$

No parallelepiped results, however, since for each of the first two representations at least one cosine exceeds unity, while for the third all cosines equal unity, and for each of the last two one face angle equals the sum of the other two. (The third is interesting as a limiting case where the vertices all lie on a line; the last two, where the vertices all lie in a plane.) The vertices of the parallelepiped (B), referred to rectangular axes through one vertex, are

$$(0, 0, 0), \quad (3, 0, 0), \quad \left(\frac{10}{3}, \frac{4\sqrt{5}}{3}, 0\right), \quad \left(\frac{1}{3}, \frac{4\sqrt{5}}{3}, 0\right),$$
$$\left(\frac{5}{3}, \frac{2\sqrt{5}}{3}, 2\right), \quad \left(\frac{14}{3}, \frac{2\sqrt{5}}{3}, 2\right), \quad (5, 2\sqrt{5}, 2), \quad (2, 2\sqrt{5}, 2).$$

A Locus Determined by Two Fixed Circles

E 526 [1942, 404]. *Proposed by R. C. Yates, Louisiana State University*

Find the locus of P if the angles formed by the tangents from P to two fixed circles are equal.

Solution by Howard Eves, Syracuse University

It is easily shown, by means of similar triangles, that the distances from P to the two centers are proportional to the respective radii. Hence P describes the

circle of similitude of the given circles.

Also solved by F. A. Alfieri, D. H. Browne, L. M. Kelly, J. Rosenbaum, P. D. Thomas, and J. B. Welchons. Browne remarks that this problem suggests another:

"An angle of given size moves so that its legs are always tangent internally (or externally) to two given circles. Find the locus of the vertex of the angle."

J. H. Butchart and A. K. Waltz interpreted the given problem in this manner, and found a locus consisting of two limaçons, each having double contact with both the given circles. By varying the size of the angle (while keeping the circles fixed), Butchart obtains a family of such limaçons; the locus of nodes is the circle of similitude of the given circles.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4070. *Proposed by P. Erdös, Princeton, N. J.*

Let ρ denote the length of the radius of the inscribed circle of the triangle ABC, let r denote the circumradius and let m denote the length of the longest altitude. Show that $\rho+r\leqq m$.

4071. *Proposed by Harry Langman, Brooklyn, N. Y.*

Setting

$$\frac{1}{1^{t+1}}+\frac{1}{2^{t+1}}+\frac{1}{3^{t+1}}+\cdots+\frac{1}{n^{t+1}}=b_t,$$

show that

$$\begin{vmatrix} b_0, & -b_1, & b_2, \cdots, & (-1)^{n-1}b_{n-1} \\ -(n-1), & b_0, & -b_1, \cdots, & (-1)^{n-2}b_{n-2} \\ 0, & -(n-2), & b_0, \cdots, & (-1)^{n-3}b_{n-3} \\ \cdot & \cdot & \cdot \ \cdots & \cdot \\ 0 & 0 & 0 & b_0 \end{vmatrix} = 1.$$

4072. *Proposed by Richard Bellman, Brooklyn College*

Show that

$$e^x = \frac{(1-x^2)^{1/2}(1-x^3)^{1/3}(1-x^5)^{1/5}\cdots}{(1-x)(1-x^6)^{1/6}(1-x^{10})^{1/10}\cdots}, \qquad |x| < 1,$$

where the exponents in the numerator are integers with an odd number of unrepeated prime factors; and those in the denominator have an even number of unrepeated prime factors.

4073. *Proposed by J. M. Feld, New York*

Let p_1 be the number of partitions of the positive integer N into an odd number of distinct positive integers and let p_2 be the number of partitions of N into an even number of distinct positive integers. For instance, for $N=8$, the odd partitions included in p_1 are 8, $5+2+1$, $4+3+1$; and the even partitions included in p_2 are $7+1$, $6+2$, $5+3$. Prove that: (1) If $N=n(3n\pm 1)/2$, where n is an odd integer, $p_1=p_2+1$. (2) If n is an even integer, $p_1=p_2-1$. (3) If N does not have the above form $p_1=p_2$.

4074. *Proposed by V. Thébault, San Sebastián, Spain*

On the sides of the given triangle ABC directly similar triangles ABC_1, etc., are constructed interiorly. Determine these latter triangles so that $(GA_1)^2+(GB_1)^2+(GC_1)^2$ is a minimum, where G is the centroid of ABC.

SOLUTIONS

Trilinear Polars of Four Points

4018 [1942, 64]. *Proposed by N. A. Court, University of Oklahoma*

If four points taken in the four faces of a tetrahedron are collinear, their trilinear polars for the respective faces cannot be (*a*) coplanar, or (*b*) hyperbolic.

I. *Solution by the Proposer*

Let P, Q, R, S be the given points in the faces BCD, CDA, DAB, ABC of the tetrahedron, and let p, q, r, s be their trilinear polars for the respective triangles.

(a) If p, q, r, s lie in the same plane μ, the harmonic pole M of μ for $DABC$ lies on each of the lines AP, BQ, CR, DS, (N. A. Court, *Modern Pure Solid Geometry*, p. 233, art. 716). Now if the points P, Q, R, S lie on a line h, the vertices of the tetrahedron $DABC$ will be coplanar, for they will all lie in the plane Mh, which is absurd.

(b) A necessary and sufficient condition for the lines p, q, r, s to be hyperbolic is that the lines AP, BQ, CR, DS be hyperbolic, *ibid.*, p. 294, ex. 14 *Mathesis*, 1926, p. 29, Q. 2278. If that is the case, a line d may be drawn through the vertex D meeting the lines AP, BQ, CR.

The line d lies in the plane DAP, hence the trace D' of d in the plane ABC is collinear with A and with the trace X of DP on the edge BC. Similarly the point D' lies on the analogous lines BY and CZ. Now the points X, Y, Z are collinear, for they lie on the line of intersection of the plane Dh with the plane ABC, hence the lines AX, BY, DZ cannot have the point D' in common. Thus the lines AP,

BQ, CR, DS are not hyperbolic, and therefore neither are the lines p, q, r, s.

Consequence: If four lines passing through the vertices of a tetrahedron are hyperbolic, their four traces in the respectively opposite faces cannot be collinear.

II. *Solution by H. S. M. Coxeter, University of Toronto*

Let the four points lie on the line whose Plücker coordinates, referred to the given tetrahedron, are (l, m, n, L, M, N). The points being distinct, we must have $lmnLMN \neq 0$. In fact, the points are

$$(0, n, -m, L), \qquad (-n, 0, l, M), \qquad (m, -l, 0, N), \qquad (L, M, N, 0),$$

and therefore their trilinear polars for the respective faces are

$$x = 0, \quad \frac{y}{n} - \frac{z}{m} + \frac{t}{L} = 0, \qquad y = 0, \quad \frac{z}{l} - \frac{x}{n} + \frac{t}{M} = 0;$$

$$z = 0, \quad \frac{x}{m} - \frac{y}{l} + \frac{t}{N} = 0; \qquad t = 0, \quad \frac{x}{L} + \frac{y}{M} + \frac{z}{N} = 0.$$

These four lines having Plücker coordinates

$$\left(0, \frac{1}{m}, \frac{1}{n}, \frac{1}{L}, 0, 0\right), \qquad \left(\frac{1}{l}, 0, \frac{1}{n}, 0, \frac{1}{M}, 0\right),$$

$$\left(\frac{1}{l}, \frac{1}{m}, 0, 0, 0, N\right), \qquad \left(0, 0, 0, \frac{1}{L}, \frac{1}{M}, \frac{1}{N}\right),$$

are linearly independent, and so cannot be coplanar or belong to a regulus.

But what *positive* statement can be made regarding them? Their mutual invariants (in the notation of P. W. Wood, *The Twisted Cubic*, Cambridge, England, 1913, p. 21) are

$$\tilde{\omega}_{14} = \tilde{\omega}_{23} = \frac{1}{mM} + \frac{1}{nN} = -\frac{lL}{mMnN},$$

$$\tilde{\omega}_{24} = \tilde{\omega}_{31} = -\frac{mM}{nNlL}, \qquad \tilde{\omega}_{34} = \tilde{\omega}_{12} = -\frac{nN}{lLmM}.$$

These, being proportional to $-(lL)^2 : -(mM)^2 : -(nN)^2$, satisfy the relation

$$(\tilde{\omega}_{23}\tilde{\omega}_{14})^{1/4} + (\tilde{\omega}_{31}\tilde{\omega}_{24})^{1/4} + (\tilde{\omega}_{12}\tilde{\omega}_{34})^{1/4} = 0,$$

which shows that *the four lines are tangents to an infinity of twisted cubics.*

Editorial Note. If one of the four points is at a vertex of $ABCD$, say P is at D, then p may be any straight line in the plane of BCD passing through D. If P is on an edge, say CD, but not at a vertex, then p contains CD and each point on CD has this same polar p with respect to BCD. Suppose now that in (a) no one of the four points is at a vertex, P lies on CD, and Q does not lie on a side of CDA. Since here the four points lie on a straight line h, the point R is on DA, S is on CA, and p, r, s are the straight lines of CD, DA, AC, while q is in the plane

of CDA. Hence in (a) we exclude the case of any one of the four points being on an edge of $ABCD$. In this case the point M of the above argument does not lie on h.

In (b) if one of the four points lies on an edge either at or not at a vertex and the four points lie on a straight line h, an examination of the various cases shows that at least two of p, q, r, s are not skew. In the case mentioned above all four are coplanar. Thus we have to consider only the case where no one of the four points lies on an edge, and we shall show that p, q, r, s cannot be a hyperbolic set. If they could be hyperbolic, then, since P does not lie on a side of BCD, the polar p cannot contain a side or pass through a vertex of BCD, and p meets BC in X' not at B or C. We then determine the point X on BC so that B, X, C, X' is a harmonic set of points, and similarly Y on CA, Z on AB. The straight line d' of X' and Y' meets s as well as p and q, and hence also r by reason of the hypothesis. Thus X', Y', Z' lie on d' and AX, BY, CZ meet in D', the pole of d' with respect to ABC. Also P, Q, R lie on DX, DY, DZ, and hence the plane Dh passes through X, Y, Z. But since D' cannot lie on a side of ABC it is impossible for X, Y, Z to lie on a straight line, and we must reject the hypothesis that p, q, r, s are hyperbolic. This completes the proof of (b) with the above limitation.

We now drop the condition that P, Q, R, S lie on a straight line but retain the above restriction on their location, and suppose that p, q, r, s are hyperbolic. then AP, BQ, CR, DS are hyperbolic. We show first that no two of the latter are coplanar. For, if two, say AP, BQ are coplanar, then their plane cuts DC in a point and the fourth harmonic of this point with respect to C and D lies on both p and q. But this is impossible since p and q are skew. Next we see from the above that the straight line $d=DD'$ meets AP, BQ, CR, DS, and similarly for the straight lines a, b, c defined in the same manner. No two of the latter four lines coincide; for, if $a=d$, then D' is at A which is impossible as indicated above. This completes the proof and shows also that a, b, c, d is a hyperbolic set. We may reverse this argument by supposing that AP, BQ, CR, DS are hyperbolic and show then that p, q, r, s are hyperbolic. Thus there is one and only one straight line d through D meeting AP, BQ, CR; denote by D' its intersection with the plane of ABC; the reader can easily supply the remaining details of the proof. The four straight lines a', b', c', d' are also hyperbolic.

If we allow one of the four points to lie on an edge then it is possible for AP, BQ, CR, DS to be hyperbolic while p, q, r, s is not a hyperbolic set. The condition imposed upon the location of the four points is equivalent to the condition that there shall be a one to one correspondence between each of the four points and its trilinear polar. We may now state the theorem.

If the points P, Q, R, S lie on the planes of the faces BCD, CDA, DAB, ABC of a tetrahedron $ABCD$ so that no one is on an edge, a necessary and sufficient condition that the straight lines of AP, BQ, CR, DS shall be hyperbolic is that the polars p, q, r, s of the corresponding points with respect of the corresponding triangles of the faces be hyperbolic.

Tangents to a Parabola

4019 [1942, 64]. *Proposed by Robin Robinson, Dartmouth College*

Given a triangle ABC. Prove that the bisectors of the interior and exterior angles at C, the side AB and its perpendicular bisector, and the perpendiculars to AC at A and to BC at B, are all tangent to a parabola. Locate its focus.

Solution by the Proposer

The conics confocal at A and B constitute a pencil of line-conics, tangent to the isotropic lines through A and B. The locus of the polar of C with respect to the conics of the pencil is the "11-line conic" of C with respect to the pencil, and is the desired parabola.

The 11-line conic of a point C with respect to the pencil of line-conics tangent to the four sides of a given complete quadrilateral contains the following 11 lines:

(*a*) The 2 fixed lines of the Sturm's Involution formed by pairs of tangents to conics of the pencil drawn from C; these fixed lines are the tangents at C to the two conics of the pencil which pass through C.

(*b*) The 3 sides of the common self-conjugate triangle of the pencil, which are the diagonals of the complete quadrilateral.

(*c*) 6 lines, each of them the harmonic conjugate, with respect to two sides of the complete quadrilateral, of the join of their common vertex to C.

In this case, where the isotropic lines through A and B constitute the complete quadrilateral, its vertices are the common foci A and B, the common imaginary foci lying on the perpendicular bisector of AB, and the circular points at infinity I and J. Hence the 11-line conic contains as tangents:

(*a*) The tangents at C to the two conics of the pencil through C, which are (because of the optical property) *the bisectors of the interior and exterior angles at* C.

(*b*) The common axes of the conics of the pencil (*AB and its perpendicular bisector*), and the line at infinity. Hence the 11-line conic is a *parabola.*

(*c*) The harmonic conjugate of AC with respect to the isotropics through A, *viz., the perpendicular to AC at A*; similarly *the perpendicular to BC at B.* The analogous lines at the imaginary foci we shall not use, but those at I and J are isotropic tangents to the parabola, and hence intersect at its focus F. Since AI and BI separate harmonically CI and FI, and similarly for J, the locus of a point whose joins to A, B and C, F form a harmonic set is a circle through A, B, C, F. If four points on a conic form a harmonic set, the tangents at one pair meet on the join of the other pair. Hence *the focus F may be located by drawing the other tangent to the circle through A, B, C from the point where the tangent at C meets AB.*

Note. It is also worth mentioning that the axis of the parabola is perpendicular to the median drawn from C, and that C and F are reciprocal points ($z'=1/z$) in an Argand diagram in which A and B are the points $z=\pm 1$.

Editorial Note. An analytic solution is as follows. Let Δ_i, $i=1, 2, 3, 4$, be four straight lines no two of which are parallel, and intersecting in six distinct points

A_{ij}, Then a parabola $y^2=4ax$ is tangent to the four lines, and we have

$$(1)\qquad \Delta_i:\ m_iy - x - am_i^2 = 0; \qquad A_{ij}:\ am_im_j,\ a(m_i + m_j).$$

Let Δ_4 be the perpendicular bisector of the segment $A_{13}A_{23}$; then A_{34} lies on the directrix since Δ_3 and Δ_4 are perpendicular tangents, and we find by computation that

$$(2)\qquad m_1 + m_2 = 2m_4 = -2/m_3.$$

Let Δ_1', Δ_2' be respectively the perpendiculars at A_{13}, A_{23} to Δ_1, Δ_2, intersecting in A_{12}'. We then find the coordinates of the latter point to be

$$(3)\qquad A_{12}':\ -a,\ 2a(m_1m_2 - 1)/(m_1 + m_2).$$

Hence the straight line through A_{34} and A_{12}' is the directrix. The slopes k_1 and k_2 of the two bisectors of angle $A_{13}A_{12}'A_{23}$ satisfy the equation

$$(4)\qquad k^2 + 2\left(\frac{m_1m_2 - 1}{m_1 + m_2}\right)k - 1 = 0.$$

The equations of the two tangents to the parabola with the slopes k_1 and k_2 may be written by means of (1) and also their intersection a/k_1k_2, $a(k_1+k_2)/k_1k_2$. But by (4) these coordinates reduce to those in (3). This proves that the two bisectors of angle $A_{13}A_{12}'A_{23}$ are tangents to the parabola.

The two circles having diameters $A_{23}A_{24}$ and $A_{13}A_{14}$ intersect in A_{34} and the focus F. It is also seen geometrically that the straight line of the centers is a tangent, since it is the perpendicular bisector of FA_{34}. Similarly, the perpendicular bisector of FA_{12}' is a tangent. Obviously the midpoints of FA_{34}, FA_{12}' lie on the tangent at the vertex.

Areas of Related Triangles

4016 [1941, 705]. CORRECTED. *Proposed by V. Thébault, San Sebastián, Spain*

The points D, E, F are taken on the sides BC, CA, AB of a triangle ABC, and the points α, β, γ are then taken on the straight lines AD, BE, CF so that $A\alpha/AD=B\beta/BE=C\gamma/CF=k$ and $\alpha D/AD=\beta E/BE=\gamma F/CF=\lambda$. Show that the area σ of triangle $\alpha\beta\gamma$ is given by

$$\sigma = (2S + s)k^2 + (1 - 3k)S = \lambda(2\lambda - 1)S + (1 - \lambda)^2s,$$

where S and s denote the areas of ABC and DEF.

Deduce from this that in a complete quadrilateral the midpoints of the three diagonals are collinear.

Solution by P. W. Allen Raine, Newport News High School

Let the letters above denote the vector coordinates of the respective points.

Then by our hypothesis, we have

$$(A-\alpha)=k(A-D),\qquad (B-\beta)=k(B-E),\qquad (C-\gamma)=k(C-F),$$

from which we get

$$(\alpha-\beta)=(1-k)(A-B)+k(D-E),$$
$$(\beta-\gamma)=(1-k)(B-C)+k(E-F).$$

Therefore

$$\begin{aligned}\sigma &= \left|\,\alpha-\beta\ \ \beta-\gamma\,\right|/2\\ &= (1-k)^2\left|\,A-B\ \ B-C\,\right|/2+k^2\left|\,D-E\ \ E-F\,\right|/2\\ &\quad -k(1-k)\left\{\left|\,A-B\ \ F-E\,\right|/2+\left|\,E-D\ \ B-C\,\right|/2\right\}\\ &= (1-k)^2S+k^2s-k(1-k)S\\ &= (2S+s)k^2+(1-3k)S\end{aligned}$$

Setting $k=1-\lambda$ we obtain the second formula.

In applying this to the complete quadrilateral, let the vertices of the quadrilateral be the points A, B, D, E, and let F, an exterior point of AB be the intersection of AB and DE, the point C being the intersection of BD and AE. Then the diagonals of the quadrilateral are AD, BE, and CF with α, β, γ their respective mid-points.

Since $s=0$, for E, F, and D are collinear, we get, by simple substitution of $k=\lambda=\frac{1}{2}$ in the above expressions, $\sigma=0$, *i.e.* α, β, γ are collinear.

Solved also by Howard Eves, using vectors, C. E. Springer, and by the proposer.

A Differential Operator

4017 [1942, 64]. *Proposed by N. H. Anning, University of Michigan*

It is easy to show that $D(D^2+16)$ will annihilate $\cos^4 x+\sin^4 x$ and also $\cos^6 x+\sin^6 x$. Show in general that the same differential operator will annihilate similar expressions with exponents $4k$ and $4k+2$, where k is any positive integer.

Solution by J. A. Bullard, University of Vermont

By well known formulae expressing powers of sin x and cos x in terms of cosines of multiple angles (see this MONTHLY, vol. 40, 1933, p. 161) we have,

$$\begin{aligned}\cos^{4k}x+\sin^{4k}x &= \frac{1}{2^{4k}}\Bigg[2\sum_{h=0}^{2k-1}{}_{4k}C_h\cos 2(2k-h)x+{}_{4k}C_{2k}\\ &\qquad +2\sum_{h=0}^{2k-1}(-1)^h{}_{4k}C_h\cos 2(2k-h)x+{}_{4k}C_{2k}\Bigg]\\ &= \frac{1}{2^{4k-1}}\left[2\sum_{h=0}^{k-1}{}_{4k}C_{2h}\cos 4(k-h)x+{}_{4k}C_{2k}\right].\end{aligned}$$

and

$$\cos^{4k+2} x + \sin^{4k+2} x = \frac{1}{2^{4k+2}}\Bigg[2 \sum_{h=0}^{2k} {}_{4k+2}C_h \cos 2(2k+1-h)x + {}_{4k+2}C_{2k+1}$$
$$- 2 \sum_{h=0}^{2k} (-1)^h {}_{4k+2}C_h \cos 2(2k+1-h)x + {}_{4k+2}C_{2k+1} \Bigg]$$
$$= \frac{1}{2^{4k+1}}\Bigg[2 \sum_{h=0}^{k-1} {}_{4k+2}C_{2h+1} \cos 4(k-h)x + {}_{4k+2}C_{2k+1} \Bigg].$$

From the theory of linear differential equations it is well known that the operator $F(D) = D(D^2+16)(D^2+64) \cdots (D^2+16k^2)$ will annihilate both these expressions containing a constant term and the cosines of the k multiple angles.

Solved also by Edward Fleisher and A. K. Waltz.

A Determinant Evaluation

4023 [1942, 127]. *Proposed by J. A. Greenwood, Duke University*

Find an expression for the determinant of order $2n$

$$\begin{vmatrix} \theta I_n, & A_n \\ A_n, & \theta I_n \end{vmatrix},$$

where θI_n is a square matrix of order n having the variable θ in the principal diagonal and zeros elsewhere, and A_n is a square symmetric matrix of order n with a for each principal diagonal element, unity for the elements in the two parallels immediately above and below this principal diagonal, and zeros elsewhere.

Solution by John Williamson, Queens College

By matrix multiplication we find that

$$\begin{pmatrix} I_n & 0 \\ -\theta^{-1} A_n & I_n \end{pmatrix}\begin{pmatrix} \theta I_n & A_n \\ A_n & \theta I_n \end{pmatrix} = \begin{pmatrix} \theta I_n & A_n \\ 0 & \theta I_n - \theta^{-1} A_n^2 \end{pmatrix}$$

and, by taking determinants of both sides of this matrix equation, that

$$(1) \qquad \Delta = \begin{vmatrix} \theta I_n & A_n \\ A_n & \theta I_n \end{vmatrix} = |\theta^2 I_n - A_n^2| = |\theta I_n + A_n|\,|\theta I_n - A_n|$$
$$= (-1)^n |A_n + \theta I_n|\,|A_n - \theta I_n| = (-1)^n f(\theta) f(-\theta),$$

where $(-1)^n f(\theta)$ is the characteristic polynomial of A_n. Therefore Δ can easily be found if its characteristic polynomial $f(\theta)$ can be calculated. In particular, if A_n has the value given in this problem and if

$$(2) \qquad P_n(a) = |A_n|,$$

$$(3) \qquad |A_n - \theta I_n| = P_n(a - \theta).$$

The determinant $P_n(a)$ is a perpetuant and its value can be calculated from the recursion formula

$$P_n(a) = aP_{n-1}(a) - P_{n-2}(a)$$

with the initial conditions

$$P_2(a) = a^2 - 1, \qquad P_1(a) = a.$$

It is now easy to show by induction that

$$P_n(a) = \sum_{i=0}^{[n/2]} (-1)^i \binom{n-i}{i} a^{n-2i}. \tag{4}$$

Hence, from (1) and (3),

$$\Delta = (-1)^n P_n(a-\theta) P_n(a+\theta), \tag{5}$$

where $P_n(a)$ is defined by (4).

Several generalizations are possible. If each unit in A_n is replaced by b, then Δ has the value

$$(-1)^n b^{2n} P_n\left(\frac{a+\theta}{b}\right) P_n\left(\frac{a-\theta}{b}\right).$$

If A_n is the matrix with a for each element in the leading diagonal, with b for each element immediately above, and with c for each element immediately below the leading diagonal, and all other elements zero,

$$A_n = P_n(a, b, c) = \sum_{i=0}^{[n/2]} (-1)^i \binom{n-i}{i} a^{n-2i} b^i c^i. \tag{6}$$

The corresponding value of Δ is

$$(-1)^n P_n(a-\theta, b, c) P_n(a+\theta, b, c).$$

The matrix whose determinant we have been considering is a particular case of what might be called a block circulant matrix.* We may write

$$R = \begin{pmatrix} \theta I_n & A_n \\ A_n & \theta I_n \end{pmatrix} = E + A_n T, \quad \text{where} \quad T = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

The matrix T is similar to

$$\begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}$$

and therefore the matrix R is similar to

$$\begin{pmatrix} I_n + A_n & 0 \\ 0 & I_n - A_n \end{pmatrix}.$$

* John Williamson. "The latent roots of a matrix of special type." Bull. Amer. Math. Soc., Aug. 1931. The matrices considered in §3 of this paper are orthogonally equivalent to block circulant matrices, since they can be reduced to that form by a re-arrangement of the rows and the same re-arrangement of the columns.

In passing to the more general case we define T to be an m-rowed square matrix whose elements are n-rowed square matrices. The element in the last row and first column of T is the unit matrix I_n, as are all elements in the diagonal above the leading one. Each other element of T is the zero matrix. If $m=3$,

$$T = \begin{pmatrix} 0 & I_n & 0 \\ 0 & 0 & I_n \\ I_n & 0 & 0 \end{pmatrix}.$$

The matrix T is an mth root of the unit matrix and is similar to the diagonal block matrix

$$[\omega_1 I_n, \omega_2 I_n, \cdots, \omega_m I_n],$$

where $\omega_1, \omega_2, \cdots, \omega_m$ are the m distinct mth roots of unity. If

$$R = R_0 T^0 + R_1 T + \cdots + R_{m-1} T^{m-1},$$

where each R_i is a square matrix of order n, R is similar to the diagonal block matrix

$$\left[\sum_{i=0}^{m-1} \omega_1^i R_i, \sum_{i=0}^{m-1} \omega_2^i R_i, \cdots, \sum_{i=0}^{m-1} \omega_m^i R_i \right]$$

and

$$(7) \qquad |R| = \prod_{j=1}^{m} \left| \sum_{i=0}^{m-1} \omega_j^i R_i \right|.$$

For example, if $m=3$,

$$R = \begin{pmatrix} R_0 & R_1 & R_2 \\ R_2 & R_0 & R_1 \\ R_1 & R_2 & R_0 \end{pmatrix}$$

and R is similar to

$$\begin{pmatrix} R_0 + R_1 + R_2 & 0 & 0 \\ 0 & R_0 + \omega R_1 + \omega^2 R_2 & 0 \\ 0 & 0 & R_0 + \omega^2 R_1 + \omega R_2 \end{pmatrix}$$

while

$$|R| = |R_0 + R_1 + R_2| \, |R_0 + \omega R_1 + \omega^2 R_2| \, |R_0 + \omega^2 R_1 + \omega R_2|.$$

If $R_2 = \theta I_n$ and R_i is obtained from A_n by replacing a by a_i,

$$\left| \sum_{i=0}^{m-1} \omega_j^i R_i \right| = \left| \theta I_n + \sum_{i=1}^{m-1} \omega_j^i R_i \right| = P_n\left(\theta + \sum_{i=1}^{m-1} \omega_j^i a_i, \sum_{i=1}^{m-1} \omega_j^i, \sum_{i=1}^{m-1} \omega_j^i \right)$$

$$P_n\left(\theta + \sum_{i=1}^{m-1} a_i, m - 1, m - 1\right) = (m - 1)^n P_n\left(\frac{\theta + \sum a_j}{m - 1}\right), \quad \omega_j = 1,$$

$$P_n\left(\theta + \sum_{i=1}^{m-1} \omega_j^i a_i, - 1, - 1\right) = (- 1)^n P_n\left(- \theta - \sum_{i=1}^{m-1} \omega_j^i a_i\right), \quad \omega_j \neq 1.$$

Hence from (7) we have

$$(8) \quad | R | = (m - 1)^n(- 1)^{n(m-1)} P_n\left(\frac{\theta + \sum_{i=1}^{m-1} a_i}{m - 1}\right) \prod_{j=2}^{m-1} P_n\left(- \theta - \sum_{i=1}^{m-1} \omega_j^i a_i\right),$$

where $\omega_2, \omega_3, \cdots, \omega_m$ are the $m-1$ mth roots of unity distinct from unity itself. If $a_i = a, i = 1, 2, \cdots, m-1$, (8) becomes

$$| R | = (m - 1)^n(- 1)^{n(m-1)} P_n\left(a + \frac{\theta}{m - 1}\right) [P_n(a - \theta)]^{m-1}.$$

In particular if $m = 3$,

$$\begin{vmatrix} I_n & A_n & A_n \\ A_n & I_n & A_n \\ A_n & A_n & I_n \end{vmatrix} = 2^n P_n(a + \tfrac{1}{2}\theta) [P_n(a - \theta)]^2.$$

C. R. Cassity gave a formula expressing the given determinant in terms of similar determinants for smaller values of n.

Similar Triangles on a Polygon

4025 [1942, 128]. *Proposed by V. Thébault, San Sebastián, Spain*

Let $A_1', A_2', \cdots, A_{2n}'$ be the vertices of equilateral triangles constructed externally (or internally) on the sides $A_1A_2, A_2A_3, \cdots, A_{2n}A_1$ of a plane polygon of $2n$ sides $(P) \equiv A_1A_2 \cdots A_{2n}$, and $M_1, M_2, \cdots, M_n$ be the midpoints of the principal diagonals $A_1A_{n+1}, A_2A_{n+2}, \cdots, A_nA_{2n}$ of (P). The midpoints $M_1', M_2', \cdots, M_n$ of the principal diagonals $A_1'A_{n+1}', A_2'A_{n+2}', \cdots, A_n'A_{2n}'$ of the polygon $(P') \equiv A_1'A_2', \cdots, A_{2n}'$ are the vertices of equilateral triangles constructed upon the sides of the polygon $(p) \equiv M_1M_2 \cdots M_n$.

Generalize by replacing the equilateral triangles by similar isosceles triangles.

I. *Solution by J. R. Musselman, Western Reserve University*

The problem may be generalized by replacing everywhere equilateral triangles by triangles similar to a given triangle $X_1X_2X_3$. If we denote the vertices of the polygon P by the complex coordinates $a_i (i = 1, 2 \cdots 2n)$; then the conditions that the triangles $A_i' \; A_{i+1} A_i$ and $A_{n+i}' A_{n+i+1} A_{n+i}$, constructed externally upon the sides $A_{i+1} \; A_i$ and $A_{n+i+1} \; A_{n+i}$ respectively, be similar to the triangle $X_1X_2X_3$ are

$$\begin{vmatrix} a_i' & a_{i+1} & a_i \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix} = 0; \quad \text{and} \quad \begin{vmatrix} a_{n+i}' & a_{n+i+1} & a_{n+i} \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

[all subscripts taken mod $(2n)$].

Adding these two determinants and dividing by 2, we have

$$\begin{vmatrix} \frac{1}{2}(a_i' + a_{n+i}') & \frac{1}{2}(a_{i+1} + a_{n+i+1}) & \frac{1}{2}(a_i + a_{n+i}) \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Now $\frac{1}{2}(a_i' + a_{n+i}')$ is the coordinate of the midpoint, M_i', of $A_i' A_{n+i}'$, $\frac{1}{2}(a_{i+1} + a_{n+i+1})$ and $\frac{1}{2}(a_i + a_{n+i})$ are the coordinates of the midpoints of $A_{i+1}A_{n+i+1}$ and A_iA_{n+i} respectively. Hence we have proved that the triangle $M_i' M_{i+1}M_i$ is similar to the triangle $X_1X_2X_3$. It is obvious that the problem is true if we replace midpoints everywhere, by points dividing their respective segments in the ratio $\lambda:\mu$. A similar argument would show the truth of the problem if the similar triangles all be constructed internally upon the sides of the polygon P.

In the special case of equilateral triangles when $n=2$ the figure $M_1' M_2M_2' M_1$ is a rhombus with angles $2\pi/3$ and $\pi/3$; when $n=3$, since M_i' are vertices of equilateral triangles constructed externally on $M_1M_2M_3$, the lines $M_1'M_3M_2'M_1$ and $M_3'M_2$ are all equal in length and intersect at angles of $2\pi/3$ at a point known as the Fermat point of the triangle $M_1M_2M_3$. The other Fermat point arises when the equilateral triangles are constructed internally on the sides of $M_1M_2M_3$. If the points corresponding to M_i' be called M'' then the hexagons formed from the six points $M_i'M''$ have been discussed by the writer. See *American Journal of Mathematics*, vol. 57 (1935), p. 504.

II. *Solution by Howard Eves, Bryan, Texas*

Let us generalize at the outset by replacing the set of equilateral triangles constructed upon the sides of (P) by a set of directly similar triangles so constructed, letting $A_1', A_2', \cdots, A_{2n}'$ be vertices of these triangles. Now, by virtue of the fundamental theorem: "If lines joining corresponding points of two directly similar curves be divided proportionally, the locus of the dividing points is a curve directly similar to the given curves," it follows that the set of triangles $M_1'M_1M_2$, etc., constructed upon the sides of (p) are all directly similar to those on the sides of (P).

Solved also by the proposer using rectangular coordinates.

Editorial Note. The proof by Musselman is essentially a proof of the above fundamental theorem, and it may be put in the following form to show this. Let C and C' be two directly similar curves in the same plane; A, A' two fixed points

on the respective curves which correspond in this similitude; and P, P' two corresponding variable points on the curves. Let the origin O of complex coordinates divide $A'A$ in the ratio $k:1$; let M be the point dividing the straight line segment $P'P$ in the same ratio; and let the complex coordinates of A, A', P, P' be a, $-ka$, $a+p$, $-ka+p'$. Then $p'=\lambda p$, where λ is a complex constant for this similitude. The coordinate of M is

$$m = [(-ka + \lambda p) + k(a + p)]/(1 + k) = [(\lambda + k)/(1 + k)]p.$$

Hence the curve described by M is similar to C.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, N. Y.

The New Jersey State Teachers College, Upper Montclair, has been designated by the United States Office of Education as one of three war information centers in its state. Various visual and teaching aids incident to a war-time program, chiefly in the secondary school field, may be obtained by writing to the Teaching Aids Service at this institution.

Dean L. P. Eisenhart of Princeton University has been appointed executive officer of the American Philosophical Society to succeed Professor Emeritus E. G. Conklin of Princeton University who is now president of the Society.

J. B. Adkins of Phillips Exeter Academy has been appointed an instructor in navigation in the Naval Reserve Midshipmen's School, New York, and Dr. D. M. Krabill of Potomac State School, West Virginia, is also an instructor in the Midshipmen's School.

Assistant Professor H. W. Bailey of the University of Illinois has been promoted to an associate professorship.

Professor B. L. Beegle of Seattle Pacific College has been appointed dean of the college.

Dr. E. E. Blanche of Michigan State College has been appointed statistical director at the Curtiss-Wright Corporation, Buffalo, N. Y.

Dr. H. E. Burns of Indiana University Extension, East Chicago, Indiana, has been promoted to an assistant professorship.

Professor J. H. Butchart of William Woods College, Missouri, has been appointed an assistant professor at Grinnell College.

Associate Professor H. V. Craig of the University of Texas has been promoted to a professorship.

R. W. Erickson of Hibbing Junior College is on leave and is a second lieutenant in the Army Specialist Corps.

Professor H. B. Evans of the University of Pennsylvania has retired with the title professor emeritus.

Associate Professor H. P. Evans of the University of Wisconsin has been promoted to a professorship.

Dr. F. A. Ficken of Cornell University has been appointed to a professorship at the University of Tennessee.

Assistant Professor Frances Harshbarger of Kent State University has been promoted to an associate professorship.

Associate Professor Sara L. Nelson of Georgia State College for Women has been promoted to a professorship and is head of the department of mathematics.

The Reverend R. E. O'Connor has been appointed to a professorship at Loyola College, Montreal, Quebec.

Dr. E. W. Paxson of Wayne University has been promoted to an assistant professorship.

Dr. W. T. Puckett of the University of California at Los Angeles has been promoted to an assistant professorship.

Assistant Professor Helen G. Russell of Wellesley College is on leave for 1942–43 and is at the University of California.

Professor W. J. Rusk of Grinnell College has retired.

Professor C. B. Upton of Teachers College, Columbia University, has retired with the title professor emeritus.

Professor D. V. Widder of Harvard University has been appointed chairman of the department, succeeding Professor M. H. Stone.

The following appointments to instructorships are announced:

University of California: Dr. Dorothy L. Bernstein
Princeton University: Paul Brock, part-time
Purdue University: N. J. Fine
Randolph Field, Texas: Dr. C. L. Seebeck
State Teachers College, Mankato, Minn.: C. J. Kirchen
Williams College: E. F. Gillette
University of Wisconsin: P. B. Norman

A recent message received through the American Red Cross announces the death in Germany on July 5, 1942, of Professor Oskar Bolza at the age of eighty-

five. He was a Reader in Mathematics at Johns Hopkins University in 1888–89, Associate at Clark University 1889–93, Associate Professor at the University of Chicago 1893–94 and Professor 1894–1910. For many years past he has been non-resident professor living in Freiburg.

As this issue was going to press word came of the death of Professor E. R. Hedrick on February 3, 1943.

THE WAR POLICY COMMITTEE

The American Mathematical Society and the Mathematical Association of America have appointed a joint War Policy Committee which will act for mathematicians in the many complex problems which will arise during the war period. Presidents W. D. Cairns and M. H. Stone, of the Association and Society respectively, will be ex officio members of the Committee and the other members are: Professors G. C. Evans, L. M. Graves, Marston Morse, Dr. Warren Weaver, and Professor G. T. Whyburn. Professor Stone will act as Chairman, at least during the initial stages of the work of the Committee.

The chief problems which will immediately concern this Committee are the deferment of mathematicians and the formulation of further plans to alleviate the serious shortage of teachers of mathematics which will confront many institutions upon the inauguration of the Army and Navy Training Program.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

The following is a list of the Sections of the Associations, with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Notre Dame, Ind., April 9–10, 1943
INDIANA, Notre Dame, April 9–10, 1943
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, Brooklyn, N. Y., May 8, 1943
MICHIGAN, Notre Dame, Ind., April 9–10, 1943
MINNESOTA
MISSOURI, Kansas City
NEBRASKA
NORTHERN CALIFORNIA
OHIO, Columbus, April 1, 1943
OKLAHOMA
PHILADELPHIA, Philadelphia, Nov. 27, 1943
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 13, 1943
SOUTHWESTERN
TEXAS, Lubbock, April, 1943
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May 7, 1943

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WHAT IS THE AREA OF A SURFACE?

TIBOR RADÓ, Ohio State University

1. The usual formula. Let $x=x(u)$, $y=y(u)$, $z=z(u)$ be a parametric representation of a curve C, where x, y, z are Cartesian coördinates.* If we assume that $x(u)$, $y(u)$, $z(u)$ have continuous derivatives of the first order in an interval $u_1 \leqq u \leqq u_2$, except possibly at a finite number of points, then the length $l(C)$ of C is given by the formula

$$(1) \qquad l(C) = \int_{u_1}^{u_2} \left[\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2\right]^{1/2} du.$$

Similarly, if $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, $u_1 \leqq u \leqq u_2$, $v_1 \leqq v \leqq v_2$, is a parametric representation of a surface S, and if $x(u, v)$, $y(u, v)$, $z(u, v)$ have continuous partial derivatives of the first order, except possibly along a finite number of smooth arcs, then the area $A(S)$ of S is given by the formula

$$(2) \qquad A(S) = \int_{u_1}^{u_2} \int_{v_1}^{v_2} \left[\left(\frac{\partial(x, y)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(y, z)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(z, x)}{\partial(u, v)}\right)^2\right]^{1/2} dudv.$$

2. Objections. For the average mathematician, who has no occasion to consider surfaces more general than those described above, the answer to the question, stated in the title, is very simple: the area of a surface is the value furnished by formula (2). While this answer is entirely satisfactory as far as it goes, it is open to at least two fundamental objections.

To illustrate the first objection, let us consider the formula $V=4\pi abc/3$ which gives the volume of an ellipsoid in terms of its axes. For most practical purposes, this formula may be taken as the *definition* of the volume of an ellipsoid, but a mathematician would surely insist that this formula represents a *theorem*. Similarly, we shall insist that the formulas (1) and (2) should be proved, on the basis of previously stated definitions of length and area.

A second fundamental objection is based on the observation that more and more mathematicians are being concerned with very general curves and surfaces, especially in Calculus of Variations. For somewhat general curves and surfaces the formulas (1) and (2) are not as reliable as may be desired. Let us consider the following example. Starting with the well-known Cantor function, it is easy to construct a function $f(u)$ with the following properties. a) $f(u)$ is continuous and strictly increasing in $0 \leqq u \leqq 1$. b) $f'(u)=0$ in $0 \leqq u \leqq 1$, with the exception of a set of measure zero. c) $f(0)=0$, $f(1)=1$. The equations $x=f(u)$, $y=f(u)$, $z=f(u)$, $0 \leqq u \leqq 1$, describe then the motion of a particle that starts at (0, 0, 0), moves continuously and steadily ahead along the line $x=y=z$, and arrives ultimately at the point (1, 1, 1). Thus the length of the path of this particle is $\sqrt{3}$. On the other hand, formula (1) yields, for this same path, the value zero (of

* We are actually interested in *surfaces*. References to curves serve the purpose of illustrating the analogies and discrepancies between arc-length and area.

course, the integral in (1) is now taken in the Lebesgue sense). A similar example may be constructed for surfaces.

In view of these remarks, we need, first of all, proper definitions for length and area. If C is a continuous curve, then the definition of its length $l(C)$ may be described by the formula

$$l(C) = \lim l(\Gamma_n), \tag{3}$$

where Γ_n designates any sequence of inscribed polygons, with maximum side-lengths converging to zero. This definition agrees with our intuition as to the way to measure length. To justify (3) mathematically, it should be shown that the limit in (3) exists and is independent of the particular choice of the sequence Γ_n. This can be done quite easily. It should be observed that for certain continuous curves C the length may be infinite, as simple examples will show.

3. Schwarz's example. In view of the striking resemblance between formulas (1) and (2), mathematicians were quite surprised, presumably, when about eighty years ago H. A. Schwarz called attention to the following example. Let S designate the lateral surface of a right circular cyclinder, with altitude 1, and with a base circle whose radius is also equal to 1. Cut S along a generator and then spread it upon a plane. We obtain a rectangle R with sides 1 and 2π. Thus the area of S is equal to 2π. Subdivide the sides of R into m and n equal parts respectively and draw lines, parallel to the sides, through the points of division. R is thus subdivided into mn small congruent rectangles. We subdivide each of these rectangles into four triangles by drawing diagonals. Let us now bend R so as to obtain the original cyclindrical surface S. We obtain then, on S, a set of $4mn$ curvilinear triangles, in each of which we replace the curvilinear sides by their respective chords. The result is an inscribed polyhedron $\Sigma_{m,n}$ with $4mn$ rectilinear triangular faces. An elementary computation yields the following formula for the area of $\Sigma_{m,n}$:

$$A(\Sigma_{m,n}) = 2n \sin \frac{\pi}{2n} + \left[\frac{1}{4} + \frac{4m^2}{n^4}\left(n \sin \frac{\pi}{2n}\right)^4\right]^{1/2} \times 2n \sin \frac{\pi}{n} \cdot$$

Let us choose $m = n^3$ and let us denote by Σ_n the inscribed polyhedron $\Sigma_{n^3,n}$. It follows that $A(\Sigma_n)$ converges, for $n \to \infty$, to infinity, and not to $2\pi = A(S)$. The interested reader will easily determine the totality of the limits that can be obtained by properly coördinating the values of m and n.

It is now clear that the areas of inscribed polyhedra, approximating a given surface, will not converge, in general, to the area of that surface.

The example of Schwarz, discussed in the preceding section, was the starting point of an extensive and fascinating literature. Still, we do not possess as yet a satisfactory *theory* of the area of surfaces, even though the subject attracted the attention of many mathematicians. Historical evidence seems to indicate that progress in this theory is closely dependent upon the stimulating influence of the demands placed upon it by current research in other fields, especially in Calculus of Variations. It is probable that the recent sustained interest in two-

dimensional problems in Calculus of Variations will lead, in a not very distant future, to decisive developments in the theory of the area of surfaces.

4. Lebesgue's definition. Even the briefest survey of the present status of the theory of the area would exceed, by far, the limitations of space that we have to observe on this occasion.* However, we shall attempt to give some information of interest to the non-specialist, who may now insist, in view of the example of Schwarz, that we show him some valid definition of the area of a surface. We shall state presently such a definition, due to Lebesgue. This definition, probably the most fruitful one of the many definitions proposed to date, may be derived, by analogy, from an alternative form of the definition of the arc-length which we shall describe first.

Let there be given continuous curves C: $x=x_n(u)$, $y=y_n(u)$, $z=z_n(u)$, $u_1 \leqq u \leqq u_2$, $n=0, 1, 2, \cdots$. Suppose $C_n \to C_0$ in the sense that $x_n(u) \to x_0(u)$, $y_n(u) \to y_0(u)$, $z_n(u) \to z_0(u)$ uniformly in $u_1 \leqq u \leqq u_2$. It is then easily proved that

$$l(C_0) \leqq \liminf l(C_n). \tag{4}$$

This relation expresses the fundamental fact that the arc-length is a *lower semi-continuous functional*. In view of the fundamental importance of such functionals in various fields, it may be desirable to possess an alternative definition of the arc-length that is based upon its property of being lower semi-continuous. An elementary reasoning yields the formula

$$l(C) = \text{gr.l.b.} \liminf l(\Gamma_n), \tag{5}$$

where the greatest lower bound is taken with respect to all sequences of polygons Γ_n (*not necessarily inscribed*), such that $\Gamma_n \to C$ in the sense explained earlier in this section.

We may now take (5) as a *definition* of the arc-length (of course (3) becomes a *theorem*). If we replace in the definition described by (5), the terms *length*, *polygon*, *curve* by the terms *area*, *polyhedron*, *surface* respectively, then we obtain the definition of Lebesgue for the area of a surface.

If we adopt the definition (5) for the arc-length and the corresponding definition of Lebesgue for the area of a surface, then we have re-established the analogy that we had a right to expect in view of the obvious resemblance of formulas (1) and (2). Of course, we should now *prove* formulas (1) and (2), at least for the curves and surfaces described in section 1. For the curves of section 1 the proof is quite simple. The literature on the Lebesgue area contains a proof of formula (2) under extremely general (although probably not final) assumptions. It may not be an easy task to derive from these proofs a presentation that would account for the surfaces of section 1 and at the same time be suitable for classroom use.

* The following publications of the writer may be used as a start in exploring the literature. a) On the problem of Plateau Ergebnisse der Mathematik, vol. 2. b) On the semi-continuity of double integrals in parametric form, Transactions of the American Mathematical Society, vol. 51, 1942, pp. 336–361.

THE SOLUTION OF p-ADIC EQUATIONS

H. S. THURSTON, University of Alabama

1. Introduction. In a paper* presented for the Slaught Memorial Volume of this MONTHLY, C. C. MacDuffee has given a brief expository discussion of Hensel's p-adic numbers. In particular, consideration was given to the solution in the field Ω_p of the equation $f(x)=0$, where $f(x)$ has rational integral coefficients. Assuming the existence of a root of the form

$$\alpha = a_0 + a_1 p + a_2 p^2 + \cdots, \qquad 0 \leqq a_i < p, \tag{1}$$

we consider this root to be known as soon as a process is determined whereby as many of the coefficients a_i as we please may be successively found. The leading coefficient a_0 having been found as a solution of the congruence $f(x) \equiv 0 \pmod{p}$, a technique was shown for solving the equation provided $f'(a_0) \not\equiv 0 \pmod{p}$. We shall refer to this as the regular case. However, if $f'(a_0) \equiv 0 \pmod{p}$, the method not only fails to yield a root α, but leaves in doubt even the existence of such a root.

The present paper develops a different method which enables us to ascertain in a finite number of steps how many roots the equation has in Ω_p, as well as to solve for any root α whose existence has been demonstrated. Throughout the paper it will be assumed that $f(x)$ is of the form

$$f(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_0,$$

since otherwise it could be reduced to this form by a transformation $x' = kx$.

2. The regular case. In order to compare the two techniques, we shall present, with slight modifications, MacDuffee's proof of

THEOREM 1. *The equation $f(x)=0$ will have a solution in Ω_p if $f(x) \equiv 0 \pmod{p}$ has a solution $x = a_0$ such that $f'(a_0) \not\equiv 0 \pmod{p}$.*

Let us assume that $f(x)=0$ has a solution α of the form (1), and set

$$\alpha_0 = a_0,\ \alpha_1 = a_0 + a_1 p,\ \alpha_2 = a_0 + a_1 p + a_2 p^2, \cdots,$$
$$\alpha_k = a_0 + a_1 p + a_2 p^2 + \cdots + a_k p^k.$$

Then $\alpha \equiv \alpha_k \pmod{p^{k+1}}$, whence $f(\alpha) \equiv f(\alpha_k) \pmod{p^{k+1}}$. Since $f(\alpha)=0$,† this implies that $f(\alpha_k) \equiv 0 \pmod{p^{k+1}}$; for $k=0$ we have

$$f(\alpha_0) \equiv 0 \pmod{p}, \tag{2}$$

which is therefore a necessary condition for a solution.

Since $\alpha_{k+1} = \alpha_k + a_{k+1}p^{k+1}$, we shall have

* C. C. MacDuffee, The p-adic numbers of Hensel, this MONTHLY, vol. 45, No. 8, pp. 500–508.

† This means that $\phi[f(\alpha)]=0$, where ϕ represents the valuation defined for the p-adic field. This paper employs the valuation defined in MacDuffee's paper: if $a = p^\nu \gamma$ where $(\gamma, p)=1$, then $\phi(a) = p^{-\nu}$.

$$\begin{aligned} f(\alpha_{k+1}) &= f(\alpha_k) + f'(\alpha_k)a_{k+1}p^{k+1} + \cdots \\ &\equiv f(\alpha_k) + f'(\alpha_k)a_{k+1}p^{k+1} \pmod{p^{k+2}} \\ &\equiv 0 \pmod{p^{k+2}}. \end{aligned}$$

Since $f(\alpha_k)\equiv 0 \pmod{p^{k+1}}$, there is an integer h_k such that

$$f(\alpha_k) = h_k p^{k+1}. \tag{3}$$

Then

$$h_k p^{k+1} + f'(\alpha_k)a_{k+1}p^{k+1} \equiv 0 \pmod{p^{k+2}},$$

whence

$$f'(\alpha_k)a_{k+1} + h_k \equiv 0 \pmod{p}. \tag{4}$$

Now $\alpha_k = a_0 + pg$ where g is a rational integer. Then

$$\begin{aligned} f'(\alpha_k) &= f'(a_0) + f''(a_0)pg + \cdots, \\ &\equiv f'(a_0) \pmod{p}, \end{aligned}$$

and from (4) we have

$$f'(a_0)a_{k+1} + h_k \equiv 0 \pmod{p}. \tag{5}$$

If $f'(a_0)\not\equiv 0 \pmod{p}$, (5) may be solved for a_{k+1}. A solution of $f(x)=0$ can then be obtained by successively solving (3) for h_k and (5) for a_{k+1} when $k=0, 1, 2, \cdots$. That a number

$$\alpha = a_0 + a_1 p + a_2 p^2 + \cdots$$

found by this procedure is actually a root of $f(x)=0$, may be seen immediately. For, since $f(\alpha)-f(\alpha_k)\equiv 0 \pmod{p^{k+1}}$, then

$$\begin{aligned} \phi[f(\alpha) - f(\alpha_k)] &\leqq p^{-k-1} \\ &< \frac{\epsilon}{2} \end{aligned}$$

for k sufficiently large. Similarly, since $f(\alpha_k)\equiv 0 \pmod{p^{k+1}}$,

$$\phi[f(\alpha_k)] < \frac{\epsilon}{2}\cdot$$

Then

$$\phi[f(\alpha)] < \epsilon$$

and $f(\alpha)=0$.

3. Another method of solution. The method of the preceding section makes use solely of the given equation $f(x)=0$ and may therefore be considered analogous to Newton's method for approximating an irrational root of an al-

gebraic equation. The analogy is further preserved in the condition imposed upon $f'(a_0)$. We shall here develop a technique comparable to Horner's method, the successive approximations being obtained by means of a chain of equations

$$f(x) = 0,\ F_1(x) = 0,\ F_2(x) = 0, \cdots, F_i(x) = 0,$$

the equation $F_i(x) = 0$ yielding the coefficient a_i.

If $f(x) = 0$ has a solution in Ω_p of the form $\alpha = a_0 + \alpha_1 p$ where $\alpha_1 = a_1 + a_2 p + a_3 p^2 + \cdots$, then

$$f(a_0 + \alpha_1 p) = f(a_0) + f'(a_0)\alpha_1 p + \cdots + \alpha_1^n p^n = 0.$$

Let $f(a_0) = k_0 p$. Then

$$p[k_0 + f'(a_0)\alpha_1 + \cdots + \alpha_1^n p^{n-1}] = 0,$$

so that α_1 is a solution of the equation

$$F_1(x) = k_0 + f'(a_0)x + \cdots + x^n p^{n-1} = 0.$$

The solution of $F_1(x) = 0$ in Ω_p is in all respects like that of $f(x) = 0$. We may thus obtain a sequence of functions $F_i(x)$, each derived from the preceding in exactly the same manner that $F_1(x)$ was derived from $f(x)$. Successive approximations to the solution of $f(x) = 0$ may be obtained from the chain of equations without returning each time to a consideration of the original equation. Thus if $\alpha = a_0 + a_1 p + a_2 p^2 + \cdots$ is a solution of $f(x) = 0$ in Ω_p, then $\alpha_1 = a_1 + a_2 p + \cdots = a_1 + \alpha_2 p$ is a solution of $F_1(x) = 0$, and in general $\alpha_i = a_i + \alpha_{i+1} p$ is a solution of $F_i(x) = 0$.

Since

$$f(\alpha) = pF_1(\alpha_1) = p^2 F_2(\alpha_2) = \cdots = p^k F_k(\alpha_k),$$

it follows that any p-adic number α determined by the above procedure will be a solution of $f(x) = 0$, since

$$\phi[f(\alpha)] \leqq p^{-k} < \epsilon$$

for sufficiently large values of k.

In solving the equation $F_1(x) = 0$, it is, of course, permissible to divide out a power of p which may be a common factor of all the coefficients. That is always the case when $f'(a_0) \equiv 0 \pmod{p}$ if the given equation has a solution. For $F_1(x) \equiv k_0 + f'(a_0)x \pmod{p}$, and if $f'(a_0) \equiv 0 \pmod{p}$ while $k_0 \not\equiv 0$, then $F_1(x) \equiv 0 \pmod{p}$ has no solution. But if p divides k_0, then $F_1(x)$ is divisible by a power of p, which may be divided out and the quotient designated as $F_1(x)$. Similarly we may simplify any of the $F_i(x)$.

4. Recurrence of the functions $F_i(x)$. Two problems present themselves for consideration: (a) the possibility of a recurrence $F_i = F_j$* for all $i > j$, (b) the possi-

* Any recurrence of the form $F_{j+k} = F_j$ for $k > 1$ would imply a non-integral rational root. Such a root cannot exist since the leading coefficient of $f(x)$ is unity.

bility that for every i we may have $f(a_0)\equiv f'(a_0)\equiv 0$, $F_i(a_i)\equiv F_i'(a_i)\equiv 0 \pmod{p}$. The latter is of considerable significance, for, although the method of the preceding section did not require the existence of an i such that $F_i'(a_i)\not\equiv 0 \pmod{p}$, we know by Theorem 1 that if such an i exists there will be a corresponding root of $F_i(x)=0$ and hence of $f(x)=0$. Such an i having been found, we may determine subsequent coefficients a_k as in Theorem 1. Problem (b) will be discussed in the next section. We shall find the answer to problem (a) in the following theorem and its corollary.

THEOREM 2. *A necessary and sufficient condition that $F_1(x)=f(x)$ is that either $f(x)=(x+1)^n$ or $f(x)=x^n$.*

If $f(a_0)\equiv 0 \pmod{p}$, we have

$$f(a_0+px) = f(a_0) + f'(a_0)px + \cdots + p^n x^n. \tag{6}$$

$F_1(x)$ being defined to be the quotient after all powers of p are divided out of the right member of (6), if $F_1(x)=f(x)$ the factor p^n must be common to every term and hence $f(a_0+px)=p^nF_1(x)$ identically. Since

$$f(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0,$$

on equating coefficients of equal powers of x we obtain

$$c_{n-1} = \frac{na_0}{p-1}$$

and, in general,

$$c_{n-k} = \binom{n}{k}\left(\frac{c_{n-1}}{n}\right)^k = \binom{n}{k}\left(\frac{a_0}{p-1}\right)^k.$$

Hence $f(x)=(x+a_0/p-1)$ and, since c_{n-k} is an integer, it follows that a_0 is either 0 or $p-1$.

The sufficiency of the condition follows immediately on forming $F_1(x)$ when $f(x)$ is x^n or $(x+1)^n$.

COROLLARY. *A necessary and sufficient condition that $F_i(x)=F_j(x)$ for every $i>j$ is that $F_j(x)=(x+1)^n$ or x^n.*

That the condition given in the corollary is exceptional is indicated in the following theorems.

THEOREM 3. *A necessary and sufficient condition that there exist a value of j such that $F_j(x)=(x+1)^n$ is that $f(x)=(x+a)^n$ where $a>0$.*

THEOREM 4. *A necessary and sufficient condition that there exist a value of j such that $F_j(x)=x^n$ is that $f(x)=(x-a)^n$ where $a>0$.*

We shall first prove the sufficiency of the condition of Theorem 3. Let a be written in the form $a=k_0+k_1p+\cdots+k_rp^r$ where $0\leqq k_i<p$, and k_m is the first

non-zero coefficient. We may successively show that for $0 \leqq i < m$ we have $a_i = 0$ and, for $i < m$,

$$F_i(x) = (x + k_m p^{m-i} + k_{m+1} p^{m-i+1} + \cdots + k_r p^{r-i})^n;$$

for $m \leqq i < r$ we have $a_i = p - k_i$ and

$$F_i(x) = (x + 1 + k_i + k_{i+1} p + \cdots + k_r p^{r-i})^n,$$

whence $F_r(x) \equiv (x+1+k_r)^n$ and $F_{r+1}(x) = (x+1)^n$.

Conversely, let $F_{r+1}(x) = (x+1)^n$, it being assumed that the leader $A_r = a_0 + a_1 p + \cdots + a_r p^r$ of the root α has been found by successively forming $F_i(x)$ for $i \leqq r$. It will be shown that $f(x) = (x+a)^n$ where $a = p^{r+1} - A_r$.

We have

$$\begin{aligned} F_r(a_r + px) &= F_r(a_r) + F_r'(a_r)px + \cdots + p^n x^n \\ &= p^n F_{r+1}(x) \\ &= p^n(x^n + nx^{n-1} + \cdots + 1). \end{aligned}$$

On equating coefficients of like powers of x, we obtain

$$F_r(a_r) = p^n, \; F_r'(a_r) = np^{n-1}, \cdots, F_r^{(n)}(a_r) = n!.$$

Let $F_r(x) = x^n + b_1 x^{n-1} + \cdots + b_n$. After successive differentiation and substitution of the above values, we obtain a system of linear equations,

$$\frac{n!}{(n-k)!} a_r^{n-k} + \frac{(n-1)!}{(n-k-1)!} b_1 a_r^{n-k-1} + \cdots + k! b_{n-k} = \frac{n!}{(n-k)!} p^{n-k},$$
$$(k = 0, 1, 2, \cdots, n-1).$$

The matrix of this system is non-singular and the system has the unique solution $b_s = \binom{n}{s}(p - a_r)^s$. Hence $F_r(x) = (x + p - a_r)^n$.

Now, in exactly the same way it may be shown that if $F_i(x) = (x+c)^n$, then $F_{i-1}(x) = (x + cp - a_{i-1})^n$. It follows that

$$\begin{aligned} F_{r-1}(x) &= [x + (p - a_r)p - a_{r-1}]^n \\ &= (x + p^2 - a_r p - a_{r-1})^n, \\ F_{r-2}(x) &= (x + p^3 - a_r p^2 - a_{r-1} p - a_{r-2})^n, \end{aligned}$$

and finally

$$\begin{aligned} F_0(x) = f(x) &= (x + p^{r+1} - a_r p^r - a_{r-1} p^{r-1} - \cdots - a_0)^n \\ &= (x + a)^n. \end{aligned}$$

The proof of Theorem 4 is in every respect like that of Theorem 3 and will be omitted.

5. Multiple roots.* It is clear that, if $f(x)=0$ has a multiple root α in Ω_p, then α is a root of $f'(x)=0$ in that field. Let $\alpha=a_0+\alpha_1 p$. Then $\alpha_i=a_i+\alpha_{i+1}p$ will be a multiple root of $F_i(x)=0$ and hence a root of $F_i'(x)=0$ for all finite values of i. However, it is easily determined if $f(x)=0$ has a multiple root in Ω_p, since the euclidean algorithm for the greatest common divisor can be applied to the polynomials $f(x)$ and $f'(x)$. If multiple roots occur the degree of the equation may be reduced. We shall therefore assume that $f(x)=0$ has no multiple roots in Ω_p.

Let us assume that

$$\alpha = a_0 + a_1 p + a_2 p^2 + \cdots$$

is a root of $f(x)=0$ in Ω_p such that $f(a_0)\equiv f'(a_0)\equiv 0$, and such that for every finite value of i, $F_i(a_i)\equiv F_i'(a_i)\equiv 0 \pmod p$.

It will be true that $f(a_0+a_1p)\equiv 0 \pmod{p^2}$. Now for every integer x,

$$f(a_0 + xp) \equiv f(a_0) + f'(a_0)xp \pmod{p^2}.$$

Since $f'(a_0)\equiv 0 \pmod p$, it follows that $f(a_0)\equiv 0 \pmod{p^2}$. That is,

$$f(a_0 + xp) \equiv 0 \pmod{p^2}.$$

Now if we define $F_1(x)$ by the equation

$$f(a_0 + xp) = p^{\beta_1}F_1(x)$$

where β_1 is maximal, it follows that $\beta_1\geqq 2$. Then

$$f'(a_0 + xp) = p^{\beta_1-1}F_1'(x),$$
$$f'(a_0 + a_1p) = p^{\beta_1-1}F_1'(a_1).$$

But we have assumed that $F_1'(a_1)\equiv 0 \pmod p$, so

$$f'(a_0 + a_1p) \equiv 0 \pmod{p^{\beta_1}}.$$

Thus a_0+a_1p is a multiple root of $f(x)\equiv 0 \pmod{p^{\beta_1}}$.

If we define $F_2(x)$ by the equation

$$F_1(a_1 + xp) = p^{\beta_2}F_2(x)$$

where β_2 is maximal, it may be shown by similar reasoning that $\beta_2\geqq 2$. Then

$$f(a_0 + a_1p + xp^2) = p^{\beta_1}F_1(a_1 + xp) = p^{\beta_1+\beta_2}F_2(x).$$

Hence

$$f'(a_0 + a_1p + xp^2) = p^{\beta_1+\beta_2-2}F_2'(x),$$
$$f'(a_0 + a_1p + a_2p^2) = p^{\beta_1+\beta_2-2}F_2'(a_2).$$

* The author is indebted to Professor MacDuffee for suggestions in the development of this section.

But we have assumed that $F_2'(a_2) \equiv 0 \pmod{p}$, so

$$f'(a_0 + a_1 p + a_2 p^2) \equiv 0 \pmod{p^{\beta_1+\beta_2-1}}.$$

Thus $a_0 + a_1 p + a_2 p^2$ is a multiple root of $f(x) \equiv 0 \pmod{p^{\beta_1+\beta_2-1}}$.

By induction it can be shown that $a_0 + a_1 p + \cdots + a_i p^i$ is a multiple root of $f(x) \equiv 0 \pmod{p^s}$ where

$$s = \beta_1 + \beta_2 + \cdots + \beta_i - 1, \qquad \beta_i \geqq 2.$$

Since $s \geqq i+1$, it follows that α is a multiple root of $f(x) = 0$ in Ω_p, contrary to our assumption.

It therefore follows that if all multiple roots of $f(x) = 0$ have been removed, there must be a first value of i, say $i = m$, such that either $F_m(x) \equiv 0 \pmod{p}$ has no solution, or else $F_m(a_m) \equiv 0$ and $F_m'(a_m) \not\equiv 0 \pmod{p}$. In the former case $f(x) = 0$ has no solution with the leader $a_0 + a_1 p + \cdots + a_i p^i$. In the latter case the congruence $F_m(x) \equiv 0 \pmod{p}$ has a root of lower multiplicity than the preceding congruences $F_i(x) \equiv 0 \pmod{p}$. Continuing, a congruence $F_k(x) \equiv 0 \pmod{p}$ is finally obtained whose roots are distinct and such that $F_k'(a_k) \not\equiv 0 \pmod{p}$. The root can therefore be found to an arbitrary number of places. We have proved

THEOREM 5. *If $f(x) = 0$ has only simple roots in Ω_p, it is possible to determine in a finite number of steps, how many solutions there are in Ω_p.*

6. Examples. (a) The equation $f(x) = x^3 + 43x^2 + 159x + 1217 = 0$ has no solution in Ω_7, for although the congruence $f(x) \equiv 0 \pmod{7}$ has the solution $a_0 = 2$, we find that $F_1(x) = x^3 + 49x^2 + 49x + 275 \equiv x^3 + 2 \equiv 0 \pmod{7}$ has no solution.

(b) In the solution of $f(x) = x^3 + 3x^2 + 9x + 338 = 0$ in Ω_3 we first find the unique solution $a_0 = 1$ of $f(x) \equiv 0 \pmod{3}$. Then $F_1(x) = x^3 + 2x^2 + 2x + 13$, and $F_1(x) \equiv 0 \pmod{3}$ has two solutions, $a_1 = 2$ and $a_1 = 1$. Since $F_1'(2) \not\equiv 0 \pmod{3}$ there is at least one solution corresponding to $a_1 = 2$ and this can be shown to be unique. Proceeding with $a_1 = 1$, we find $F_2(x) = 3x^3 + 5x^2 + 3x + 2$ and, since $F_2 \equiv 0 \pmod{3}$ has no solution, $f(x) = 0$ has in Ω_3 but one solution, namely $\alpha = 1 + 2\cdot 3 + 1\cdot 3^2 + 2\cdot 3^3 + 0\cdot 3^4 + \cdots$.

(c) In solving $f(x) = x^3 + 3x^2 + 300x + 5590 = 0$ in Ω_3 we find successively $a_0 = 2$, $F_1 = x^3 + 3x^2 + 36x + 230$, $a_1 = 1$, and $F_2 = x^3 + 2x^2 + 5x + 10$. Now the congruence $F_2 \equiv 0 \pmod{3}$ has the two solutions $a_2 = 1$ and $a_2 = 2$ and, since $F_2'(2) \not\equiv 0 \pmod{3}$, there is a corresponding solution. Proceeding with $a_2 = 1$, we find $F_3 = 3x^3 + 5x^2 + 4x + 2$, $a_3 = 2$ and $F_4 = 9x^3 + 23x^2 + 20x + 6$. The congruence $F_4 \equiv 0 \pmod{3}$ has two solutions, $a_4 = 0$ and $a_4 = 2$. Since $F_4(0) \not\equiv 0 \pmod{3}$ and $F_4(2) \not\equiv 0 \pmod{3}$, there is a solution corresponding to each of these values of a_4. Thus $f(x) = 0$ has in Ω_3 the three solutions

$$\alpha = 2 + 1\cdot 3 + 1\cdot 3^2 + 2\cdot 3^3 + 0\cdot 3^4 + \cdots,$$
$$\beta = 2 + 1\cdot 3 + 1\cdot 3^2 + 2\cdot 3^3 + 2\cdot 3^4 + \cdots,$$
$$\gamma = 2 + 1\cdot 3 + 2\cdot 3^2 + 0\cdot 3^3 + 2\cdot 3^4 + \cdots.$$

CERTAIN MATHEMATICAL ACHIEVEMENTS OF JAMES GREGORY

MAX DEHN, Illinois Institute of Technology
E. D. HELLINGER, Northwestern University

For a long time the light of James Gregory did not shine as brightly as did that of John Wallis, Isaac Barrow and Isaac Newton, the other three great British mathematicians of the seventeenth century. Only recently, through the endeavours of several Scottish mathematicians, especially E. T. Whittaker, G. A. Gibson and H. W. Turnbull, Gregory's genius is revealed and fills with admiration all those interested in the development of modern mathematics.

The "*James Gregory Tercentenary Memorial Volume*," edited by H. W. Turnbull [1], contains Gregory's momentous scientific correspondence, mostly with J. Collins. An extremely important supplement is the large number of Gregory's hitherto unpublished notes, recording his mathematical ideas and calculations. These notes were found in a collection of documents in the University of St. Andrews Library, written on the blank spaces of letters to Gregory. This material affords the possibility of studying his achievements and ideas.

In this paper we shall discuss Gregory's expansions of general and particular functions into series. In addition, we shall exhibit the ideas which are set forth in his first mathematical publication "*Vera circuli et hyperbolae quadratura*" [2]. These ideas are concerned, to some extent, with the associated problem of constructing by certain limiting processes the functions which measure the areas of circles and conics.

1. The "Taylor's series". In a letter of February 15, 1671 to J. Collins (see "*Memorial*" [1], pp. 170 ff.) Gregory gives the power series for seven important functions, each with 5 or 6 terms. These functions are, if for the sake of brevity we may use modern notations,

$$\text{arc tan } x,\ \tan x,\ \sec x,\ \log \sec x,\ \log \tan\left(\frac{x}{2}+\frac{\pi}{4}\right),$$

$$\text{arc sec } (\sqrt{2}\, e^x), \qquad 2 \text{ arc tan}\left(\tanh \frac{x}{2}\right).$$

He mentions without further explanation that he had some knowledge of Newton's "universal method." Hereby, he refers to some series which Newton had discovered and which Collins had but recently communicated to him.

We may surmise that he obtained the arc tangent series in a way analogous to that by which three years earlier N. Mercator [3] had found the series for $\log(1+x)$. He may have considered arc tan x as the area under the curve $y=(1+x^2)^{-1}$, transformed $(1+x^2)^{-1}$ by formal division into a power series and finally integrated this infinite sum. However, there is no possibility of obtaining the other series in a similar way.

On the blank space of a letter to Gregory, dated January 29, 1671, Turnbull found a group of computations about just these seven functions [4]. The com-

parison of these computations with Gregory's expansions indicates the way of his thoughts. First, they include almost without exception, as many of the successive derivatives of the functions, as would be needed in finding the 5 or 6 numerical coefficients of the series by successive differentiation. Second, all coefficients in Gregory's series are correct with the exception of a single coefficient in both the expansions for tan x and for log sec x. (The second error is a consequence of the first since he obviously obtained the log sec series by integrating the tangent series.) Finally, all derivatives in Gregory's notes are correct with the exception of a single numerical error in the derivatives of tan x, which was probably due to miscopying one number. However, using this erroneous value one finds exactly the erroneous coefficients in the series for tan x and log sec x. From these two facts, Turnbull argues conclusively that Gregory used the tables of the derivatives for the construction of his power series.

We see two possibilities for such a construction. On the one hand, we may imagine that Gregory applied in each particular case something like the "method of undetermined coefficients" together with successive differentiation. That he mentions "Newtons universal method" immediately before giving his series may be considered as supporting this assumption. In fact, if we look upon the whole of Newton's work we are justified in assuming that Gregory thought of this combined method as "Newtons universal method," even though the idea had been sketched as early as 1637 by Descartes in his "géométrie," and had since been applied by many other mathematicians. Nevertheless, Gregory's remark must be considered as a mere guess based upon the few results from Newton's still unpublished investigations which Collins had communicated to him with no hint about Newton's method.

On the other hand, we may suppose that Gregory could have applied the same process for an unspecified function and could have obtained the general expression for the nth coefficient of the expansion. Thus he would have anticipated Taylor's classical expansion by forty-four years. Neither the letters nor the other material, so far as published, substantiate the latter possibility. From all these facts, we may conclude that Gregory possessed a method for finding the Taylor expansion of any *particular* function, but we cannot affirm that he possessed Taylor's formula for an *unspecified* function.

It may be interesting that the second man, C. Maclaurin, whose name is closely associated with this series, deduced it seventy years later, in his "*Treatise of Fluctions*" (1742) by a reasoning similar to that of Gregory. Of course he applied it at once to an unspecified function. He quotes Taylor's book for the formula but could not have known Gregory's discovery then buried in the correspondence.

2. The interpolation formula. For the independent discovery by Gregory of a famous interpolation formula, full evidence is given in a letter of his published long ago. Nevertheless, nobody seems to have realized this fact until E. T. Whittaker brought it to general notice. In the letter to Collins [5] of November 23, 1670 Gregory stated explicitly a formula which interpolates for a

function $y=f(x)$ when its values at equidistant points $0, c, 2c, 3c, \cdots$ are given. This formula is identical with the famous formula

$$(1)\ f(x) = f(0) + \frac{x}{c}\Delta f(0) + \frac{x(x-c)}{c\cdot 2c}\Delta^2 f(0) + \frac{x(x-c)(x-2c)}{c\cdot 2c\cdot 3c}\Delta^3 f(0) + \cdots,$$

which Newton made known some years later [6] and which mostly bears his name. It is not essential that Gregory assumes here $f(0)=0$. Further, we may note that, of course, he did not have for the differences the notation $\Delta f(0)$, $\Delta^2 f(0)$, $\Delta^3 f(0)$, $\cdots$. This came into use much later under the influence of Leibniz's symbolism. He takes single letters $d, f, h, \cdots$ for these values, carefully defined by forming the sequences of the 1st, 2nd, 3rd, $\cdots$ differences. Newton uses almost the same notation as Gregory.

In the correspondence on this formula between Collins and Gregory [7], there is mentioned the procedure which Briggs had used in extending his table of logarithms to subintervals. Briggs took differences, generalizing the older method of linear interpolation. His procedure can be considered in some way as the predecessor of the interpolation formula. However, Briggs does not state such a formula nor does he give any motivation of this procedure. Gregory's formula was given in answer to a question raised by Collins for such a motivation.

Of course, Gregory also states his formula without a proper proof, but it is obvious that he could and did verify the formula for polynomials. The same is true for Newton's first publications, although later, in the "*methodus differentialis*," he sketches a way to derive the formula. It is interesting that the interpolation of tables is only *one* aim of Gregory's statement; he emphasizes strongly its use for the problem of approximate quadrature of curves and gives various formulas in this connection. Incidentally Newton [8] makes the same application of the interpolation formula.

The infinite process which is involved in this interpolation formula implies a serious mathematical difficulty which even its discoverers may have felt semiconsciously. The polynomial $P_n(x)$ of the nth degree which is given by the first $n+1$ terms of the formula (1) takes on the values of $f(x)$ at the equidistant points $0, c, 2c, \cdots, nc$, and is determined by this property. This, obviously, is the essential fact which was discovered and communicated by Gregory and Newton. Yet they tacitly assumed that for other unspecified values of x the successive polynomials $P_n(x)$ yield an approximation to $f(x)$ which can be improved by increasing n. Apparently, they thought only of such values of x which are located *between* $0, c, \cdots, nc$, that is to say, they considered only the proper problem of *interpolation*. Here the fact of the steadily improved approximation looks rather evident although a precise formulation and an exact proof was not within the range of these early developments. Things are different if one turns to the problem of *extrapolation*, considering values x *outside* the interval of the multiples of c. The published material gives no evidence that Gregory used his formula for extrapolation. And Newton in the "*Philosophiae Naturalis Prin-*

cipia" [6] applies the interpolation formula not in order to find the place of a comet at any time beyond the range of the observations, but only for intermediate moments.

It is important to realize this situation since the way from the interpolation formula to the Taylor series goes through a sort of extrapolation. Assuming c infinitely small, one concentrates $0, c, 2c, \cdots$ in an arbitrarily small neighborhood of a fixed value and one seeks an expression for $f(x)$ at another fixed value at a finite distance. This can be done formally by applying the usual symbols of the difference and differential calculus. One has only to replace, corresponding to this limiting process, the nth difference quotient $\Delta^n y/\Delta x^n$ in Newton's formula by the nth derivative $d^n y/dx^n$. But in doing so one leaps over a very serious difficulty, using the symbols without regard to their original meaning. In fact, the higher derivatives are defined originally by iteration of the differentiation process (limit of first difference quotient) and their connection with the higher difference quotients is not trivial. And still more difficult for a critical mathematician is the whole limiting process from the interpolation formula to the infinite series. Perhaps such difficulties make us understand why Gregory did not state any connection between his two great results and why Newton, so far as we know, never formulated the Taylor series.

The first to dare to leap over these gaps was Brook Taylor in 1715 [9]. He could do so, since he obviously knew not only Newton's methods but also the concepts and notations introduced in the mean time by Leibniz. He did not use the symbols of Leibniz, but, adapting them to Newton's language, he developed a notation of his own which may, of course, appear a little awkward to us. He applied this symbolism without being influenced by the intrinsic difficulties mentioned above. Thus he came automatically from the interpolation formula to his general series by this purely formal procedure which later on was often performed unscrupulously with the help of the suggestive notation of Leibniz.

3. The binomial series. In an enclosure [10] with the letter to Collins of November 23, 1670, Gregory deals with the problem of finding the "number" of a given logarithm x; that is to say, if we denote the base by $1+d$, of finding $y=(1+d)^x$. For the sake of brevity, we again use modern notations without changing anything else. Gregory gives the solution as follows:

$$(1)\qquad (1+d)^x = 1 + xd + \frac{x(x-1)}{1\cdot 2}d^2 + \frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}d^3 + \cdots,$$

which is of course the binomial series. The comparison of Gregory's formula and notation with the statement of the interpolation theorem in the principal part of the same letter [5] shows clearly that he found his result by applying the theorem to the function $f(x)=(1+d)^x$ using the known values at $x=0, 1, 2, \cdots$. Indeed, since the first difference of this function turns out to be

$$(2)\qquad \Delta f(x) = f(x+1) - f(x) = (1+d)^{x+1} - (1+d)^x = d\cdot f(x),$$

the values of its successive differences at $x=0$ become

$$f(0) = 1,\ \Delta f(0) = d,\ \Delta^2 f(0) = d^2,\ \Delta^3 f(0) = d^3, \cdots .$$

Thus, the interpolation formula **2,** (1) yields immediately the binomial series (1).

The correspondence of Gregory and Collins gives full evidence that this discovery of Gregory was entirely independent of Newton's investigations in the binomial theory. Gregory knew at this time only a single one of Newton's results, namely the series for the "zone of the circle," *i.e.* the series for the function $\int_0^x (R^2-x^2)^{1/2}dx$. Collins had communicated the mere statement of the latter to him seven months previously [**11**]. In fact, Newton had found this series by integrating term by term the expansion of the binomial $(R^2-x^2)^{1/2}$. Having Collins' communication, Gregory tried hard but without success to prove the result directly. Obviously, his discovery of the general binomial theorem was in no way influenced by this knowledge and he did not guess any connection. Afterwards, he recognized suddenly that Newton's series was a simple consequence of his own theorem and, in a letter of December 19 [**12**], complains much of "his own dullness," not to have noticed the fact before. Besides, Newton's binomial theorem did not become generally known before 1676, when, about ten years after he had found it, he communicated it to Oldenburg in the two famous letters [**13**] (June 6 and October 4).

It is interesting to compare the way in which Newton had discovered his theorem, as he describes it in the second of these letters, with Gregory's deduction. We mention only the most important points, simplifying the notation as before. Newton computes first the powers $(1+d)^n$ for the lowest integers $n=2, 3, 4, \cdots$, and discusses how to find directly the numerical coefficients of $d, d^2, d^3, \cdots$ in each of these expressions. He then makes the important remark that these coefficients in the expansion of $(1+d)^n$ can be generated by *multiplication* of the numbers $(n-0)/1$, $(n-1)/2$, $(n-2)/3, \cdots$, that is to say, that the coefficient of d^m in the expansion of $(1+d)^n$ is equal to

$$\frac{n(n-1)\cdots(n-m+1)}{1\cdot 2\cdot\cdots\cdot m}. \tag{3}$$

Of course, equivalent multiplicative relations for actually the same integers had been discovered a few years before by Pascal who defines them as elements of his "arithmetical triangle," without reference to the binomials.

From this statement Newton proceeds in an extremely audacious way. He got the idea from the procedure by which J. Wallis had developed his famous product formula for π by considering the successive integrals $\int_0^1 (1-x^2)^{n/2}dx$ for $n=0, 1, 2, \cdots$. (As a matter of fact, Newton starts in that letter with the consideration of these integrals instead of with the binomial itself.) He applies the same formula (3) also for the intermediate values $n=1/2, 3/2, 5/2, \cdots$ in order to obtain expressions for $(1+d)^n$ with these fractional values of the exponent, although he now has to write infinite series instead of finite sums. Further generalizations enable him to state the theorem for arbitrary values of the exponent.

To be sure, neither Gregory's nor Newton's deduction is an exact proof in the modern sense. In some respects, Gregory's way may seem to us more satisfactory: he deduces the result from a general theorem, the interpolation formula, and from a characteristic property of the function $(1+d)^x$, namely the difference equation (2). On the other hand, Newton makes this almost adventurous generalization of a finite algebraic identity, deduced for integral exponents only, into an infinite series for fractional exponents. Nevertheless, there is some internal connection between the two procedures. In his investigation, Newton considers the powers of a binomial as a function of the *exponent* as does Gregory, and not as a function of the second term d of the binomial. Thus, the procedures are not so different in their essence as they are in their execution. If one compares them with the usual modern proofs of the binomial theorem, one may remark that the latter are based on the consideration of $(1+d)^x$ as a function of d and that they use the successive *derivatives* with respect to d and the Taylor series instead of the successive *differences* with respect to x and the interpolation formula.

Newton realized the necessity of showing the way in which his consideration may be completed by a proper proof. As an example, he verifies by direct multiplication that the square of his series for $(1+d)^{1/2}$ is equal to $1+d$. Neither Gregory nor Newton tried to prove the convergence of the series. Such a proof was not, at this time, believed to be necessary; but certainly they had the feeling that these infinite sums determined definite numbers.

In this connection, it is interesting to find in a somewhat later letter of Gregory, dated April 9, 1672 [**14**] an early attempt to estimate the remainder of an infinite series by comparing it with the geometrical series. Here, he approximates the logarithmic series $x+x^3/3+x^5/5+x^7/7+\cdots$ by expressions such as

$$x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{9x^7}{7\cdot 9-7\cdot 7x^2}$$

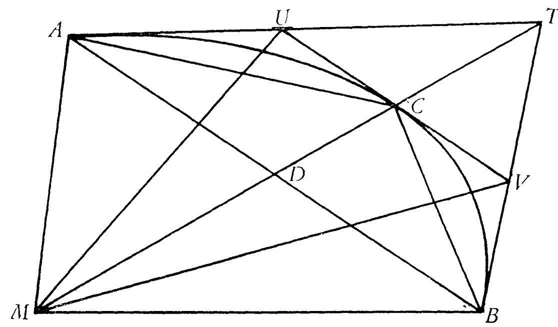
and emphasizes that the analogous expressions formed by using more terms of the original series will give a better approximation. Obviously, this estimate is obtained by comparison with the geometric series

$$x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}\left(1+\frac{7x^2}{9}+\left(\frac{7x^2}{9}\right)^2+\cdots\right).$$

Thus, we see here the first step on the way which, more than a century later, led Cauchy to his convergence tests.

4. Gregory's "Vera Quadratura." Gregory's "*Vera Circuli et Hyperbolae Quadratura*" [**2**], a small volume, contains extremely interesting and original ideas which are, to be sure, a little remote from the mathematics of his time. Even if his mathematical technique was not always sufficient to get a complete solution of the problems he saw, even if he sometimes makes incomplete deductions and wrong conclusions, the investigations show an immense creative power.

He follows in some way the classical procedure of Archimedes, but reveals the algebraic content of the method. Besides, instead of calculating the perimeter of the circle as Archimedes did, he operates on areas. This enables him to deal simultaneously with the sectors of the circle, ellipse and hyperbola.



Let M be the center of a conic ACB, let AT and BT be the tangent lines at A and B, respectively, and let the straight line MT intersect AB at D and the conic at C. Gregory concludes first from fundamental properties of the conics the relations [15]:

$$AD = DB, \qquad MC^2 = MD \cdot MT. \tag{1}$$

Now he draws the tangent line at C which intersects AT at U and BT at V, and compares the following pairs of polygonal areas which are inscribed in or circumscribed about the sector $MACB$: on the one hand he compares the inscribed triangle $i_0 = MAB$ with the circumscribed quadrangle $I_0 = MATB$, on the other hand the inscribed polygon $i_1 = MACB$ with the circumscribed polygon $I_1 = MAUCVB$. The polygon i_1 is composed of two equal triangles MAC and MCB; the polygon I_1 of two equal quadrangles $MAUC$ and $MCVB$. Then, elementary properties of the conics, especially the relations (1), enable him to deduce easily two equations between these four areas as follows:

$$i_1 = \sqrt{i_0 I_0}, \qquad I_1 = \frac{2 i_1 I_0}{i_1 + I_0}\,.$$

Now, operating on the triangles MAC and MCB, and on the quadrangles $MAUC$ and $MCVB$ in the same way as he had operated on the triangles MAB and the quadrangle $MATB$, he gets four triangles of equal areas $i_2/4$, inscribed in the sector $MACB$, and four quadrangles of equal areas $I_2/4$ circumscribed about the same sector. Obviously, he obtains:

$$i_2 = \sqrt{i_1 I_1}, \qquad I_2 = \frac{2 i_2 I_1}{i_2 + I_1}\,.$$

Repeating the same operation n times, he constructs for each successive $n = 3, 4, \cdots$ an inscribed polygonal area i_n, composed of 2^n equal triangles,

and a circumscribed one I_n, composed of 2^n equal quadrangles. The successive areas are given by:

$$(2) \qquad i_{n+1} = \sqrt{i_n I_n}, \qquad I_{n+1} = \frac{2i_{n+1}I_n}{i_{n+1} + I_n} = \frac{2i_n I_n}{i_n + \sqrt{i_n I_n}} \qquad (n = 0, 1, 2, \cdots).$$

Geometrically it is obvious that the area S of the sector $MACB$ lies between each pair i_n, I_n, and that, if n increases indefinitely, these areas will approach S as closely as one desires, one sequence increasing from below, the other decreasing from above. But Gregory is not satisfied with this visual evidence. He recognizes in the successive construction of the i_n, I_n a new arithmetic operation which yields the value S, and therefore he feels a necessity to *prove* what we call the convergence of the limiting processes

$$(3) \qquad \lim_{n\to\infty} i_n = \lim_{n\to\infty} I_n = S.$$

In fact, with that high degree of exactness which we find in the classical Greek mathematics, he first shows that

$$|I_{n+1} - i_{n+1}| < \tfrac{1}{2}|I_n - i_n|$$

and then concludes that $|I_n - i_n|$ becomes smaller than any given number if n is sufficiently large.

To realize the mathematical importance of Gregory's method we may state that, for the circle and ellipse where $I_0 > i_0$, the area S can be expressed as follows:

$$(4) \qquad S = I_0\sqrt{\frac{i_0}{I_0 - i_0}} \text{ arc tan } \sqrt{\frac{I_0 - i_0}{i_0}}\,.$$

For the circle, the first factor is simply $\frac{1}{2}MA^2$, the second the angle $\theta = BMA$. For the hyperbola where $I_0 < i_0$, we have only to interchange I_0 and i_0 and to replace the arc tangent function by the inverse of the hyperbolic tangent function. If we use imaginary numbers, we recognize that we have the same analytic function, since $\tanh ix = i \tan x$. But Gregory has discovered, without applying imaginary numbers, that the same analytical process—the approximation by the formulas (2), (3)—yields the area of the hyperbola as well as the area of the ellipse. In other words, he has found, for the first time in history, the analytical connection between the quadrature of sectors of the ellipse (or of the circle) and the quadrature of sectors of the hyperbola.

The history of these quadratures is interesting. We may assume that astronomical practice originally suggested the introduction of the arc of a circle as independent variable and the coordinates of the point on the circumference as dependent variables, that is to say, the introduction of the circular functions sine, cosine, and so on. This development may be connected with the fact that Archimedes investigated primarily the rectification of the circle instead of the quadrature. But the rectification of the general conics is an entirely different

and much more difficult problem. In considering the *area* of the circular sectors Gregory was able to find one single analytical process for the quadrature of all conics.

Now, it has been known since the middle of the 17th century that the quadrature of the hyperbola is connected with the logarithmic function. Therefore, it was obvious to Gregory himself that he had found *one* analytical process for getting from algebraic expressions to logarithmic functions as well as to inverses of the circular functions.

This discovery is generally ascribed to Euler who, some seventy years later, arrived at the connection between the exponential function and the circular functions by using formal operations in the field of complex numbers. It is doubtful whether Euler considered hyperbolic functions as analogous to circular functions and whether he used, in this respect, the analytical analogy between the processes of quadrature of circular and hyperbolic sectors.

The comparison of Euler's and Gregory's achievements may enhance our admiration for Gregory's genius. Indeed, it is not easy to connect in the field of real numbers the two integrals

$$\int \sqrt{1-x^2}\,dx \quad \text{and} \quad \int \sqrt{1+x^2}\,dx, \quad \text{or} \quad \int \frac{dx}{1+x^2} \quad \text{and} \quad \int \frac{dx}{1-x^2}.$$

As we have seen, this was achieved by Gregory.

In his "*appendicula ad veram circuli et hyperbolae quadraturam*" of 1668 [16] Gregory gives an array of linear combinations of the first i_n and I_n with definite numerical coefficients which yield much better approximations to the area S than do i_n and I_n themselves. Gregory was extremely offended that Huygens did not acknowledge his work to be an essential improvement over his older methods. Therefore he tried to make obvious the strength of the new theory by stating numerous new and surprising results without revealing how he had found them. Turnbull [17] has verified that, for the circle, one gets exactly Gregory's approximations if one first expresses i_n and I_n in terms of trigonometric functions of the angle θ, then expands these expressions in power series in θ, and finally forms such linear combinations of them which begin with the term θ and contain afterwards as many vanishing coefficients as possible. Analogous considerations are valid for the hyperbola. If Gregory operated in this manner he must have known the first terms of the power series for trigonometric and hyperbolic functions as early as 1668. Indeed, it is possible that he got this knowledge without using differentiation, but the published material does not seem to contain anything to corroborate this.

There are two other points in Gregory's speculations which particularly reveal the range of his mathematical ideas with respect to the actual later development of our science. First, the recurrent construction of the areas i_n, I_n is with him only *one* example of a very general, new analytic process which he coordinates as the "sixth" operation along with the five traditional operations

(addition, subtraction, multiplication, division, and extraction of roots). In the introduction, he proudly states "ut haec nostra inventio addat arithmeticae aliam operationem et geometriae aliam rationis speciem, ante incognitam orbi geometrico." This operation is, as a matter of fact, our modern limiting process. Clearly, his idea is, if we formulate it in modern language without changing the notions, to investigate two sequences of quantities $a_1a_2, \cdots$ and $b_1, b_2, \cdots$, defined by the recurrent equations

$$a_{n+1} = \phi(a_n, b_n), \qquad b_{n+1} = \chi(a_n, b_n) \qquad (n = 1, 2, 3, \cdots). \tag{5}$$

He uses the word "convergent" for these sequences, very probably for the first time in history, if for each n

$$0 < b_{n+1} - a_{n+1} < b_n - a_n.$$

Of course, this definition does not conform completely to our precise notion of convergence; but in applying his notion he proves in most cases the correct and sufficient inequality

$$0 < b_{n+1} - a_{n+1} < \rho(b_n - a_n)$$

where $\rho < 1$ is independent of n. (In his original problem, he has, as seen previously, $\rho = \frac{1}{2}$.) Then he concludes that the "last convergent terms" of the sequences a_n and b_n are equal, and he calls them *terminatio* of the sequences. In his original problem this terminatio is the area S.

From his further examples we may mention the following ones:

$$a_{n+1} = a_n + \alpha(b_n - a_n), \qquad b_{n+1} = b_n - \beta(b_n - a_n) \tag{6}$$

and

$$a_{n+1} = \frac{2a_nb_n}{a_n + b_n}, \qquad b_{n+1} = \frac{a_n + b_n}{2}. \tag{7}$$

Here he succeeds in finding the terminatio by an ingenious and simple idea: he determines an invariant expression $F(a_n, b_n)$ such that

$$F(a_{n+1}, b_{n+1}) = F(a_n, b_n); \tag{8}$$

then, the terminatio t will satisfy the equation

$$F(a_1, b_1) = F(t, t), \tag{9}$$

which gives the value t in terms of a_1 and b_1. For the examples (6), (7) he can state immediately the invariant expressions $F(a_n, b_n) = \beta a_n + \alpha b_n$ and $F(a_n, b_n) = a_n \cdot b_n$, respectively, and he finds as the termatio, using (9):

$$t = \frac{\beta a_1 + \alpha b_1}{\beta + \alpha} \quad \text{and} \quad t = \sqrt{a_1 b_1},$$

respectively.

One may remark that Gregory investigated in (2) and (7) different combinations of arithmetical, geometrical and harmonical means. One could imagine that he tried to treat other combinations of these means, but that he could not find out an algebraic expression or a geometric interpretation. In the following century the relation between the arithmetical-geometrical mean and the elliptic integrals was discovered by Lagrange, Legendre and Gauss. We know especially that Gauss studied these means in his early youth before he had any knowledge of the calculus, and that these means, later on, showed him the way to the elliptic integrals [**18**]. We know moreover that Pfaff, the teacher of Gauss, investigated sequences closely related to Gregory's sequence (2) [**19**]. Thus, we could guess that we have here an influence of Gregory's work on one of the most important theories of modern analysis, but we have no definite evidence of such connections.

The second point may be still more momentous. Gregory attempts to prove that the terminatio S of the polygons i_n, I_n cannot be expressed by using the traditional five "elementary" operations on i_0 and I_0. In the preface he puts particular emphasis on this phenomenon. From his exposition we may suppose that he first had tried to "square the circle," *i.e.* to find such an "elementary" expression for S. But he was critical enough to recognize that the difficulties in this search could not be overcome. And realizing that the task of algebra and analysis consists as well in solving a problem as in proving, if necessary, the "impossibility" of a certain solution, he dared to try such a proof, although he did not find any pattern for doing it. He emphasizes that since Euclid's classification of the usual irrationalities in his tenth book, nothing of this kind has even been attempted. Of course, Leonardo Pisano had shown [**20**] about 1200 A.D., that a certain cubic equation cannot be solved by Euclid's irrationalities. However, Gregory could not have had any knowledge of this investigation since it was not published before the nineteenth century. It is a testimony to Gregory's surprising intuition that he mentions further as problems impossible in the same sense just these two: to solve the general algebraic equation and to get an nth root by solving quadratic equations.

To be sure, Gregory does not prove that it is impossible to square the circle, although this is in his mind. He approaches only a much easier problem: to prove that the area of an arbitrary circular sector S cannot be expressed in terms of the areas i_0 and I_0 by the five elementary operations—or, in modern language, that the arc tangent function as given by (4) and defined by the limiting process (2), (3), is not a combination of such algebraic functions. The foundation of his proof is the remark that two sequences (2) yield the same terminatio S whether we begin the process with i_0, I_0 or with i_1, I_1; therefore S depends upon i_0 and I_0 in the same way as upon i_1 and I_1. To put it in modern language, the function satisfies the algebraic functional equation:

$$S(i_0, I_0) = S(i_1, I_1) = S\left(\sqrt{i_0 I_0}, \frac{2 i_0 I_0}{i_0 + \sqrt{i_0 I_0}}\right), \tag{10}$$

i.e. $S(i_0, I_0)$ can be transformed algebraically into itself. He tries to prove that the identity (10) is impossible for any function formed only by the five elementary operations. First he removes the irrationality, introducing two suitable new variables u, v by the equations

$$i_0 = u^2(u + v), \qquad I_0 = v^2(u + v).$$

Then (2) shows that

$$i_1 = uv(u + v), \qquad I_1 = 2uv^2,$$

and the identity (10) becomes

$$S(u^2(u + v), v^2(u + v)) = S(uv(u + v), 2uv^2). \tag{11}$$

Now he states two properties of this identity from which he is going to deduce its impossibility for functions S of the above described algebraic type: 1) The arguments of S on the left side contain u up to the third power, while those on the right side contain u only up to the second power. 2) On the left side, both arguments are binomial, while on the right side the second one is only monomial.

Of course, Gregory is able to prove correctly by this procedure that the identity (11) cannot be satisfied by a rational integral function S of its two arguments, and even, with slightly more difficulty, that it cannot be satisfied by any rational function. However, we do not believe that the facts he offers are sufficient to furnish the proof that S is not an irrational function built up in using extraction of roots. Indeed, the algebraic factor $I_0\sqrt{i_0}/\sqrt{I_0-i_0}$ of (4) satisfies, itself, an identity which differs from (10) only by a factor 2 in the left member, and Gregory's considerations could be applied equally well to the modified identity. The point is that the identity (10), used as basis for his proof implies an intrinsic difficulty: it is equivalent to the algebraic relation between $\tan\theta$ and $\tan 2\theta$ and, moreover, Gregory thinks of it only as valid in the restricted interval $0<\theta<\frac{1}{2}\pi$.

Today, we would conclude the transcendental character of $\tan\theta$ (and, simultaneously, of the inverse function arc tangent) immediately from the periodicity of that function ($\tan\theta=\tan(\theta+\pi)$). Although such a conclusion seems to us extremely simple, it may have been difficult and remote at Gregory's time.

A modern mathematician will highly admire Gregory's daring attempt of a "proof of impossibility" even if Gregory could not attain his aim. He will consider it a first step into a new group of mathematical questions which became extremely important in the 19th century. However, the contemporary echoes of Gregory's undertaking were in no way favorable. First of all, Huygens criticized [**21**] the "*Vera Quadratura*" in an extremely unfavorable manner. Gregory had sent him one of the first copies. He expected his discoveries to be fully appreciated by this great mathematician who himself had done very important work on the problem of the quadrature of conics and the circle. But, unfortunately, Huygens was apparently angry that those earlier investigations were not mentioned. Thus, he put more emphasis on some claims of priority and on

some objections against Gregory's deductions than on the importance of Gregory's new ideas and results. There is no need to report here on the unpleasant discussion which arose from this criticism [22]. We mention only the single point of importance where Huygens showed a profounder insight. He says: even if the area of an arbitrary circular sector cannot be expressed algebraically in terms of the areas i_0, I_0, one can still imagine such an expression to be possible for particular sectors, for example, for the whole circle itself. Gregory, obviously, had overlooked this possibility in his original publication. In his answer he tried to deduce the result for the "particular case" from that for the arbitrary sector. These endeavors could not but fail; it took more than two centuries before mathematics had developed the necessary means to prove the transcendency of π.

5. Conclusion. Surveying the importance of all these discoveries and ideas of Gregory, and realizing that the total range of his scientific work is by no means covered by our report, one may wonder why this great man did not exert more influence on the actual development of mathematics. The reason can be found in some unfortunate, almost tragical facts in Gregory's life which hampered his activity as well as the effectiveness of his work. After some short sojourns in London (1663 and 1668), and several years of inspiring studies in Italy (1664–1668), mostly in Padua, he was appointed Professor of Mathematics at the Scottish University of St. Andrews. At this old school, still living entirely in medieval traditions, the young scholar was rather isolated. There he was the only one who knew of the new development of mathematics. He himself abounded with new ideas, but there was no possibility to discuss or to teach them. Moreover, hardly any literature was available. Only through his correspondence with Collins whom he had met in London and who had become his close friend, could he learn what the great mathematicians in England and abroad were planning and completing.

Thus, his ideas could not find the response they deserved and he himself did not develop them as far as it might have been possible in closer contact with mathematicians of equal rank. Still worse consequences may have been involved in the lack of appreciation of his first important publication, the *Vera Quadratura*, and especially in the unkind and unjust criticism of Huygens which we have mentioned above.

Apparently, these experiences impressed the proud young Scotchman so deeply that he abandoned entirely the trend of ideas he had started so successfully. We can imagine that otherwise he might have applied his "convergent" pairs of sequences, as defined by recurrence formulas, to various problems and that he might have brought this important process to greater prominence in the early analysis. In fact, he afterwards used the infinite series, probably influenced by the reports he got, scantily, on Newton's work. Yet, also here, fate did not favor him. For he was not given time and opportunity to complete and publish his investigations; and his great merits were darkened by Newton's glory who, meanwhile, could finish his work.

Besides, Gregory had inaugurated research on differential and integral calculus without knowing what his eminent competitors were doing simultaneously in this field. He was even the first to publish, as early as 1668, a proof [**23**] of the "fundamental theorem," that the two characteristic problems of the calculus, namely, to determine the slopes and to determine the areas, are inverse to one another. Also here he met misfortune; immediately afterwards there appeared Barrow's great work "lectiones geometricae," which went much farther and won all fame. A few years later, Newton's and Leibniz's momentous results on the calculus became known and made obsolete the work of all their predecessors.

Gregory did not live to see this development. He had eventually taken over a professorship at the University of Edinburgh, which granted him better working opportunities. But only one year later, in the fall of 1675, he suddenly fell ill and died in his thirty-seventh year. Most of his discoveries and ideas were buried in his letters and notes or lost through his death.

References

1. Published for the Royal Society of Edinburgh, London, 1939.

2. Pataviae, 1667; reprinted as appendix to J. Gregory's Geometria pars Universalis, Venetiae, 1668, and again in Chr. Huygens, Opera varia, vol. II, Lugduni Batavorum, 1724, pp. 407–462. Our report in no. 4 is based on our essay in the Memorial [**1**], pp. 468–478.

3. N. Mercator, Logarithmotechnica, Londini, 1668.

4. Published with a comprehensive commentary of H. W. Turnbull in the Memorial [**1**], pp. 350–359.

5. Memorial [**1**], pp. 118–122, especially p. 119 f.; *cf.*, Turnbull's commentary, *ibid.*, p. 124. With regard to the earlier publications of that letter, one may compare *ibid.*, pp. 25 and 29.

6. It is mentioned first, but not formulated, in Newton's letter to Oldenburg of October 24, 1676 (Newton, Opuscula Mathematica I, Lausannae et Genevae, 1744, pp. 328–357; see particularly p. 340) and completely stated in his Philosophiae Naturalis Principia, 1687, book III, lemma V, and in his Methodus Differentialis, 1711 (Opuscula [**6**], p. 271 ff.)—at both places for equidistant and non-equidistant ordinates.

7. See Memorial [**1**], p. 58, and Turnbull's note, p. 59.

8. In the letter quoted in [**6**], p. 341.

9. Methodus Incrementorum, Londini, 1715.

10. See Memorial [**1**], p. 131 f., and Turnbull's commentary, p. 132 f.

11. Letter of March 24, 1670, Memorial [**1**], p. 88; *cf.*; Gregory's answer, *ibid.*, p. 92.

12. Memorial [**1**], p. 148; *cf.*, Turnbull's commentary, p. 150 f.

13. Newton, Opuscula I [**6**], pp. 307–322, especially p. 307 f., and pp. 328–357, especially pp. 329 ff.

14. Memorial [**1**], p. 230.

15. In our essay in the Memorial [**2**], p. 469, the second of these formulas is misprinted. We may correct here some other minor misprints in that essay: p. 469, last line, read I_2 instead of $\sqrt{I_2 \cdot i_2}$; p. 471, last formula, read $b_{n+1}=\chi(a_n, b_n)$ instead of $b_{n+1}=\phi(a_n, b_n)$; p. 478, note 7, read tanh instead of tan on one side of the formula.

16. First part of Gregory's Exercitationes Geometricae, Londini, 1668.

17. See Memorial [**1**], p. 461 ff.

18. *Cf.*, L. Schlesinger, Gauss' Fragmente zur Theorie des arithmetisch-geometrischen Mittels, Nachrichten der Goettinger Gesellschaft der Wissenschaften, 1912, and his essay in Gauss, Werke, vol. X, part 2, Berlin, 1933, Abhandlung 2.

19. *Cf.*, Pfaff's letters to Gauss in Gauss, Werke, vol. X, part 1, Leipzig, 1917, p. 234 ff., and H. Geppert, Mathematische Annalen, vol. 108, 1933, p. 205 ff.

20. Published first by B. Boncompagni, tre scritti inediti di Leonardo Pisano, Firenze, 1854. The proof to which we refer is reviewed comprehensivly by F. Woepke, Journal de Mathématiques pures et appliquées, vol. 19, 1854, p. 401 ff.

21. First in a review in the Journal des Sçavans, Paris, July, 1668; *cf.*, the references in [**22**].

22. One may compare, also for further literature, our essay [**2**] and the comprehensive report of E. J. Dijksterhuis, Memorial [**1**], pp. 478–486. The most important parts of the discussion are reprinted in Chr. Huygens, Opera varia II [**2**], pp. 463 ff., and again in the Oeuvres complètes de Christiaan Huygens, vol. VI, La Haye, 1895, pp. 228 ff.

23. Contained in the Geometriae pars universalis, Venetiae, 1668. *Cf.*, the essay of A. Prag on this work in the Memorial [**1**], pp. 487 ff.

THE CLOSURE OF SYSTEMS OF ORTHOGONAL FUNCTIONS

G. E. ALBERT, Ohio State University

1. Introduction. A system of functions $\phi_n(x)$, $n=0, 1, 2, \cdots$ is said to be *orthonormal* on the finite interval (a, b) provided that

$$\int_a^b \phi_n(x)\phi_m(x)dx = \begin{cases} 0 & \text{if} \quad n \neq m, \\ 1 & \text{if} \quad n = m. \end{cases}$$

With any integrable function $g(x)$ there is an associated generalized Fourier series

$$g(x) \sim \sum_{n=0}^{\infty} a_n\phi_n(x), \qquad a_n = \int_a^b g(x)\phi_n(x)dx. \tag{1}$$

An orthonormal system is said to be *closed in the class H* of functions if the series (1) associated with an arbitrary function $g(x)$ in H converges in the mean to $g(x)$; that is, if

$$\lim_{n\to\infty} \int_a^b \{g(x) - s_n(x)\}^2 dx = 0 \tag{2}$$

where $s_n(x)$ denotes the sum of the first n terms of (1).

The importance of the concept of closure in teaching courses involving orthogonal series is quite generally recognized. Various conditions for the validity of the property are known. Unfortunately, the application of those conditions to specific orthogonal systems is, even in the simplest cases, somewhat abstruse for presentation to a class composed of, say, seniors and first year graduate students in physics and engineering. In the second section of this paper a new criterion for closure is given which can be applied directly to verify the property for a number of classical orthogonal systems. In the third section the application of the condition is indicated for the system of Legendre polynomials and the system of trigonometric functions. The entire procedure may be shown to students having no more preparation than a course in Advanced Calculus.

GRAPHICAL SOLUTIONS OF CUBIC, QUARTIC, AND QUINTIC

T. R. RUNNING, University of Michigan

1. Statement of the problem. It is proposed in this paper to develop a method for the graphical solution of equations of the 3rd, 4th, and 5th degrees with real coefficients. The roots are found as abscissas of the points of intersection of two curves whose equations are easily obtained from the equation to be solved. We will begin with the solution of the quartic in x. The cubic may be solved as a quartic after multiplying through by x. The solution of the quintic will be taken up after the quartic has been solved. It will be seen that the method may be applied to the solution of equations of higher degree provided they have no more than four complex roots.

2. Basis of solution. Let the roots of the quartic be represented by $a \pm bi$ and $c \pm di$. The equation may then be written

$$\text{(A)} \quad \begin{aligned} &x^4 - 2(a+c)x^3 + (a^2+b^2+c^2+d^2+4ac)x^2 \\ &\qquad - 2[a(c^2+d^2)+c(a^2+b^2)]x + (a^2+b^2)(c^2+d^2) = 0. \end{aligned}$$

This is identically

$$\begin{aligned} &[x^2 - (a+c)x + \tfrac{1}{2}(d^2+b^2+2ac)]^2 \\ &\qquad - [(d^2-b^2)(a-c)x + \tfrac{1}{4}(d^2-b^2)^2 + (b^2c-ad^2)(a-c)] = 0. \end{aligned}$$

Transposing the second bracket and extracting the square root of both members we see that equation (A) is satisfied by the abscissas of the points of intersection of the two parabolas

$$(1) \qquad y = x^2 - (a+c)x + \tfrac{1}{2}(d^2+b^2+2ac) \quad \text{and}$$

$$(2) \qquad y^2 = (d^2-b^2)(a-c)x + \tfrac{1}{4}(d^2-b^2)^2 + (b^2c-ad^2)(a-c).$$

The parabolas (1) and (2) will be referred to as auxiliary curves. To obtain the right hand member of (1) extract the square root of the left hand member of equation (A) stopping at the third term. The square of the right hand member of (1) minus the left hand member of (A) gives the right hand member of (2).

3. Real roots. If the auxiliary curves intersect in four points the four roots are read as abscissas of the points of intersection. If they intersect in only two points the two real roots are read from the diagram and the complex roots are easily obtained.

4. Complex roots. If the auxiliary curves do not intersect the roots are complex. These roots will be obtained by employing a new set of auxiliary curves. These curves will give us the value $a-c$. Rewrite equations (1) and (2) in the form

$$(3) \qquad y = x^2 + Ax + B,$$

$$(4) \qquad y^2 = Cx + D.$$

By equating the coefficients in (3) and (4) to those in (1) and (2) and eliminating b^2 and d^2 we obtain the sextic (5) in $a-c$.

$$(a-c)^6+(4B-A^2)(a-c)^4+(4D-2AC)(a-c)^2-C^2=0 \quad \text{or} \tag{5}$$

$$z^3+(4B-A^2)z^2+(4D-2AC)z-C^2=0 \tag{6}$$

where

$$z=(a-c)^2, \qquad b^2=B-ac-\frac{C}{2(a-c)}, \qquad d^2=B-ac+\frac{C}{2(a-c)}.$$

It is easily shown that equation (6) has one and only one positive root. Multiplying (6) through by z we have the quartic equation

$$z^4+(4B-A^2)z^3+(4D-2AC)z^2-C^2z=0 \tag{7}$$

to solve for the value of $(a-c)^2$. The auxiliary curves obtained for the solution of (7) are

$$y=z^2+(2B-\tfrac{1}{2}A^2)z+2D-AC-\tfrac{1}{2}(2B-\tfrac{1}{2}A^2)^2 \quad \text{and} \tag{8}$$

$$y^2=\{(2B-\tfrac{1}{2}A^2)[4D-2AC-(2B-\tfrac{1}{2}A^2)^2]+C^2\}z+[2D-AC-\tfrac{1}{2}(2B-\tfrac{1}{2}A^2)^2]^2. \tag{9}$$

As we are concerned with only the positive root of (7) the diagram in any particular case will be quite simple. The introduced root zero is disregarded. It is well to observe that if $C=0$, $a-c$ is not necessarily zero. Either $a=c$ or $d^2=b^2$ will make $C=0$. If $C=0$ and D negative the value of $(a-c)^2$ is found from the equation $z^2+(4B-A^2)z+4D=0$.

Example. Solve $x^4-2x^3+3x^2-4x+5=0$.

The auxiliary curves are

$$y=x^2-x+1, \qquad A=-1, \qquad C=2, \tag{10}$$

$$y^2=2x-4, \qquad B=1, \qquad D=-4. \tag{11}$$

From (7) we have

$$z^4+3z^3-12z^2-4z=0. \tag{12}$$

The roots of equation (12) are the abscissas of the points of intersection of the two auxiliary curves

$$y=z^2+\tfrac{3}{2}z-7\tfrac{1}{8} \quad \text{and}$$

$$y^2=-17\tfrac{3}{8}z+50\tfrac{49}{64}.$$

From the graphs of the two equations z is found to be 2.48. The complex roots are then found in the following way:

$$(a-c)^2=2.48 \qquad c=-0.29$$

$$a-c=1.57 \qquad ac=-0.37$$

$$a+c=1.00 \qquad b^2=0.74$$

$$a = 1.29 \qquad d^2 = 2.00$$

$$x = 1.29 \pm 0.86i \quad \text{or} \quad -0.29 \pm 1.42i.$$

The solution of the cubic is illustrated by the solution of equation (12) and needs no further explanation.

The solution of the quintic is best set forth by an

Example. Solve $x^5-4x^4+7x^3-10x^2+13x-10=0$. Introducing the factor x we obtain the equation

$$x^6 - 4x^5 + 7x^4 - 10x^3 + 13x^2 - 10x = 0.$$

By a process similar to the one used in obtaining equations (1) and (2) it becomes evident that the roots of the sextic are the abscissas of the points of intersection of the auxiliary curves

$$y = x^3 - 2x^2 + \tfrac{3}{2}x - 2 \quad \text{and} \tag{13}$$

$$y^2 = -\tfrac{11}{4}x^2 + 4x + 4. \tag{14}$$

From the graphs of these equations it is seen that (omitting the introduced value zero) there is only one point of intersection. The abscissa of this point is 2. Reducing the degree of the given quintic we obtain the quartic just solved.

It might be remarked that in the case of double roots the auxiliary curves are tangent to each other, and in the case of triple roots they cross with a common tangent. The graphs of the auxiliary curves can only lead us to suspect the presence of double or triple roots if they exist. Nearly equal roots cannot be separated graphically.

5. Equations of higher degree. The preceding naturally suggests the possibility of a graphical solution of equations of higher degree with real coefficients proved they have no more than four complex roots. Let the general equation be written

$$x^{2n} + A_1x^{2n-1} + A_2x^{2n-2} + A_3x^{2n-3} + A_4x^{2n-4} + \cdots + A_{2n-1}x + A_{2n} = 0.$$

If an equation given for solution is of an odd degree it must be made of an even degree by introducing the factor x. The equations of the auxiliary curves, obtained in a way similar to the way (1) and (2) we obtained, are

$$y = x^n + B_1x^{n-1} + B_2x^{n-2} + B_3x^{n-3} + \cdots + B_{n-1}x + B_n \quad \text{and} \tag{15}$$

$$y^2 = C_1x^{n-1} + C_2x^{n-2} + C_3x^{n-3} + \cdots + C_{n-1}x + C_n. \tag{16}$$

To obtain the right hand member of (15) extract the square root of the left hand member of the general equation stopping with the $(n+1)$th term, then proceed as before.

Example. Let it be required to solve the equation

$$x^8 - 10x^6 + 33x^4 - 40x^2 - \tfrac{2}{9}x + 15\tfrac{4}{9} = 0.$$

The auxiliary curves found from the given equation are

$$y = x^4 - 5x^2 + 4, \tag{17}$$

$$y^2 = \tfrac{2}{9}x + \tfrac{5}{9}. \tag{18}$$

By carefully graphing equations (17) and (18) on cross section paper, say twenty divisions to the unit, it will be seen that the abscissas of the eight points of intersection are -2.03, -1.97, -1.10, -0.90, 0.85, 1.15, 1.90, and 2.07. The last figures in the roots are somewhat in doubt. In the case of complex roots, not more than four, the solution of the quintic illustrates the method of procedure. Only rough approximations to the roots can be obtained graphically.

ON THE PRODUCT OF LINEAR FORMS

GORDON PALL, McGill University

1. Introduction. We shall give in this note a new proof, differing in its essentials from any yet published, of Minkowski's

THEOREM. *If a_{ij} $(i, j=1, 2)$ and b_1, b_2 are real numbers, and if $D=|a_{ij}|\neq 0$, then there exist integers x_i such that*

$$|(a_{11}x_1 + a_{12}x_2 - b_1)(a_{21}x_1 + a_{22}x_2 - b_2)| \leqq \tfrac{1}{4}|D|. \tag{1}$$

Several proofs of this result have been published, and two proofs of its extension to $n=3$. References through 1936 are given in Koksma's "*Diophantische Approximationen*," Ergebnisse der Mathematik, 4, 1936, p. 19. A new proof for $n=2$ was given recently by Mordell (*J. London Math. Soc.*, 16, 1941, 86–88); and a very simple proof for $n=3$ was given by Davenport (ibid., 14, 1939, 47–51). The conjecture for any n is that if $L_1, \cdots, L_n$ are n real homogeneous linear forms in $x_1, \cdots, x_n$ with determinant D not zero, and $b_1, \cdots, b_n$ are n real numbers, then there exist integers x_i such that $\prod|L_i-b_i|\leqq 2^{-n}|D|$. No complete proof has yet been given for any $n\geqq 4$. However proofs have been given by Tschebotareff,* Siegel,† Davenport,† and Mordell[1] that there exists a constant γ_n, somewhat larger than 2^{-n}, which would serve in place of the 2^{-n}.

The generalization of our present proof seems to require a generalized euclidean algorithm, to be used in assuring that certain inequalities like our (12) shall work out successfully.

2. Permissible transformations. If any of the a_{ij} is zero, the theorem is immediate. For example if a_{12} is zero, then $D=a_{11}a_{22}\neq 0$, and we choose x_2 so that $|a_{22}x_2-b_2|\leqq\frac{1}{2}|a_{22}|$, and for this x_2 choose x_1 so that $|a_{11}x_1+a_{12}x_2-b_1|\leqq\frac{1}{2}|a_{11}|$.

The following transformations of the forms $\phi_i=a_{i1}x_1+a_{i2}x_2$ are *permissible*, provided the theorem can be proved for all b_i with respect to the derived pair:

* N. G. Tschebotareff, Sci. Notes Kazan Univ. 94, 14–16, 1934; and Vierteljschr. Naturforsch. Ges. Zürich, 85 Beiblatt, 27–30, 1940. Mordell's result is also in this Beiblatt, pp. 47–50.

† C. L. Siegel, in a letter to Mordell, October 10, 1937. At Siegel's suggestion, H. Davenport published a similar proof in the Acta Arithmetica, 2, 1937, 262–265.

(a) any transformation $x_i = t_{i1}y_1 + t_{i2}y_2$ $(i=1, 2)$ of determinant ± 1 and integral coefficients t_{ij}; *e.g.* sign-changes and interchanges of the x_i;

(b) interchanging the forms ϕ_1 and ϕ_2;

(c) multiplying either form by a non-zero real number, *e.g.* -1.

LEMMA 1. *By permissible transformations we can replace ϕ_1, ϕ_2 by forms in which, either, one of the a_{ij} is zero, or*

$$a_{11} > 0, \qquad a_{12} > 0, \qquad a_{21} < 0, \qquad a_{22} > 0. \tag{2}$$

Supposing no a_{ij} zero, we can obviously secure either (2) or

$$a_{ij} > 0 \qquad (i, j = 1, 2). \tag{3}$$

If (3) holds write $a_{12} = a_{11}q_1 + a_{13}$, $a_{22} = a_{21}q_2 + a_{23}$, where the q_i are integers, $0 \leqq a_{13} < a_{11}$, $0 \leqq a_{23} < a_{21}$. If now $q_1 > q_2$ (and similarly if $q_1 < q_2$), either some a_{ij} is zero or we obtain (2) on replacing x_1 by $x_1 - q_1x_2$; for a_{11} and a_{21} are unaltered, $a_{12} \to a_{13}$, $a_{22} \to a_{23} - a_{21}(q_1 - q_2) < 0$. If $q_1 = q_2$ we interchange columns and continue the process, thus obtaining for the q's the successive quotients in the continued fractions for a_{12}/a_{11} and a_{22}/a_{21}. The quotients must differ at some step unless $a_{12}/a_{11} = a_{22}/a_{21}$, that is $D = 0$.

LEMMA 2. *If* (2) *holds a further transformation secures*

$$a_{11} > a_{12} > 0, \qquad a_{22} > -a_{21} > 0. \tag{4}$$

A simple interchange and sign-change evidently allows us to assume $a_{11} > a_{12} > 0$, $a_{21} < 0$, $a_{22} > 0$. If now $-a_{21} \geqq a_{22}$, let $-a_{21} = a_{22}q + r$, $0 \leqq r < a_{22}$, $q \geqq 1$, and replace x_2 by $x_2 + qx_1$. This replaces a_{11} by $a_{11} + qa_{12}$ *a fortiori* greater than a_{12}, and $-a_{21}$ by r.

To simplify our further argument we divide through by a_{11} and a_{22} respectively, and have

LEMMA 3. *In proving Theorem* 1 *we can restrict ourselves without loss of generality to the forms*

$$x + \rho y, \qquad -\sigma x + y, \qquad (0 < \rho < 1, 0 < \sigma < 1). \tag{5}$$

We shall presently require the additional condition $2\sigma\rho + \rho \geqq \sigma$, which can be achieved, if $\rho < \sigma$, by interchanging the forms, changing the signs of one form, and replacing x by y, y by $-x$.

3. The fundamental lemma. *If $0 < \rho < 1$, $\sigma > 0$, κ and λ are real, and $2\sigma\rho + \rho \geqq \sigma$, then we can choose integers x and y to satisfy simultaneously*

$$-\tfrac{1}{2} < x + \rho y - \kappa \leqq \tfrac{1}{2}, \tag{6}$$

$$\left| (x + \rho y - \kappa)(-\sigma x + y - \lambda) \right| \leqq (1 + \rho\sigma)/4. \tag{7}$$

There are solutions of (6) with y as small and x as large as we please, and hence such that $-\sigma x + y$ is arbitrarily small algebraically. For to any y, (6)

corresponds a unique x. For any solution (x_0, y_0) of (6) there is a unique r such that

$$(x_0,\ y_0 + i) \qquad (i = 0, \cdots, r) \tag{8}$$

are solutions of (6), but (x_0, y_0+r+1) is not. Abbreviating $\epsilon \equiv x_0+\rho(y_0+r)-\kappa$, we have

$$-\tfrac{1}{2} < \epsilon \leqq \tfrac{1}{2}, \qquad \epsilon + \rho > \tfrac{1}{2}. \tag{9}$$

Further, the values (8) give to $-\sigma x+y$ the values

$$-\sigma x_0 + y_0,\ -\sigma x_0 + y_0 + 1, \cdots, -\sigma x_0 + y_0 + r \equiv \eta,$$

and hence take care in respect to (7) of all values λ from a little less than $-\sigma x_0+y_0$ to $\eta+\mu$, where μ is defined by

$$|\epsilon| \cdot \mu = (1 + \rho\sigma)/4. \tag{10}$$

Next, (x_0-1, y_0+r+1) is likewise a solution of (6), giving to $x+\rho y-\kappa$ the value $\epsilon+\rho-1$, and to $-\sigma x+y$ the value $\eta+1+\sigma$; for by (9), $-\frac{1}{2}<\epsilon+\rho-1\leqq\frac{1}{2}$. These values take care of values λ from $\eta+1+\sigma-\nu$ to $\eta+1+\sigma+\nu$, where ν is defined by

$$|\epsilon + \rho - 1| \cdot \nu = (1 + \rho\sigma)/4. \tag{11}$$

A forward step will have been completed if $\mu+\nu\geqq 1+\sigma$, which can be expressed as follows:

$$\frac{1}{|\epsilon|} + \frac{1}{|\epsilon + \rho - 1|} \geqq \frac{4(1+\sigma)}{1+\rho\sigma}. \tag{12}$$

It remains only to verify that (12) holds in all cases.

I. *Case* $\epsilon<0$. Write $\theta=1-\epsilon$. Then by (5) and (9), $\rho<1<\theta<\rho+\frac{1}{2}$. Since the right hand side of (12) is $\leqq 4/\rho$ since $\rho<1$, (12) follows from $1/(\theta-1)+1/(\theta-\rho) \geqq 4/\rho$, and hence ($\theta$ being given its maximum value $\rho+\frac{1}{2}$) from

$$2/(2\rho - 1) + 2 \geqq 4/\rho, \qquad 4(\rho - 1)^2 \geqq 0. \tag{13}$$

II. *Case* $1-\rho<\epsilon\leqq\frac{1}{2}$. Then (12) holds if $1/\epsilon+1/(\epsilon+\rho-1)\geqq 4/\rho$, which reduces to (13) on giving ϵ its maximum value $\frac{1}{2}$.

III. *Case* $0<\epsilon<1-\rho$. Then (12) holds if

$$1/\epsilon + 1/(1 - \rho - \epsilon) \geqq 4(1 + \sigma)/(1 + \rho\sigma).$$

The left hand side is least when $\epsilon=(1-\rho)/2$, and the inequality reduces to $2\rho\sigma+\rho\geqq\sigma$.

A SURVEY COURSE FOR TEACHERS

J. W. QUERRY, Sam Houston State Teachers College

Much has been said and written concerning so called "survey" courses in high schools and colleges. In mathematics such courses are usually called General Mathematics and are offered to freshmen. In college, the students enrolled in these courses often have had meager high school training and in general are poorly prepared. They seem to possess neither the desire to broaden their mathematical horizon, nor the capacity for the proper appreciation. One noteworthy exception is the course on The Significance of Mathematics at the University of Buffalo.*

After teaching a general mathematics course for college freshmen with results that were none too encouraging we decided to experiment with a survey course for mathematics majors offered to seniors and students beginning graduate work. Consequently, in the summer session of 1941 we offered a course called "Mathematics for High School Teachers," its general aim or objective being stated in its description in the college catalogue: "Selected Topics from Higher Algebra and Geometry Designed to Strengthen Academically Teachers of High School Mathematics."

Recognizing that the study of mathematics begins somewhere in the "middle" and proceeds both toward its beginnings or foundations and toward abstract generalizations, an effort was made to take the class in both directions as far as time and abilities would warrant.

The outline for the course included the following topics:†

1. Definitions of mathematics.
2. Evolution of the number system: development of the number system from basic postulates; other number systems; elementary number theory.
3. Algebra and logic; Boolean algebra and other algebras.
4. Fundamental theorem of algebra.
5. Infinite series.
6. Criterion for ruler and compass constructions; trisection of an angle, duplication of the cube, squaring the circle.
7. Statistics, probability, interpolation, extrapolation.
8. Geometries: Lobachevskian, Riemannian, projective, modern.
9. The n nth roots and logarithms of complex numbers.
10. Meaning of topology.
11. Mathematics for recreation; mathematical puzzles.
12. Selected topics from the history of mathematics; famous mathematicians, historical and contemporary.

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* Harriet F. Montague, A course on the significance of mathematics, this MONTHLY, vol. 48, 1941, pp. 681–684.

† Because of lack of time some of these had to be omitted. It is recommended that the topics omitted and the ones stressed be dependent upon the needs, desires and abilities of the individual students.

13. Applications of mathematics* to music, to physics, to biology, to agriculture, to industry; role of mathematics in modern warfare.

14. Relativity theory.

Such reference books as *Fundamentals of Mathematics* by Richardson and *Introduction to Mathematics* by Cooley, Gans, Kline, and Wahlert were quite valuable. However, these texts were written primarily for freshmen and therefore many omissions of elementary materials and supplementations of more advanced reading were necessary. For example, the materials concerning analytic geometry and calculus were omitted because the students had credits in these subjects; on the other hand proofs of the impossibility of the trisection of a given angle, *etc.*, were given in class, while these texts omit such proofs.

As the class progressed, many subjects for term papers and term reports presented themselves, the ideas for them growing out of topics briefly discussed in class. In this connection each individual was encouraged to select some phase of the work and explore it as far as time and facilities would permit as a sort of a miniature research project. Class time was given for reports on progress and many interesting bits of information concerning the history, philosophy, and applications of mathematics as well as pure mathematics were exchanged among the members. Although this definitely was not a methods course we felt free to connect certain contents of the course with their practical teaching problems. An effort was made to develop an awareness on the part of students to the possibility of a direct transfer to their own classroom situations.

As is so often the case, objective data concerning the lasting values which students obtain from a course are difficult to obtain. But in terms of subjective evaluation the results were quite pleasing. A high degree of interest was shown by each student. Some persons from other subject matter fields attended class regularly and took part in discussions without being enrolled for credit. Some students expressed disgust at the manner in which they had been teaching certain topics to high school classes because of "ignorance" on their part. One decided to experiment with a unit on Boolean algebra in her second year algebra class as a basis for a thesis topic. Another is planning some non-Euclidean experiments in plane geometry.

Obviously such a course as this might seem trivial for a university where research proper is stressed, but for a teachers college, the majority of whose students are not looking forward to the doctorate, it seems that a need is being met. At least, the results were pleasing enough to justify further experimentation with a survey course on an advanced level.

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* The class was encouraged to submit applications suitable for the inclusion in the 17th Yearbook of the National Council of Teachers of Mathematics.

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AN INDUCTIVE PROOF OF DESCARTES' RULE OF SIGNS

A. A. ALBERT, University of Chicago

1. Introduction. The so-called proofs of Descartes' Rule generally given in college algebras are merely verifications of special cases. The only rigorous direct proof in the literature known to me is that of L. E. Dickson's *First Course in the Theory of Equations*. This proof uses a rather complicated notation and has always seemed to me to be difficult to follow.

I have recently discovered an alteration in the proof which enables the omission of any consideration of permanences. This results in a fundamental simplification of the notation. I present it here in full detail.

2. Sequence of real numbers. Let n be a positive integer and consider a sequence

$$a_0, a_1, \cdots, a_n \tag{1}$$

of $n+1$ real numbers a_i. We shall be interested in pairs of *consecutive non-zero* terms a_i, a_j of our sequence, and shall call a_i and a_j *consecutive* either if $j=i+1$ or if $j>i+1$ but every intervening a_k, with $i<k<j$, is zero.

A *variation* in sign of our sequence is a pair of consecutive non-zero terms of opposite sign. Let us count the number of variations and designate the integer so obtained by

$$V(a_0, a_1, \cdots, a_n). \tag{2}$$

It is the number of changes of sign if we take as our first sign that of the first non-zero a_i (reading from left to right) and then pass to the right through all the signs of its non-zero terms. What then if the first and last non-zero terms have the same sign? In this case an even number of changes of sign must have

taken place, $V(a_0, a_1, \cdots, a_n)$ must be even. Similarly an odd number of changes of sign will result in a first and a last non-zero term which have opposite signs. We state a form of this result as

LEMMA 1. *Suppose that* $a_0a_n \neq 0$. *Then* $V(a_0, a_1, \cdots, a_n)$ *is even or odd according as* a_0a_n *is positive or negative.*

From this result we shall derive immediately

LEMMA 2. *Let* $r_0, \cdots, r_{n-1}$ *be a sequence of positive real numbers and derive a second sequence* $b_0, \cdots, b_n$ *from* (1) *by the definitions*

$$b_0 = a_0, \qquad b_j = a_j + r_{j-1}b_{j-1} \qquad (j = 1, \cdots, n. \tag{3}$$

Then if a_0, a_n, *and* b_n *are all not zero, and if* $V(a_0, \cdots, a_n) = V(b_0, \cdots, b_n)$, *the signs of* a_n *and* b_n *are the same.*

For our hypothesis and Lemma 1 imply that a_0a_n and a_0b_n have the same signs. Hence so do a_n and b_n.

If we adjoin a term a_{n+1} to our sequence we will have $V(a_0, a_1, \cdots, a_n) = V(a_0, a_1, \cdots, a_{n+1})$ either if $a_{n+1}=0$ or if the sign of a_{n+1} is the same as that of a_n. It is also clear that in all cases

$$V(a_0, a_1, \cdots, a_n) \leqq V(a_0, a_1, \cdots, a_{n+1}) \leqq V(a_0, a_1, \cdots, a_n) + 1. \tag{4}$$

We shall use these remarks in an inductive proof of our principal.

LEMMA 3. *Define the sequence* $b_0, b_1, \cdots, b_n$ *as in* (3). *Then* $V(b_0, b_1, \cdots, b_n) \leqq V(a_0, a_1, \cdots, a_n)$, *and if* a_0 *and* a_n *are not zero but* $b_n = 0$ *we have* $V(b_0, b_1, \cdots, b_n) < V(a_0, a_1, \cdots, a_n)$.

For if $n=1$ the number of variations in the sequence a_0, a_1 is zero or one according as a_0a_1 is not or is negative. Since $b_0b_1 = a_0a_1 + r_0a_0^2$ we see that if a_0a_1 is non-negative so is b_0b_1, the number of variations of sign is zero in both cases. Moreover if $a_0 \neq 0$ then b_0b_1 is positive, $b_1 \neq 0$. If a_0a_1 is negative then $V(a_0, a_1) = 1 \geqq V(b_0, b_1)$ and $V(a_0, a_1) > V(b_0, b_1) = 0$ if $b_1 = 0$.

Assume now that our result is true for all pairs of related sequences of our type with $n = m$, and consider two sequences $a_0, a_1, \cdots, a_{m+1}$, and $b_0, b_1, \cdots, b_{m+1}$. If $a_0 = 0$ then $b_0 = 0$, $b_1 = a_1 + r_0a_0 = a_1$ the sequences $a_1, a_2, \cdots, a_{m+1}$ and $b_1, b_2, \cdots, b_{m+1}$ have precisely the same numbers of variations in sign as our original two sequences and are related sequences for $n = m$. Our conclusion then follows from the hypothesis of our induction. If some other $a_j = 0$ then $b_j = a_j + r_{j-1}b_{j-1} = r_{j-1}b_{j-1}$ has the same sign as b_{j-1} and we may delete a_j from our first sequence and b_j from our second sequence without changing the number of variations of sign in either case. Moreover the new sequences are related in the prescribed fashion since $b_{j+1} = a_{j+1} + r_jb_j = a_{j+1} + (r_{j-1}r_j)b_{j-1}$. Hence our result follows again from the hypothesis of our induction.

There remains only the case where no one of the numbers $a_0, a_1, \cdots, a_{m+1}$ is zero. We note the trivial relations

$$(5) \qquad V(b_0, b_1, \cdots, b_m) \leqq V(a_0, a_1, \cdots, a_m) \leqq V(a_0, a_1, \cdots, a_{m+1}).$$

If $b_{m+1}=0$ then the hypothesis $V(b_0, b_1, \cdots, b_m)<(a_0, a_1, \cdots, a_m)$ implies that $V(b_0, b_1, \cdots, b_{m+1})<V(a_0, a_1, \cdots, a_{m+1})$. However if $V(b_0, b_1, \cdots, b_m) = V(a_0, a_1, \cdots, a_m)$ the hypothesis of our induction implies that $b_m \neq 0$ and has the same sign as a_m by Lemma 2. But then $b_{m+1}=a_{m+1}+r_m b_m=0$ only if a_{m+1} and b_m have opposite signs. Then a_{m+1} and a_m will have opposite signs, the sequence $a_0, a_1, \cdots, a_{m+1}$ has one more variation in sign than the sequence $a_0, a_1, \cdots, a_m$ and one more than $b_0, b_1, \cdots, b_{m+1}$ as desired. Finally let $b_{m+1}\neq 0$. From (4) if $V(b_0, b_1, \cdots, b_m)<V(a_0, a_1, \cdots, a_m)$ then we have our desired inequality. The only possibility for it not to hold is indeed when $V(b_0, b_1, \cdots, b_m) = V(a_0, b_1, \cdots, a_m)$, when a_m and a_{m+1} have the same signs, and when b_m and b_{m+1} have opposite signs. But by Lemma 2 a_m and b_m have the same signs and if $b_m b_{m+1}=b_m^2 r_m+a_{m+1}b_m$ is negative so is $a_{m+1}b_m$. Then $a_{m+1}a_m$ is negative. This proves the lemma.

3. Polynomials with real coefficients. We shall assume, as is usual, the analytic result stating that if $f(x)$ is a polynomial with real coefficients and $a<b$ then there is an odd number or an even number of real roots in the interval $a<x<b$ according as $f(a)\cdot f(b)$ is negative or positive. We then have

LEMMA 4. *Let $a_0, \cdots, a_n$ be real numbers such that $a_0 a_n \neq 0$, and $f(x)=a_0x^n + \cdots + a_n$. Then $f(x)$ has either an odd or an even number of positive roots according as a_0a_n is negative or positive.*

For the sign of a_0a_n is the same as that of $c_n=a_0^{-2}(a_0a_n)=a_0^{-1}a_n$. But c_n is the constant term of $\phi(x)=a_0^{-1}f(x)=x^n+c_1x^{n-1}+ \cdots +c_n$. This polynomial has the same roots as $f(x)$ and $\phi(0)=c_n$, $\phi(h)>0$ for $h>g$ where g may actually be taken to be any number greater than all the numbers $1+|c_j|$. Then the positive roots of $f(x)$ lie in the interval $0<x<h$ and, by the result assumed, there is an odd number of such roots if $c_n>0$, an even number if $c_n<0$.

We now arrive quickly at

DESCARTES' RULE OF SIGNS. *Let $f(x)=a_0x^n+a_1x^{n-1}+ \cdots +a_n$ have real coefficients and t be the number of positive roots of $f(x)=0$. Then the difference*

$$V(a_0, a_1, \cdots, a_n) - t,$$

is a non-negative even integer.

We may clearly assume that a_0 and a_n are not zero. If ρ is a positive real root of $f(x)$ we may write $f(x)=(x-\rho)\phi(x)$ where $\phi(x)=b_0x^{n-1}+ \cdots +b_{n-1}$, the b_i are computed as in (3) with the $r_j=\rho$, $b_n=0$. By Lemma 3 we have $V(b_0, \cdots, b_{n-1}) \leqq V(a_0, a_1, \cdots, a_n)-1$. After t such steps we obtain $f(x)=(x-\rho_1)\cdots(x-\rho_t)\psi(x)$, where $\psi(x)=c_0x^{n-t}+c^{n-t-1}+ \cdots +c_{n-t}$, $0\leqq V(c_0, c_1, \cdots, c_{n-t}) \leqq V(a_0, a_1, \cdots, a_n)-t$. Hence $V(a_0, a_1, \cdots, a_n) \geqq t$ as desired. The evenness of their difference follows from the fact that the criteria for evenness of t and $V(a_0, a_1, \cdots, a_n)$ in Lemmas 4 and 1 are the same.

DISTRIBUTION OF POINTS IN n-SPACE

L. M. BLUMENTHAL AND B. E. GILLAM, University of Missouri

1. Introduction. Numerous writers have been concerned with the relation satisfied by the $(n+1)(n+2)/2$ mutual (positive) distances of $n+2$ pairwise distinct points of a euclidean n-space E_n. Known to Lagrange for the case of five points p_1, p_2, p_3, p_4, p_5 in E_3, this relation was established anew by Cayley in his first published paper and exhibited in determinant form

$$D(p_1, p_2, p_3, p_4, p_5) = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & p_1p_2^2 & \cdots & p_1p_5^2 \\ 1 & p_2p_1^2 & 0 & \cdots & p_2p_5^2 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & p_5p_1^2 & p_5p_2^2 & \cdots & 0 \end{vmatrix} = 0,$$

(where $p_ip_j = p_jp_i$ denotes the distance of the points p_i, p_j) together with the relations $D(p_1, p_2, p_3) = 0$, $D(p_1, p_2, p_3, p_4) = 0$ satisfied by the mutual distances of three and four points in a line and plane, respectively.*

If $V(p_0, p_1, \cdots, p_k)$ denotes the volume of the euclidean simplex $\{p_0, p_1, \cdots, p_k\}$, Cayley's method is readily extended to yield

$$V^2(p_0, p_1, \cdots, p_k) = \frac{(-1)^{k+1}}{2^k(k!)^2} D(p_0, p_1, \cdots, p_k);$$

and hence if $n+2$ points $p_0, p_1, \cdots, p_{n+1}$ are in E_n, their determinant $D(p_0, p_1, \cdots, p_{n+1})$ clearly vanishes since the volume of the *degenerate* simplex $\{p_0, p_1, \cdots, p_{n+1}\}$ is evidently zero. This formula gives an additional important relation which is not mentioned by some of the early writers; for as $V^2(p_0, p_1, \cdots, p_{n+1})$ is non-negative, it follows that the determinant D formed for *any euclidean m-tuple either vanishes or has the sign of* $(-1)^m$. Since these facts are basic in Menger's characterization of the euclidean metric among all semimetrics (the structure of the distance matrix (p_ip_j) of a semimetric m-tuple which is congruently imbeddable in a euclidean space being described in Menger's development in terms of the signs of the determinants D formed for subsets of the m-tuple) we are prompted to refer to D as the Cayley-Menger determinant.†

It is noted that any *bordered* principal minor of $D(p_0, p_1, \cdots, p_{n+1})$ is itself a Cayley-Menger determinant $D(p_{i_0}, p_{i_1}, \cdots, p_{i_k})$ and hence (for $p_0, p_1, \cdots, p_{n+1}$ a euclidean $(n+2)$-tuple) either vanishes or has the sign of $(-1)^{k+1}$. Further, it is well-known that the *unbordered* principal minor $C(p_0, p_1, \cdots, p_{n+1}) = |p_ip_j^2|$,

* A. Cayley, On a theorem in the geometry of position, Cambridge Mathematical Journal, 2, 1841, pp. 267–271; Collected Papers, vol. 1, pp. 1–4. See also J. J. Sylvester, On Staudt's theorems concerning the contents of polygons and polyhedrons, Collected Works, pp. 382–391.

† See Chapter III of L. M. Blumenthal, Distance Geometries, University of Missouri Studies, vol. 13, 1938.

$(i, j=0, 1, \cdots, n+1)$, has for a euclidean $(n+2)$-tuple the sign of $(-1)^{n+1}$ or vanishes, with the latter case holding if and only if $p_0, p_1, \cdots, p_{n+1}$ are on a hyperspherical surface $S_{n-1,r}$ (which may, in particular, be a hyperplane E_{n-1}). Hence the classical theory interprets geometrically the signs of the cofactors of all elements in the *principal diagonal* of the Cayley-Menger determinant of $n+2$ points in E_n.

The purpose of this note is to complete the theory by exhibiting the geometrical significance of the signs of the cofactors of the remaining elements of the determinant $D(p_0, p_1, \cdots, p_{n+1})$, where, in order not to unduly complicate the discussion, we suppose this euclidean $(n+2)$-tuple not to lie in a hyperplane E_{n-1}. The results, obtained without the use of coördinates, are believed to be new. They lead to a complete description of the distribution of $n+2$ points in E_n in terms of the mutual distances of the points. We offer, further, an application of the results, and the statement of analogous theorems proved for $(n+2)$-tuples in hyperbolic and spherical n-dimensional spaces.

2. Cofactors of bordering elements. We have to examine cofactors of the elements 1 in the bordering row and column of D (since we are concerned with non-principal cofactors) and take the cofactor of the element in the first row and last column as typical. Denote this cofactor by $\gamma(p_0, \cdots, p_n; p_{n+1})$.

THEOREM 2.1. *Let $\gamma(p_0, \cdots, p_n; p_{n+1})$ be different from zero. Then* (1) *the points $p_0, p_1, \cdots, p_n$ are in E_n, not in E_{n-1}, and* (2) *p_{n+1} is outside or inside the sphere circumscribing $p_0, p_1, \cdots, p_n$ if and only if the sign of $\gamma(p_0, \cdots, p_n; p_{n+1})$ is $(-1)^n$ or $(-1)^{n+1}$, respectively.*

Proof. Since $D(p_0, p_1, \cdots, p_{n+1})=0$, we have

$$(*)\quad D(p_0, p_1, \cdots, p_{n-1}, p_n)\cdot C(p_0, p_1, \cdots, p_n, p_{n+1}) - \gamma^2(p_0, \cdots, p_n; p_{n+1}) = 0,$$

and as $\gamma(p_0, p_1, \cdots, p_n; p_{n+1})\neq 0$ neither of the determinants in the first term of (*) vanishes. Hence the points $p_0, p_1, \cdots, p_{n-1}, p_n$ are not in E_{n-1} and the $n+2$ points are not on a hypersphere $S_{n-1,r}$.

Denote by o the center of the circumsphere of $p_0, p_1, \cdots, p_{n-1}, p_n$ and let r be its radius. Subtracting from the last row (column) of $D(p_0, p_1, \cdots, p_{n+1}, o)$ the product of the first row (column) by r^2 and applying the Cauchy development to the resulting determinant gives

$$\begin{aligned}(op_{n+1}^2 - r^2)^2 D(p_0, \cdots, p_n) + 2(op_{n+1}^2 - r^2)\gamma(p_0, \cdots, p_n; p_{n+1})\\ + C(p_0, \cdots, p_{n+1}) = 0.\end{aligned}$$

Substituting for $C(p_0, p_1, \cdots, p_{n+1})$ its value from (*), and recognizing that $op_{n+1}^2 - r^2$ is the power $P(p_{n+1})$ of p_{n+1} with respect to the circumsphere, we obtain

$$P(p_{n+1}) = -\gamma(p_0, \cdots, p_n; p_{n+1})/D(p_0, \cdots, p_n).$$

Since sgn $D(p_0, \cdots, p_n)=(-1)^{n+1}$, $P(p_{n+1})$ is positive or negative if and

only if sgn $\gamma(p_0, \cdots, p_n; p_{n+1})$ is $(-1)^n$ or $(-1)^{n+1}$, respectively, and the theorem is proved.

In case $\gamma(p_0, \cdots, p_n; p_{n+1})=0$ then either $D(p_0, \cdots, p_{n-1}, p_n)$ or $C(p_0, \cdots, p_n, p_{n+1})$ vanishes. In the latter case the $n+2$ points are on a sphere $S_{n-1,r}$, which might be degenerate (*i.e.*, a hyperplane E_{n-1}), while in the former case (with $C(p_0, \cdots, p_n, p_{n+1})\neq 0$) the $n+1$ points $p_0, p_1, \cdots, p_{n-1}, p_n$ are in an E_{n-1} which, according to our assumption on the $(n+2)$-tuple, does not contain p_{n+1}. These $n+1$ points may be examined in the light of the above and later results.

3. Cofactors of non-bordering elements. We are concerned here with cofactors $[p_ip_j^2]$ of elements $p_ip_j^2$ $(i\neq j)$, of D. Selecting the cofactor of $p_np_{n+1}^2$ as typical, we have

THEOREM 3.1. *Let $[p_np_{n+1}^2]$ be different from zero. Then* (1) *the points $p_0, p_1, \cdots, p_{n-1}$ determine an $(n-1)$-dimensional hyperplane and* (2) *p_n and p_{n+1} are on the same side or on opposite sides of this hyperplane if and only if the sign of $[p_np_{n+1}^2]$ be $(-1)^n$ or $(-1)^{n+1}$, respectively.*

Proof. Since $D(p_0, \cdots, p_n, p_{n+1})=0$ we have

$$(**)\qquad D(p_0, \cdots, p_n)\cdot D(p_0, \cdots, p_{n-1}, p_{n+1}) - [p_np_{n+1}^2]^2=0.$$

The non-vanishing of $[p_np_{n+1}^2]$ implies that neither of the determinants whose product forms the first term in (**) vanishes, and hence the points $p_0, p_1, \cdots, p_{n-1}$ are in E_{n-1}, not in E_{n-2}, and determine a hyperplane $\pi(p_0, \cdots, p_{n-1})$ which does not contain either p_n or p_{n+1}.

Replacing p_{n+1} in (**) by any point x of E_n, it is seen that $[p_nx^2]$ vanishes if and only if x is a point of π and hence sgn $[p_np^2]=$sgn $[p_nq^2]$ *for any two points p, q on the same side of π.*

If, now, p_n and p_{n+1} are on the same side of π then sgn $[p_np_{n+1}^2]=$sgn $[p_np_n^2]$, and since $[p_np_n^2]=-D(p_0, \cdots, p_n)$ we have in this case sgn $[p_np_{n+1}^2]=(-1)^n$.

On the other hand, suppose that p_n and p_{n+1} are on opposite sides of π, and denote the reflection of p_n in π by p_n^*. Then sgn $[p_np_{n+1}^2]=$sgn $[p_np_n^{*2}]$. Inspection of the vanishing determinant $D(p_0, \cdots, p_n, p_n^*)$ gives

$$(***)\qquad [p_np_n^{*2}] = -\left\{D(p_0, \cdots, p_{n-1}, p_n) + (p_np_n^{*2})\cdot D(p_0, \cdots, p_{n-1})\right\},$$

while from (**) we have

$$[p_np_n^{*2}]^2 = D^2(p_0, \cdots, p_{n-1}, p_n).$$

It follows that $[p_np_n^{*2}]=D(p_0, \cdots, p_{n-1}, p_n)$ and hence sgn $[p_np_n^{*2}]$ $=(-1)^{n+1}$. Thus for p_n and p_{n+1} on opposite sides of π sgn $[p_np_{n+1}^2]=(-1)^{n+1}$.

To establish the converse it suffices to show that if sgn $[p_np_{n+1}^2]=(-1)^n$ then p_n, p_{n+1} are on the same side of π. This is immediate, for then sgn $[p_np_{n+1}^2]$ $\neq$sgn $[p_np_n^{*2}]$ and hence p_n^*, p_{n+1} are not on the same side of π. Since none

of the points p_n, p_{n+1}, p_n^* lies in π, it follows that p_n and p_{n+1} are on the same side of π, and the theorem is proved.

COROLLARY. *The point p_{n+1} is inside the simplex with vertices $p_0, p_1, \cdots, p_n$ if and only if each cofactor $[p_i p_{n+1}^2]$, $(i=0, 1, \cdots, n)$, in $D(p_0, \cdots, p_{n+1})$ has the sign of $(-1)^n$.*

Theorems 2.1 and 3.1 may be combined to characterize the regions bounded by $S_{n-1,r}$ and the "faces" of an inscribed simplex.

4. An application. In an article dealing with pseudo-planar quintuples it was found necessary to establish the fact that if four points are in a plane at least one of them is inside or on the circumcircle of the other three.* As an immediate application of Theorem 2.1 we have

THEOREM 4.1. *If $p_0, p_1, \cdots, p_n, p_{n+1}$ are $n+2$ points of E_n at least one of these points is inside or on an $(n-1)$-dimensional spherical surface $S_{n-1,r}$ passing through the others.*

Proof. If the contrary be supposed then each of the points $p_0, p_1, \cdots, p_{n+1}$ is outside any $S_{n-1,r}$ containing the remaining $n+1$ points. It follows from Theorem 2.1 that each of the cofactors $\gamma(p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{n+1}; p_i)$, $(i=0, 1, \cdots, n+1)$, has the sign of $(-1)^n$ whenever it does not vanish. Since

$$\sum_{i=0}^{n+1} \gamma(p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{n+1}; p_i) = D(p_0, \cdots, p_{n+1}) = 0,$$

all of these cofactors must vanish. But this implies that either $C(p_0, \cdots, p_{n+1}) = 0$ (and hence the $n+2$ points are on an $S_{n-1,r}$) or $D(p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{n+1}) = 0$, $(i=0, 1, \cdots, n+1)$, and the $n+2$ points are in an E_{n-1} (a degenerate $S_{n-1,r}$). This contradiction yields the theorem.

Closely connected with this is the following theorem (well-known at least for the case $n=3$);†

THEOREM 4.2. *Let $p_0, p_1, \cdots, p_{n+1}$ be $n+2$ non-cospherical points of E_n, with no $(n+1)$-tuple in an E_{n-1}. Then the sum of the reciprocals of the power of each point with respect to the sphere circumscribing the remaining $n+1$ points is zero.*

We propose to give a new proof of this theorem, utilizing the expression for the power of a point developed in Theorem 2.1.‡

Since the $n+2$ points are non-cospherical, $C(p_0, \cdots, p_{n+1})$ is not zero, and since no $n+1$ of the points are in an E_{n-1}, each $(n+1)$-tuple has a non-vanishing determinant D. It follows that each of the cofactors

* P. M. Pepper, Concerning pseudo-planar quintuples, Reports of a Mathematical Colloquium, No. 2, 1940, p. 28–32.

† See, for example J. L. Coolidge, Treatise on the Circle and Sphere, p. 254. The reader is there asked to prove the theorem for 5 points in E_3 by applying the Darboux-Frobenius Identity.

‡ It is worth remarking that for planar quadruples a stronger theorem is valid, for there does not exist a plane set p_0, p_1, p_2, p_3 with p_3, p_2, p_1 inside the circumcircles of p_0, p_1, p_2; p_0, p_1, p_3; p_0, p_2, p_3, respectively.

$\gamma(p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{n+1}; p_i)$, $(i=0, 1, \cdots, n+1)$, in $D(p_0, \cdots, p_n, p_{n+1})$ is different from zero, and that the power $P(p_i)$ with respect to the sphere circumscribing $p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{n+1}$ is, from Theorem 2.1, given by

$$P(p_i) = -\gamma(p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{n+1}; p_i)/D(p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{n+1}).$$

Since

$$D(p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{n+1})\cdot C(p_0, \cdots, p_{n+1}) - \gamma^2(p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{n+1}; p_i) = 0,$$

we have

$$\begin{aligned} \sum_{i=0}^{n+1} 1/P(p_i) &= [-1/C(p_0, \cdots, p_{n+1})]\cdot \sum_{i=0}^{n+1} \gamma(p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{n+1}; p_i), \\ &= -D(p_0, \cdots, p_n, p_{n+1})/C(p_0, \cdots, p_{n+1}), \\ &= 0, \end{aligned}$$

and the theorem is proved.

5. Analogous theorems for spherical and hyperbolic spaces. If $p_0, p_1 \cdots, , p_{n+1}$ are $n+2$ points of the n-dimensional spherical surface $S_{n,r}$ (the "surface" of a sphere of radius r in euclidean $(n+1)$-space, with geodesic (shorter arc) metric), then it is well-known that the determinant $\Delta(p_0, \cdots, p_{n+1}) = |\cos(p_ip_j/r)|$ $(i, j=0, 1, \cdots, n+1)$, vanishes, and has each principal minor non-negative. The non-vanishing of a principal minor $\Delta(p_{i_0}, p_{i_1}, \cdots, p_{i_k})$ signifies that the $k+1$ points $p_{i_0}, \cdots, p_{i_k}$ form an independent set (*i.e.*, they do not lie in a great hypersphere $S_{k-1,r}$. Concerning the geometrical significance of the signs of the non-principal minors, we have established the following theorem:

THEOREM 5.1. *Let $p_0, p_1, \cdots, p_{n+1}$ be $n+2$ points of $S_{n,r}$ with the cofactor $[\cos(p_np_{n+1}/r)]$ of the element $\cos(p_np_{n+1}/r)$ in $\Delta(p_0, \cdots, p_{n+1})$ not zero. Then* (1) *the points $p_0, p_1, \cdots, p_{n-1}$ determine an $(n-1)$-dimensional great hypersphere $S_{n-1,r}$ and* (2) *the points p_n and p_{n+1} lie on the same or on opposite sides of $S_{n-1,r}$ if and only if $[\cos(p_np_{n+1}/r)]$ be negative or positive, respectively.*

As an obvious corollary we have a metric characterization of the interior of a spherical simplex.

Denoting the n-dimensional hyperbolic space of space constant μ by $H_{n,\mu}$, we can establish

THEOREM 5.2. *Let $p_0, p_1, \cdots, p_{n+1}$ be $n+2$ points of $H_{n,\mu}$ with the cofactor $[\cosh(p_np_{n+1}/\mu)]$ of the element $\cosh(p_np_{n+1}/\mu)$ in the determinant $|\cosh(p_ip_j/\mu)|$,* $(i, j=0, 1, \cdots, n+1)$, *not zero. Then* (1) *the points $p_0, p_1, \cdots, p_{n-1}$ determine an $(n-1)$-dimensional hyperbolic hyperplane $H_{n-1,\mu}$ and* (2) *the points p_n, p_{n+1} are on the same or on opposite sides of $H_{n-1,\mu}$ if and only if $[\cosh(p_np_{n+1}/\mu)]$ has the sign of $(-1)^{n+1}$ or $(-1)^n$, respectively.*

These theorems furnish the basis for a metric characterization of euclidean, spherical, and hyperbolic simplices among all semimetric spaces.

THE CURTAIN ROD PROBLEM

W. A. BLANKINSHIP, University of Virginia

The following problem was proposed by Professor E. J. McShane: "Determine the shape that a rod of uniform cross-section, density, and elasticity will assume if forced to pass through the three non-collinear points, $(a, 0)$, $(-a, 0)$, and $(0, b)$."

This problem, as the name suggests, arose from trying to bend a curtain rod across an arched window. It is of interest in that it is one of the few calculus of variations problems arising from physical circumstances which has a solution in terms of well-known and tabulated functions. Note that the problem is phrased so as not to imply a fixed length of the rod, otherwise there would arise forces of compression which would considerably disturb the assumptions we are about to make.

The situation, then, is this. In whatever shape the rod happens to be, there will be a certain amount of potential energy stored in the rod due to its bending. Thus the rod will assume the shape for which the least amount of potential energy is stored, subject to the conditions that it still pass through the required points. It can be shown* that the potential energy stored in the rod when it assumes a given shape is proportional to the integral with respect to arc length of the square of the curvature, the constant of proportionality being determined by the moment of inertia of a cross-section and by the elasticity of the rod. Thus, in seeking to minimize a certain integral, we have a calculus of variations problem.

It can also be seen from physical considerations that the minimizing curve will be symmetric to the y-axis. Thus we need only consider the right-hand part of the curve, replacing the left-hand end condition by the condition that the tangent to the curve at $(0, b)$ be parallel to the x-axis.

At this point we could set up the problem in non-parametric form, assuming the equation of the curve to be given in the form, $y=y(x)$. However, this straightforward procedure leads to Euler equations which are rather difficult to handle. Let us then take arc length, s, as an independent variable, and replace the symbols x, y by y_1, y_2 respectively. The curve will then be given by equations of the form:

$$y_i = y_i(s) \qquad i = 1, 2; s_1 \leqq s \leqq s_2$$

with

$$(y_1')^2 + (y_2')^2 - 1 = 0.$$

The curvature of the curve will then be given by

$$K = y_1' y_2'' - y_1'' y_2',$$

and the integral of the square of this function is that which we wish to minimize. To treat this integrand we introduce the two new variables, y_3 and y_4, in the places of y_1' and y_2', resp. This substitution makes our integrand become

* See Lord Rayleigh's "Theory of Sound," vol. I, p. 255.

$$K^2 = (y_1' y_4' - y_2' y_3')^2$$

and introduces the differential equations.

$$y_1' - y_3 = 0 \quad \text{and} \quad y_2' - y_4 = 0$$

to be satisfied.

Thus we have a Lagrange problem. Although it appears that we are using a parametric formulation, we must remember that s is arc length and is to be regarded as independent variable. Moreover our differential equations are not homogeneous and would not retain validity under change of parameter. We shall assume that arc length is measured from the left-hand end point.

Summing up the problem, then, we have the following: We wish to minimize the integral,

$$\int_{s_1}^{s_2} (y_1' y_4' - y_2' y_3')^2 ds,$$

in the class of C^2 curves,

$$y_i = y_i(s) \qquad i = 1, 2, 3, 4;\ s_1 \leqq s \leqq s_2,$$

satisfying the differential equations,

$$\phi_1 \equiv (y_1')^2 + (y_2')^2 - 1 = 0,$$
$$\phi_2 \equiv y_1' - y_3 = 0,$$
$$\phi_3 \equiv y_2' - y_4 = 0,$$

and the end conditions,

$$s_1 = y_1(s_1) = y_2(s_1) - b = y_3(s_1) - 1 = y_4(s_1) = 0,$$
$$y_1(s_2) - a = y_2(s_2) = 0.$$

Then the Lagrange multiplier rule states that there exist functions, $\lambda_0(s) = \text{const.} \geqq 0$, $\lambda_1(s)$, $\lambda_2(s)$, and $\lambda_3(s)$, not all zero, such that if we define

$$F(y, y', \lambda) \equiv \lambda_0 K^2 + \lambda_1 \phi_1 + \lambda_2 \phi_2 + \lambda_3 \phi_3,$$

then on the minimizing set, $y_i^0(s)$, the function F satisfies the Euler equations

$$\frac{d}{ds} F_{y_i'} - F_{y_i} = 0, \qquad i = 1, 2, 3, 4$$

$$\frac{d}{ds} (F - y_i' F_{y_i'}) - F_s = 0,$$

and the transversality conditions,

(1) $$F_{y_3'}(s_2) = 0$$

(2) $$F_{y_4'}(s_2) = 0$$

(3) $$F(s_2) - y_i' F_{y_i'}(s_2) = 0.$$

Hereafter we shall omit the superscript from y_i^0, and this shall be understood to be the minimizing curve.

Since F_{y_1}, F_{y_2}, and F_s are all zero, the first two and the last of the Euler equations are immediately integrable, and our complete set is:

$$(4)\qquad 2\lambda_0 K y_4' + 2\lambda_1 y_1' + \lambda_2 = 2c_1,$$

$$(5)\qquad -2\lambda_0 K y_3' + 2\lambda_1 y_2' + \lambda_3 = 2c_2,$$

$$(6)\qquad \frac{d}{ds}(-2\lambda_0 K y_2') = -\lambda_2,$$

$$(7)\qquad \frac{d}{ds}(2\lambda_0 K y_1') = -\lambda_3,$$

$$(8)\qquad F - y_i' F_{y_i'} = \text{const.}$$

Now we note that if λ_0 is zero, so are λ_2 and λ_3, in which case the solution to the Euler equations is a straight line, which is impossible unless $(0, b)$ is the origin. Thus we may assume that λ_0 is 1. Substituting the values for λ_2 and λ_3 from (6) and (7) into (4) and (5), we get

$$(9)\qquad 2Ky_4' + \lambda_1 y_1' + K' y_2' = c_1,$$

$$(10)\qquad -2Ky_3' + \lambda_1 y_2' - K' y_1' = c_2.$$

Multiplying (9) and (10) by y_2' and y_1' respectively, and subtracting, and making use of the equations,

$$\phi_1 = \phi_2 = \phi_3 = 0,$$

and also

$$\tfrac{1}{2}\phi_1' = y_1' y_3' + y_2' y_4' = 0,$$

we obtain

$$(11)\qquad K' = c_1 y_2' - c_2 y_1',$$

which integrates to

$$(12)\qquad K = c_1 y_2 - c_2 y_1 + d.$$

Now from (3) and (8) we have

$$F - y_i' F_{y^i} \equiv 0$$

or

$$(13)\qquad 4K^2 + 2\lambda_1(y_1'^2 + y_2'^2) + \lambda_2 y_1' + \lambda_3 y_2' - F = 0.$$

But from (4) and (5)

$$2\lambda_1(y_1'^2 + y_2'^2) + \lambda_2 y_1' + \lambda_3 y_2' = 2c_1 y_1' + 2c_2 y_2' - 2K^2,$$

which with (13) and $\phi_1 = \phi_2 = \phi_3 = 0$ gives

$$(14)\qquad K^2 + 2c_1 y_1' + 2c_2 y_2' = 0.$$

From (11) and (14) we can eliminate y_1' and y_2' and we get the differential equation:

$$K^4 + 4K'^2 = 4(c_1^2 + c_2^2) \equiv \alpha^2,$$

which may be written

$$\tfrac{1}{2}ds = \frac{dK}{\sqrt{\alpha^2 - K^4}}.$$

This equation can be integrated and we get

$$\tfrac{1}{2}(s - \sigma)\sqrt{2\alpha} = cn^{-1}(K/\sqrt{\alpha}, \tfrac{1}{2}\sqrt{2})$$

or

$$K = \sqrt{\alpha}\, cn(\sqrt{\tfrac{1}{2}\alpha}(s - \sigma), \tfrac{1}{2}\sqrt{2})$$

where σ is a constant of integration.

We also need to evaluate

$$2\psi(s) \equiv \int^s - K^2 ds = - 2\sqrt{2\alpha}\, E(\sqrt{\tfrac{1}{2}\alpha}(s - \sigma), \tfrac{1}{2}\sqrt{2}) + \alpha(s - \sigma),$$

where E is defined by

$$E(u, k) \equiv \int_0^u [1 - k^2 sn^2(u, k)]du.$$

Thus equations (12) and (14) become

$$c_1 y_2 - c_2 y_1 + d = K(s) \tag{15}$$

$$c_1 y_1 + c_2 y_2 + e = \psi(s). \tag{16}$$

These two equations may now be solved for y_1 and y_2 explicity in terms of s, and moreover all the functions involved are well-tabulated. The transversality conditions, (1) and (2), are

$$- 2K(s_2)y_2'(s_2) = 0 \quad \text{and} \quad 2K(s_2)y_1'(s_2) = 0,$$

which, with

$$y_1'^2 + y_2'^2 = 1,$$

yields $K(s_2) = 0$.

Thus the transversality condition, $K(s_2) = 0$, and the end conditions,

$$- K^2(0) = 2c_1 \qquad \text{from (14)}$$

$$K(0) = bc_1 + d,\ K(s_2) = - ac_2 + d \qquad \text{from (12)}$$

$$\psi(s_2) = ac_1 + e, \quad \psi(0) = bc_2 + e \qquad \text{from (16)}$$

together with the equations (15) and (16) constitute a complete solution of the problem

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The department of Discussions and Notes is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

SQUARE ROOTS FROM A TABLE OF COSINES

W. S. H. CRAWFORD, University of Minnesota

The three real roots of the reduced cubic equation

$$y^3 + py + q = 0 \qquad (p^3/27 + q^2/4 < 0)$$

are usually found* by making the transformation $y = nz$ and then comparing the resulting equation

$$z^3 + \frac{p}{n^2} z + \frac{q}{n^3} = 0$$

with the trigonometric identity

$$\cos^3 A - \tfrac{3}{4} \cos A - \frac{\cos 3A}{4} = 0.$$

This same scheme of relating an algebraic equation to a trigonometric identity, when applied to the quadratic equation $x^2 = k$, gives a method for finding the square root of a positive real number by means of a little mental arithmetic and a table of cosines. We let $x = ny$, where n is a positive number, and compare $y^2 = k/n^2$ with the trigonometric identity

$$\cos^2 A = (1 + \cos 2A)/2.$$

We evidently wish to take

$$k/n^2 = (1 + \cos 2A)/2,$$

which gives

(1) $$\cos 2A = 2k/n^2 - 1.$$

We will then have $y = \cos A$, or

(2) $$x = n \cos A.$$

In order for an angle A to exist such that (1) holds it is necessary and sufficient that $2k/n^2 - 1 \leqq 1$, which gives

(3) $$n \geqq \sqrt{k}.$$

* For example, see Dickson, First Course in the Theory of Equations, p. 49.

This is the only restriction on n; in the case of the cubic, n is uniquely determined. However, a large value of n will cause a correspondingly large error in the determination of x from (2), and so it is best to use the smallest convenient value of n satisfying (3). The computations are simplest when n is taken to be a power of 10. Some sample computations, to four places, are given below.

k	n	$2k/n^2-1$	$2A$	A	$n\cos A$
8739	100	.7478	41°36′	20°48′	93.48
.3661	1	−.2678	105°32′	52°46′	.6051
1.21	10	−.9758	167°22′	83°41′	1.100

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Allegheny College, Meadville, Pa.

Mathematics Club of Washington Square College, New York University

The club held regular meetings throughout the year on the average of one every two weeks. The lectures on the following topics were all well attended: *Mathematics for defense* by Professor Frederick W. John, *Non-euclidean geometry* by Dr. Morris Kline, *Geometrical transformations* or *Pushing points around* by Mr. David Gans, *Insurance* by Mr. Howard Wahlert, *Vector analysis* by Mr. John Rohrbaugh, *Actuarial mathematics* by Professor Tilley, *Mathematical orientation with respect to scientific methodology* by Professor Thomas Jenkins, *Non-dimensional variables* by Mr. Melvin Lax. Four socials were held during the year, including the annual boat ride. At one of these the winners of the Freshman Mathematics Contest, selected from a field of over fifty entrants, were awarded their prizes—a $25 defense bond to the winner, Mr. William Sollfrey, and copies of *What is mathematics* by Courant and Robbins to Miss Marion King, Mr. Joseph Allison, and Mr. Harvey Goldstein, the runners up. In the William Lowell Putnam contest, the New York University team composed of Mr. Melvin Lax, Mr. Henry Shenker, and Mr. Harold Lewis received honorable mention. The club publication, *Math X*, was completely sold out in an edition of 400 copies. Officers for 1941–42 and for 1942–43 were as follows. President, Mr. Melvin Lax (41–42), Mr. Marvin Forray (42–43); Vice-President, Miss Norma Ornstein (41–42), Miss Alice Press (42–43); Social Secretary, Miss Gladys Fleishman (41–42), Miss Eileen Press (42–43); Corresponding Secretary, Mr. Julius Ozick (41–42), Miss Marian Lipschutz (42–43); Treasurer, Miss Helen Sigel (41–42), Mr. Stanley Braun (42–43); Editor *Math X*, Mr. Frank Grace (41–42), Mr. Harold Lewis (42–43); Associate Editor, Mr. Maxwell Levy (41–42), Mr. Hal Cooper (42–43); Head Coach, Mr. Levy (41–42), Mr. Lew Fite (42–43).

Mathematics Society, Brooklyn College

In the fall term of 1941 talks were presented on *Probability* and on *Geometry*. The titles were *The introduction to probability* by Leonard Greenstone, *The introduction to probability, continued,* by Bernice Schwartz, *Some paradoxes of probability* by Dr. James Singer, *A Euclidean finite geometry* by Mr. Bernard Greenspan, *Regular polygons, and regular and reguloid solids* by Professor H. F. MacNeish. The semi-annual integration contest of the club was conducted in December, with Julius Vogel as individual high scorer (16 out of 18 correct) and with a winning team from Professor Merle Bishop's class consisting of Leon Baker, Shirley Bloom, Bernice Hyman, Arthur Howick,

and Sylvia Teich. The annual Christmas party on December 23 concluded the program for the first semester. Officers were: President, Frank Bausch; Vice-President, Jay Weinstein; Secretary, Aida Kalish; Treasurer, Peter Chiarulli; Social director, Phyllis Cohen; Publicity Director, Arthur Zeichner. In the Spring term of 1942, lectures were given on *Mathematics in defense* by Professor H. F. MacNeish, *Vector fields* by Dr. James Singer, and *Graphical solution of the imaginary roots of the cubic equation* by Dr. Jack Wolfe. Officers for the Spring term were: President, Peter Chiarulli; Vice-President, Harvey Casson; Secretary, Frances Brand; Social Director, Bernice Schwartz; Publicity Director, Albert Blank.

Kappa Mu Epsilon, State Teachers College, Wayne, Nebraska

Following the organization in September and a *Kappa Mu Epsilon* party in October, seven program meetings were held once a month at which the following topics were presented: *Mathematics used in surveying* by Don Strahan, *Mathematical geography* by Earl Prousse, *Mathematics in defense* by Bob Dale, *History of the slide rule* by Richard Hedglin, *The dozen numbering system* by Margie Morgan (a particularly successful talk after which the audience worked problems using the new system), *Mathematics of gunnery* by LeRoy Thomsen, and *Mathematics as used by the Navy* by Ensign Jim Ahern. The last speaker is a past president of our Nebraska Alpha Chapter, just returned from his study at Annapolis, and his talk provoked plenty of questions from the boys. The program of the year closed in May with a joint banquet with *Lambda Delta Lambda.* Officers for the year were: President Leibnitz, Margie Morgan; Vice-President Archimedes, Don Strahan; Secretary Galileo, Homer Scace; Treasurer Einstein, Russell Vlaanderen; Reporter Gauss, Marjorie Johnson; Historian Pascal, Ruth Lundberg; Faculty Sponsor and Corresponding Secretary, E. Marie Hove.

Mathematics Club, Oberlin College

The program for the year included regular meetings with one or two speakers each, a Christmas party, and the annual banquet. Topics were presented as follows: *Transformations by paper folding* by Roselyn Siegel, *The Chicago meetings* by Professor Mary E. Sinclair, *Revolving numbers* by Dora Sherman, *Groups, a mathematical merry-go-round* by William Fishback, *Algebra of the ninth century* by Doris Miller, *Complex numbers* by Daniel Cowgill, *Spherical trigonometry and applications* by Bolton Strauch, *Taking a line apart* by Dr. R. W. Wagner, *Hyperbolic functions and applications* by Richard Hayden, *Triangles with sixty degree angles* by Arthur Oshlag, *Infinity* by Elizabeth Miller, *Quaternions* by Robert Kelner, *Finite geometries* by Allen Strehler, *Dimensional analysis* by Edgar Everhart, *Line coordinates* by Frederick Grannis, *Things that go up must come down* [ballistics] by Professor L. W. Taylor. Officers for the club were: President, Allen Strehler; Vice-President, Roselyn Siegel; Treasurer, Richard Hayden; Secretary, Margaret Sigler; Faculty Adviser, Professor Marie M. Johnson.

Junior Mathematics Club, The Iowa State College

The *Junior Mathematics Club* at the Iowa State College meets two or three times per quarter. Its aim is to promote interest in mathematics. A means to this end is finding interesting speakers (usually undergraduate) to discuss some mathematical topic at the freshman and sophomore level. Mailing cards and posters are used to announce the meetings, emphasizing a striking feature of each meeting. More than sixty students attended an informal draft party at the first meeting. They were assigned in three groups to different classrooms, and subjected in turn to three short mathematical tests entitled *K. P. duty, Maneuvers,* and *Medical department.* A colonial lighting arrangement of candles prevailed during the examinations. For faithful performance of duty each participant was granted an honorable discharge from the meeting, entitling the bearer to a fair share of refreshments. Prizes were awarded to the winners. The titles for the other meetings included *Magic squares, Primitive counting, abacus, Napier's bones, Statistical meeting, Ciphering match, Readings, the workers A-B-C, Musical meeting, Speedy computation,* and *Computation short-cuts.* The attendance varied between forty and eighty. The program committee consisted of the

following undergraduates: Misses Lucille Neff, Frances Wilson, Eleanor Hoeflin and Glee Barth, and the faculty adviser, Mr. Fred Robertson.

Junior Mathematical Club, University of Chicago

During the past year the following papers were presented by students: *Cardinal numbers* by Jack Indritz, *A test for goodness of fit* by Charles Stein, *The algebraic solution of equations* by W. C. Carter, *Some applications of classes to probabilities* by George Platzman, *Singularities on plane curves* by Alice Turner, *Linear diophantine equations* by Daniel Zelinsky. To Miss Turner was awarded a copy of Coxeter's *Non-Euclidean geometry* as a prize for the best student paper in content and presentation. In addition to the student papers, the following talks on pure and applied mathematics were presented to the club: *Nerve-fiber network in steady state activity* by A. S. Householder, (department of biophysics, University of Chicago), *Some mathematical aspects of cryptography* by Professor A. A. Albert, *Some applications of mathematics to psychology* by R. K. Meister (department of psychology, University of Chicago), *A mathematical analysis of extra-tropical storms* by F. L. Martin (Institute of Meteorology, University of Chicago), and *Some applications of groups* by Professor A. C. Lunn. In addition to the teas given before the talks, the Club sponsored two parties during the autumn and winter quarters, and a picnic in the spring quarter. The officers for the year 1941–42 were: President, Roy Dubisch; Social Chairman, Florence Dorfman; Treasurer, Jack Indritz; Committee, Albert Cahn, Anne Lewis, William Massey, Richard Schafer, Alice Turner, Ernest Wilkins, Jr., and Daniel Zelinsky. The officers elected for 1942–43 were: President, Anne Lewis; Social Chairman, Janet McDonald; Treasurer, Daniel Zelinsky; Committee, A.R. Jacoby, Hyman Zimmerberg.

The Cooper Union Mathematics Club

The program for the year included two moving pictures on mathematical topics and three other papers: *Rate of change*, a moving picture, *Number theory* by Harold Grad, *Calculus of variations* by S. Roth, instructor, *Forced vibrations*, a moving picture, *Calculus of finite differences* by J. M. Diamond. A book was awarded to W. Pepper, winner in the freshman contest in mathematics sponsored by the Club, and a *K. & E.* slide rule was awarded to Bernard Levine for excellence in first year mathematics. A team composed of the Club members Murray Klamkin, Harold Grad and Kenneth Robinson received honorable mention in the Putnam Mathematics Contest of 1942, and with the additional members D. R. Frankl and J. M. Diamond placed second in the Metropolitan Intercollegiate Mathematics Contest of 1942. Officers for the year were: President, Murray Klamkin; Vice-President, Kenneth Robinson; Secretary, Harold Grad; Treasurer, Milton Rabinowitz; Faculty Adviser, J. K. L. MacDonald.

Echols Mathematics Club, University of Virginia

Topics presented at the four meetings were: *Continued fractions* by Mr. Paul White, *Some properties of continued fractions* by Mr. Frank Myers, *An application of continued fractions* by Dr. G. A. Hedlund, *Some applications of circular inversion* by Mr. Walter Gottschalk. The officers for the year were: President, T. P. Botts; Vice-President, Mariano Garcia; Secretary-Treasurer, R. R. Bernard.

Pi Mu Epsilon, University of Missouri

At the first of eight monthly meetings a series of odd mathematical problems were presented to the twelve members present. At later meetings talks were given as follows: *The tautochrone* by Mr. Francis Abel, *Cryptography* by Dr. G. E. Schweigert, *The application of trigonometric series to X-ray analysis* by Mr. Paul Sharrah of the Physics Department, *The theory of dimensionability* by Dr. Ray Dufford of the Physics Department, *The theory of scientific concepts* by Professor Lewis Hahn of the Philosophy Department. The two May meetings were devoted to the election of new members and to the annual banquet, at which Professor W. D. A. Westfall was toastmaster. After the banquet the following officers were elected for the year 1942–43: Director, Paul Lerret; Vice-

Director, William Becker; Recording Secretary, Frances Jackson; Corresponding Secretary, Eugene Jackson; Treasurer, W. R. Utz.

Mathematics Club, Butler University

At three of the meetings the following papers were presented: *The fourth dimension* by Joseph Berry, *Mathematics and art* by Jane Gibson, *A survey of the background and trends in plane geometry textbooks* (thesis) by Maribelle Foster. Other programs included a report by Robert Stump on the part mathematics plays in our nation's war effort, a showing of a group of slides on the history of arithmetic (at two meetings), and the annual Christmas party which featured mathematical games and puzzles and a gift exchange. Officers for the year 1941–42 were: President, Maribelle Foster; Vice-President, Robert Stump; Secretary, Jane Gibson; Treasurer, Joseph Berry; Faculty Adviser, Mrs. Juna L. Beal.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

Plane and Spherical Trigonometry. By P. R. Rider. New York, The Macmillan Company, 1942. 9+275 pages. $2.00.

In writing this text the author has accomplished his objective. "The primary purpose of this book is to present in a sound pedagogical manner the usual course in trigonometry as offered in colleges and technical schools." The book is written in flexible units so that it can be adapted to different types of courses. But more important than this is the clear and careful exposition, well selected illustrative examples, and excellent lists of exercises.

The first quarter of the book contains material based on the trigonometric functions of acute angles. Geometrical, physical, and mechanical applications of right triangles are numerous. A chapter on approximate numbers and computation is fortunately included. The introduction of logarithms and their use in the solution of right triangles concludes this part.

The definitions of the trigonometric functions of a general angle are followed by the reduction formulas and solution of oblique triangles. The proofs of the laws are well chosen and check formulas are prominent. A course in computational trigonometry could end here.

Both the radian and the mil measurement of angles are given. Then the line values and graphs of the trigonometric functions are presented. The inverse functions are discussed fully as to principal values and illustrated graphically. Trigonometric formulas, identities, and both single and simultaneous equations are treated comprehensively. A chapter on complex numbers completes the analytic part of plane trigonometry.

The fifty pages devoted to spherical trigonometry start with definitions and statements of propositions from solid geometry. Napier's rules are proved and then used in the proof of the laws of sines and cosines. Right spherical triangles and six cases of oblique triangles are considered with the usual formulas for solution and check. Applications refer to the terrestrial and celestial spheres.

The printing and set-up is attractive and the diagrams are well drawn. Answers are included for odd numbered exercises. A protractor and four-place tables of trigonometric functions are provided while the text may be obtained with the complete Macmillan Logarithmic and Trigonometric Tables bound in the back.

MARIE M. JOHNSON

Military and Naval Maps and Grids. By W. W. Flexner and G. L. Walker, New York, The Dryden Press. 1942, 96 pages. $1.00.

The properties of five maps are discussed, namely, Gnomonic (or Great Circle), the Mercator, the Lambert Conformal Conic with Two Standard Parallels, the Stereographic, and the American Polyconic. No mathematical equipment for the reader beyond plane trigonometry is presupposed, and only basic mathematical properties of the maps are treated. Emphasis is upon the practical use of these maps for solving naval and military problems including aerial and terrestrial great circle navigation, determination of position by radio bearings, radio location of aircraft, military grid systems, and aerial photography. An extensive and well selected set of problems is included. Altogether this little book presents a brief and elementary organization of material that is interesting in itself and extremely valuable for the prosecution of the war.

R. E. GILMAN

A Review of Arithmetic. By Z. L. Smith. The Institute of Military Studies, University of Chicago, 1942, 37 pages planographed. 15¢.

The topics covered comprise addition, subtraction, multiplication, division, common fractions, decimals, ratio and proportion together with exercises and answers. The exposition is brief but clear. Anyone who has forgotten arithmetic will find this inexpensive little book a useful aid in recapturing at least the elements.

R. E. GILMAN

Calculus. By A. L. Nelson, K. W. Folley, and W. M. Borgman. Boston, D. C. Heath and Company, 1942. 10+356 pages. $2.75.

The authors have written a carefully planned and usable textbook for the beginning calculus. The outstanding feature of the text is the early introduction of integration. This is done by devoting the first three chapters to functions which are powers of x and polynomials. The differentiation and integration of these functions together with the usual applications to geometry and physics are thus given to the student before he needs to develop technique for more

complicated functions. Consequently, the student may concentrate in the early part of the course on principles and applications, and the calculus presents an immediate usefulness to him.

Definitions and theorems are carefully worded and displayed so that they may be found easily. The list of theorems on limits in the first chapter is fuller than usual and will help the student with some of his difficulties. No attempt is made to prove theorems which are beyond the capacity of the average beginning student, but the need for proof is pointed out and references for such proofs are given. The illustrations are good and many illustrative examples are worked in the text. There is an ample supply of exercises, which casual sampling show to be well chosen. Useful notations such as the use of capital letters to denote the principal values of the inverse trigonometric and the inverse hyperbolic functions and the use of the logarithm of the absolute value of a function aid in preventing the student from making errors in applications.

The text contains the usual subjects allotted to a first course in the calculus. The authors claim that the material may be covered in one year by a class that meets four times a week. The book varies perhaps in having chapters on curve tracing, hyperbolic functions, and differential equations. The chapter on series is quite full, the elementary theory of partial differentiation is given, and the chapter on differential equations gives the most common elementary forms of the ordinary differential equation together with their applications. The book closes with tables of integrals, natural logarithms, exponential and hyperbolic functions, and the trigonometric functions for radian measure.

MARIE J. WEISS

Basic Mathematics. By W. W. Hart. Boston, D. C. Heath and Company, 1942. 6+456 pages. $1.52.

In his preface the author introduces *Basic Mathematics* as a survey course in secondary mathematics for either classroom use or self-instruction. He suggests that the second half of the book is designed as a refresher course, but the reviewer is inclined to think that the entire book is useful for that purpose alone. The book is definitely condensed. It treats arithmetic, plane and solid geometry, algebra (through the binomial theorem), logarithms, and trigonometry of the right triangle all in the space of 450 pages. The condensation is, in general, well done. The wording is quite precise; important statements are italicized; and only material relevant to the main line of development is included. It is for these very reasons that the reviewer finds the book suitable only for review work. There is not sufficient drill on varied interpretations of the ideas introduced to form the basis of a thorough classroom course, and the style is too concise for the book to serve as a basis for self-instruction by a student who has not covered the material before. However, the book should prove valuable as a refresher course and reference book on secondary mathematics.

One interesting feature is the author's selection of exercises. He makes a definite attempt to show the application of the mathematics involved to prac-

tical problems which have come into prominence in connection with the war effort. Problems deal with the loaded weight of bombers rather than the number of eggs in a basket, with mensuration of airplane wings and machine parts rather than rooms and tables, with triangulation to obtain artillery ranges rather than to find the height of monuments. It is impossible to escape the impression that quite a number of these examples serve not so much to increase the value of the book as to popularize it by giving the impression that it is "up to the minute," but undoubtedly the book gains by the author's choice of exercises.

Several specific faults appear, the most glaring of which are as follows. Logarithms are introduced on page 177, and some 20 pages are devoted to their theory and applications. However, the laws of exponents are not discussed thoroughly until page 415. A second unfortunate point occurs on page 398. A recapitulation is given there in the form of an outline of the types of numbers. From this outline it appears that all real numbers are algebraic and that real numbers are not complex numbers.

The reviewer would recommend chapters VII and VIII on synthetic plane and solid geometry as the soundest portion of the book. The author proves congruence of triangles by superposition, but he is sufficiently broadminded to note that the congruence theorems can well be introduced as postulates and that this is frequently done. The section on the geometry of the sphere is to be commended particularly. Sufficient preparation in geometry is given there to enable the student to handle spherical trigonometry once he has mastered plane trigonometry. Unfortunately, the author cannot resist the temptation to discuss the circumference and area of circles without giving an adequate discussion of the notion of a limit.

M. E. Munroe

General Trade Mathematics. By E. P. Van Leuven. New York, McGraw-Hill Book Co., Inc., 1942. 10+575 pages.

This book presents in a clear, concise fashion the arithmetic and mathematics required by the carpenter, machinist, and electrician. It is directed to students in technical high schools but is well adapted to home study by workmen with little training. The academic contents, consisting of arithmetic, elementary geometry, and simple equations, are applied to a wide variety of problems which include construction costs, gear speeds, screw threads, and indexing of milling machines. The last chapters, dealing with mechanics and electricity, are on the level of a high-school physics course.

Within the chapters, material is divided into easily understood unit topics, in which the order of presentation is: definitions, rule, examples, exercises, problems. Only at times is a derivation of the rule indicated. About half of the chapters are concluded by review problems. It is mentioned that the type of work determines the accuracy that may be expected, but the number of decimal places rather than the number of significant figures is given as a criterion. The

use of bold face type and boxing of examples contribute in making the book useful as a reference as well as a text.

E. L. CROW

Spherical Trigonometry with Naval and Military Applications. By L. M. Kells, W. F. Kerns, and J. R. Bland. New York and London, McGraw-Hill Book Company, Inc., 1942. 13+163 pages. $1.50. (With tables $2.40.)

This timely text for a pre-induction course in mathematics consists of a reprint of certain sections from the authors' *Plane and Spherical Trigonometry*, second edition (1940)* augmented by further applications to navigation and related topics. The extent of the text is indicated by the chapter headings: I. Logarithms; II. Review of plane trigonometry; III. The right spherical triangle; IV. Elementary applications; V. The oblique spherical triangle; VI. Applications; and the appendices: A. The mil; B. The range finder; C. Stereographic projections; D. Vectors and relative movement problems. Chapters I, III, V, VI and the first three appendices in the main coincide with subdivisions of the same name in the *Plane and Spherical Trigonometry* and have been adequately treated in reviews of that book.

In regard to the new material, Chapter III includes an additional section treating an oblique spherical triangle by solving two right spherical triangles. Chapter IV defines the navigational terms used and discusses the length of an arc on a parallel of latitude, plane sailing, middle latitude sailing, and the Mercator chart. Chapter VI has been augmented by a mention of the Ageton Method and the Dreisonstok method of solving oblique spherical triangles, a treatment of the Summer line of position method for making a fix, and an essay on aerial navigation. Appendix D applies vector addition to certain problems in the alignment and relative movement of a group of ships.

The text is well written. Students' difficulties are anticipated and warnings of possible pitfalls are given. An important feature is an emphasis on the form of computation. A more than ample collection of exercises of varying difficulty, many well illustrated, is included.

G. L. WALKER

Essentials of Astronomy. By J. C. Duncan, New York and London, Harper and Brothers, 1942. 179 pages. $1.85.

The ideal way to teach astronomy would be to have each student observe and record the daily, monthly, seasonal and yearly phenomena from early childhood to his college years. If such a student could also have the advantage of making his observations at several places, from a far northern to a far southern latitude, so much the better.

* Reviewed in this MONTHLY, vol. 47, 1940, pp. 703–704 by J. M. Feld. The first edition of the same book was also reviewed in this MONTHLY, vol. 43, 1936, p. 39 by J. M. Feld.

A college teacher who had such a class with such a background, would have an ideal opportunity to develop a course which should give a clear and full appreciation of the scientific method, as applied to the observational data at the disposal of his students.

In the brief book before us Professor Duncan has provided such a background of observational material in chapter one. In the second chapter the observational material is interpreted.

So skillfully woven into this material are the definitions and principles that the student is led to their use as the natural and easy way of expressing the new ideas which unfold in his mind.

Chapter three gives a brief account of the discovery of the laws of motion and the law of gravitation and the importance of these laws in explaining all of the phenomena of motion in the solar system.

Chapter four deals with radiation and introduces the student to some of the instruments which enable us to interpret the messages which the radiations from the heavenly bodies bring. The elementary principles of the telescope and the spectroscope are especially well presented.

In the brief space allotted to discussing gravitation and radiation in chapters three and four, the author gives a fine presentation of the part they play in producing the phenomena we observe in the universe.

The remaining three chapters give a brief account of the various celestial bodies which modern instruments have brought within our field of observation. The book concludes with a presentation of our present conception of the structure of the universe.

The appendices contain fourteen tables of useful data and information, followed by the very excellent star maps, which have been such a valuable feature of Professor Duncan's "Astronomy."

The publishers are to be commended for their part in producing this book, with its fine reproductions of astronomical photographs and diagrams. The few typographical errors which it contains will undoubtedly be corrected in a second reprint.

The relative brevity of the section dealing with astrophysical subjects, together with the excellent star maps, should recommend the book as a desirable text for courses in pre-aviation astronomy.

The writer was charmed with this book when it was first read, and the charm was further enhanced after using it as a text in his introductory course in Astronomy given in the summer session at Cornell during July and August 1942.

S. L. BOOTHROYD

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 561. *Proposed by Howard Eves, Syracuse University*

Given two triangles inscribed in the same circle and such that the Simson lines with respect to one triangle of the vertices of the other are concurrent (as in E 535), prove that the Simson lines with respect to the two triangles of a point on the common circumcircle are parallel.

E 562. *Proposed by V. Thébault, San Sebastián, Spain*

Find a number of the form $ab0cd$ whose square contains the nine digits 1, 2, 3, 4, 5, 6, 7, 8, 9.

E 563. *Proposed by N. A. Court, University of Oklahoma*

Let A', B', C', D' be the antipodes of the circumcenter O of a tetrahedron $ABCD$ on the respective spheres $OBCD$, $OCDA$, $ODAB$, $OABC$. Show that the lines AA', BB', CC', DD' are generators of a quadric. May this quadric be a cone?

E 564. *Proposed by Ivan Niven, Purdue University*

Let a, b, and n be any positive integers such that n divides $a^n - b^n$. Prove that n divides $(a^n - b^n)/(a-b)$.

E 565. *Proposed by H. W. Becker, Mare Island Submarine Base*

Show that the number of ways n men can be divided up in crews is N_n, in the notation of E 461 [1941, 701]. What is the number of ways if two particular men cannot stand the sight of each other, and must be kept in different crews? What is it if one must be segregated from each of m others?

SOLUTIONS

Exponential Limits

E 528 [1942, 404]. *Proposed by R. A. Rosenbaum, Reed College*

Prove and generalize the identity

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{k^2}\cdot e^{-k} = e^{-1/2}.$$

I. *Solution by A. M. Peiser, Cornell University.* This is easily proved by noting that

$$\log\left\{\left(1+\frac{1}{k}\right)^{k^2}\cdot e^{-k}\right\} = -k + k^2\left\{\frac{1}{k} - \frac{1}{2k^2} + o\left(\frac{1}{k^2}\right)\right\}$$
$$= -\tfrac{1}{2} + o(1)$$

as $k\to\infty$.

One immediate generalization is

$$\lim_{k\to\infty}\left(1+\frac{x}{k}\right)^{k^2}\cdot e^{-kx} = e^{-x^2/2}.$$

II. *Solution by Alexander Beckerman, New York City.* The generalization

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{k^n}\exp\left(\sum_{r=1}^{n-1}\frac{(-1)^r k^{n-r}}{r}\right) = \exp\left(\frac{(-1)^{n-1}}{n}\right)$$

may be proved by noting that

$$k^n\log\left(1+\frac{1}{k}\right)+\sum_{r=1}^{n-1}\frac{(-1)^r k^{n-r}}{r} = k^n\sum_{r=n}^{\infty}\frac{(-1)^{r-1}k^{-r}}{r}$$
$$= \frac{(-1)^{n-1}}{n} + o(1)$$

as $k\to\infty$.

Also solved by B. H. Bissinger, L. M. Kelly, H. D. Larsen, Melvin Lax, J. Rosenbaum, and P. D. Thomas. Lax combined the two generalizations, proving that

$$\lim_{k\to\infty}\left(1+\frac{z}{k}\right)^{k^n}\exp\left(\sum_{r=1}^{n}\frac{(-1)^r}{r}k^{n-r}z^r\right) = 1.$$

An In- and Circum-scriptible Hexagon

E 529 [1942, 404]. *Proposed by J. Rosenbaum, Bloomfield, Conn.*

Construct an irregular hexagon which shall be both inscriptible and circumscriptible.

Solution by the Proposer. CONSTRUCTION. On a circle take two diametrically opposite points A and D, and another point B, with the restriction that AB be unequal to the radius. Then locate a fourth point C, on the semicircle ABD, such that the sum of AB and DC is equal to the diameter. Finally, locate E and F, the images of C and B by reflection in AD. The hexagon $ABCDEF$ will fulfil the conditions of the problem.

Proof. The essential feature is that the bisectors of the angles ABC and BCD meet on AD, so that (on account of the symmetry of the figure) all the angle bisectors are concurrent, and the hexagon has an incircle.

NOTE. After this special (symmetrical) hexagon has been constructed, and its circumcircle and incircle drawn, then by means of Poncelet's Porism the most general such hexagon can be constructed. [See this MONTHLY, 1942, 364.]

The Six Radical Spheres

E 530 [1942, 404]. *Proposed by P. D. Thomas, Southeastern State College, Okla.*

Is there a sphere orthogonal to the six radical spheres determined by four given spheres whose centers are not coplanar? (The radical sphere of two spheres is the locus of a point whose two powers have zero sum.)

Solution by the Proposer. The answer is Yes. The intersection of the six radical planes of the four given spheres is their radical center, which, having equal powers with respect to the four spheres, is the center of a sphere, S, orthogonal to them. But the radical sphere of any two of the four given spheres is coaxal with those two. Also a sphere orthogonal to two spheres is orthogonal to every sphere of the coaxal pencil determined by them. Thus the sphere S is orthogonal to the six radical spheres as well as to the four original spheres. (See N. A. Court, *Modern Pure Solid Geometry*, 1935, pp. 179, 186, 201, 202.)

The Meigs Hall Problem

E 531 [1942, 475]. *Proposed by P. R. Hill, University of Georgia*

Suppose six students be standing an examination in a row of seats with an aisle at each end. If they finish in random order, what is the probability that a student will have to pass over one or more other students in order to reach an aisle?

I. *Solution by R. K. Allen, Montpelier, Vermont.* The students may finish the examination in 6! or 720 ways. In order that they may finish so that no student has to pass over any other student the first one to finish must be one of the two on the ends, which may be done in either of two ways. The second to finish must be one of the two then sitting on the ends, and so on. There are 2^5 or 32 ways in which no student will have to pass over some other student. The chance that some student will have to pass over some other student is accordingly

$$(720 - 32)/720 = 43/45.$$

II. *Solution by Howard Eves, Syracuse University.* Given n students, the probability that the first man finished will not have to pass over any of the others in gaining the aisle is obviously $2/n$. Hence the required probability for our problem is

$$1 - \frac{2}{6}\,\frac{2}{5}\,\frac{2}{4}\,\frac{2}{3} = \frac{43}{45}\cdot$$

Also solved by D. F. Barrow, H. W. Becker, and W. E. Buker.

Five Consecutive Digits Forming a Square

E 532 [1942, 475]. *Proposed by V. Thébault, San Sebastián, Spain*

Find a perfect square whose digits form one of the permutations of five consecutive digits.

Solution by W. E. Buker, Pittsburgh Public Schools. If N is a perfect square, then $N \equiv 0, 1, 4$, or $7 \pmod 9$. So N must be one of the permutations of either

$$01234 \text{ or } 34567.$$

Of the 120 permutations of 01234, we omit those whose first digit is 0, and those whose last digit is 0, 2, or 3. Of the 36 remaining possibilities, we note that all squares ending in 1 or 4 have an even number in the tens place. This eliminates twelve more, leaving 24 numbers to look up in a table. We find that

$$23104 \text{ and } 32041$$

are the squares of 152 and 179.

Of the permutations of 34567, we note that squares do not end with 3 or 7, nor with 5 unless preceded by 2. Thus we have 48 possibilities. But squares ending in 4 have an even number in the tens place, while those ending in 6 have an odd number in the tens place. There are again 24 numbers to look up, and it is found that none of these are squares.

Also solved by R. K. Allen, Howard Eves, R. V. Heath, Aida Kalish, and the proposer. Allen remarks that the only values of n less than 1000 such that the digits of n^2 are a permutation of consecutive integers are 18, 24, 66, 74, 152, and 179.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4075. *Proposed by N. A. Court, University of Oklahoma*

If the radical center of four spheres coincides with the Monge point of the tetrahedron (T) determined by their centers, the tetrahedron (S) formed by the four radical planes of the given spheres with their orthogonal sphere is orthological to the twin tetrahedron (T') of (T) (*i.e.*, the perpendiculars dropped from the vertices of (S) upon the corresponding faces of (T') are concurrent).

4076. *Proposed by Harold E. Gove, Major, Washington, D. C.*

Given the triangle $A_1A_2A_3$ show how to construct the triangle $B_1B_2B_3$ so that the triangles $B_jB_iA_k$ will be equilateral and exterior to $B_1B_2B_3$.

4077. *Proposed by V. Thébault, San Sebastián, Spain*

If four spheres, each passing through a corresponding vertex of a tetrahedron $ABCD$, intersect in pairs on the corresponding edge, the four spheres are concurrent in a point M (S. Roberts). Show that: (1) The points A', B', C', D' diametrically opposite to A, B, C, D on the corresponding spheres are in a plane (P) passing through M (R. Bouvaist). (2) The plane (P) is a Simson plane of the tetrahedron $A_1B_1C_1D_1$ formed by the planes parallel to the planes of BCD, CDA, DAB, ABC and passing respectively through A', B', C', D'.

SOLUTIONS

Twin Tetrahedrons

4024 [1942, 128]. *Proposed by N. A. Court, University of Oklahoma*

Given two twin tetrahedrons $(T)\equiv ABCD$, $(T')\equiv A'B'C'D'$ (see the proposer's *Modern Pure Solid Geometry*, p. 58, art. 191), consider the tetrahedron (A') formed by the face BCD of (T) and the three planes forming the trihedral angle A' of (T'); let (B'), (C'), (D') be the analogous tetrahedrons for the vertices B', C', D' of (T'). Show that the twelve-point spheres of the tetrahedrons (A'), (B'), (C'), (D') (*ibid.*, p. 251, art. 764) are tangent to the twelve-point sphere of (T).

Solution by the Proposer. The diagonal AA' of the common circumscribed parallelepiped of (T) and (T') is trisected internally by the face BCD of (T), say, in L, and the point L is the centroid of the triangle BCD.

On the other hand, the edges of the trihedral angle A' of (T') pass through the mid-points of the sides of the triangle BCD. Consequently, the two tetrahedrons (A'), (T) correspond to each other in the homothecy $(L, -2)$ having L for homothetic center and -2 for homothetic ratio.

Now the twelve-point spheres of the two tetrahedrons (A'), (T) correspond to each other in the homothecy $(L, -2)$, and both spheres pass through the point L, hence they touch each other at this point. Similarly for the tetrahedrons (B'), (C'), (D').

Note. The above proposition is an extension to space of the following proposition in the plane.

The parallels to the sides of a triangle ABC through the respectively opposite vertices form a triangle $A'B'C'$. Show that the nine-point circles of the triangles $A'BC$, $B'CA$, $C'AB$ are tangent to the nine-point circle of the triangle ABC (*Nouvelles Annales de Mathématiques*, 1865, p. 322, Q. 722).

THE MATHEMATICAL ASSOCIATION OF AMERICA

Because of the cancellation of the New York meeting of the Mathematical Association of America at the request of the Office of Defense Transportation, the business of the Board of Governors for 1942 was completed by mail vote. The outgoing Board voted

1. To elect B. W. Jones Associate Secretary for a term of five years.
2. To adopt the following motion:

   Members who are in active war service may, on their request, be exempt from dues, not receiving the MONTHLY but otherwise being members in full standing. This is particularly appropriate when they are not able to receive the MONTHLY or the official announcements.
3. To transfer $2,000 in Defense Bonds to General Endowment, this to be grouped with "Investments."
4. To adopt the following motion jointly with the American Mathematical Society Council:

   That a War Policy Committee of the A.M.S. and the M.A.A. be appointed jointly by the incoming presidents, and a sub-committee to handle at once the problem of allocation of teachers, the work of this sub-committee to be absorbed with that of the War Policy Committee; and that President-elect Stone act as representative of the two organizations in Washington until such time as the War Policy Committee will function.
5. To elect to membership the following thirty-eight persons on applications duly certified:

P. R. ANNEAR, M.S. (Case) Asst. Prof., Acting Head of Dept., Math. and Astr., Baldwin-Wallace Coll., Berea, Ohio

A. V. BAEZ, A.M. (Syracuse) Asst. Prof., Math. and Physics, Wagner Coll., Staten Island, N. Y.

C. W. BAILEY, Reporter, Cleveland Press, Cleveland, Ohio

HELEN P. BEARD, A.M. (Pennsylvania) Instr., Sophie Newcomb Coll., New Orleans, La.

F. P. BEER, Ph.D. (Geneva, Switzerland) Instr., Univ. of Kansas City, Kansas City, Mo.

R. L. BEINERT, A.B. (Hobart) Part-time Instr., Cornell Univ., Ithaca, N. Y.

S. G. BOURNE, B.S. (Rutgers) Jr. Instr., Johns Hopkins Univ., Baltimore, Md.

HERMAN BRANSON, Ph.D. (Cincinnati) Asst. Prof., Physics, Howard Univ., Washington, D. C.

R. C. BUCK, A.M. (Cincinnati) Jr. Prize Fellow, Harvard Soc. of Fellows, Cambridge, Mass.

W. B. CATON, Ph.D. (Yale) Instr., Illinois Inst. of Tech., Chicago, Ill.

LOUISE H. CHIN. Student, Univ. of California, San Francisco, Calif.

D. E. CHRISTIE, Ph.D. (Princeton) Instr., Physics, Bowdoin Coll., Brunswick, Me.

E. L. CROW, Ph.D. (Wisconsin) Math. Consultant, Research and Development Div., Bur. of Ordnance, Navy Dept., Washington, D. C.

BENJAMIN EPSTEIN, Ph.D. (Illinois) Physicist, Frankford Arsenal, Philadelphia, Pa.

H. W. EVES, A.M. (Harvard) Asst. Prof., Syracuse Univ., Syracuse, N. Y.

J. W. GIVENS, JR., Ph.D. (Princeton) Asst. Prof., Northwestern Univ., Evanston, Ill.

V. H. HAAG, A.M. (Duke Univ.) Instr., Hershey Jr. Coll., Hershey, Pa.

R. W. HAMMING, Ph.D. (Illinois) Instr., Univ. of Illinois, Urbana, Ill.

CECIL HASTINGS, JR., B.S. (Florida) Computer, Franklin Inst., New York, N. Y.

B. K. HOVEY, Ph.D. (Göttingen) Instr., Elec. Engineering, Univ. of Pittsburgh, Pittsburgh, Pa.

S. W. HOWELL, A.M. (South Dakota) Instr., Math. and Physics, Univ. of South Dakota, Vermillion, S. D.

MRS. LOUISE S. HUNTER, Ed.M. (Harvard) Asst. Prof., Virginia State Coll., Ettrick, Va.

IRVING KAPLANSKY, Ph.D. (Harvard) Instr., Harvard Univ., Cambridge, Mass.

V. O. MCBRIEN, Ph.D. (Catholic Univ.) Mathematician, U. S. Government, Washington, D. C.

LEONARD MCFADDEN, Ph.D. (Brown Univ.) Instr., Virginia Poly. Inst., Blacksburg, Va.

INGO MADDAUS, JR., Ph.D. (Michigan) Asst. Prof., Univ. of Oregon, Eugene, Ore.

N. A. PEEBLES. Statistical Clerk, Office of President, Atlantic Coast Line RR Co., Wilmington, N. C.

W. G. POLLARD, Ph.D. (Rice Inst.) Asso. Prof., Physics, Univ. of Tennessee, Knoxville, Tenn.

MRS. ANNIE N. ROWLAND, M.S. (Texas Tech.) Instr., Texas Tech. Coll., Lubbock, Texas

REV. B. M. RUSSELL, A.B. (St. Viator's) Head of Dept., Fournier Inst., Lemont, Ill.

W. G. SCOBERT, A.B. (U.C.L.A.) Grad. Asst., Univ. of Oregon, Eugene, Ore.

MAURICE SINGER, B.S. (Tulane) Grad. Asst., Illinois Inst. of Tech., Chicago, Ill.

H. W. STEINHAUS, Ph.D. (Göttingen) Chief, Research Div., Group Dept., Equitable Life Assur. Soc., New York, N. Y.

E. C. STEWART, B.A. (British Columbia) Teacher, Math. and Science, Wells Barkerville School, Wells, B. C., Canada

H. I. TREIBER, A.B. (Brooklyn Coll.) Signal Corps Inspector, Newark Signal Corps Inspection Zone. *Res.* Brooklyn, N. Y.

Mrs. LOIS K. TURNER, M.S. (Virginia Poly. Inst.) Instr., University High School, Univ. of Minnesota, Minneapolis, Minn.

W. R. UTZ, JR., A.M. (Missouri) Instr., Univ. of Missouri, Columbia, Mo.

J. W. WILEY, A.M. (Michigan) Head of Dept., Math. and Physics, Anderson Coll., Anderson, Ind.

W. D. CAIRNS, *Retiring Secretary-Treasurer*

The election of officers for 1943 by the membership of the Association and by the Board of Governors was conducted by means of mail ballots. The results of the elections were as follows:

President for a two-year term: W. D. Cairns, Oberlin College.

Second Vice-President for a two-year term: C. C. MacDuffee, Hunter College.

Governors at large for three-year terms: Saunders MacLane, Harvard University, and E. J. Moulton, Northwestern University.

On the recommendation of the Editor-in-Chief, L. R. Ford, the following associate editors of the MONTHLY have been elected for the year 1943:

L. M. Blumenthal	Marjorie J. Groves
N. B. Conkwright	M. R. Hestenes
H. S. M. Coxeter	B. W. Jones
W. M. Davis	R. E. Langer
Otto Dunkel	J. R. Musselman
B. F. Finkel	C. V. Newsom
J. S. Frame	Virgil Snyder
Orrin Frink, Jr.	Marie J. Weiss

W. B. CARVER, *Secretary-Treasurer*

Final Report of the Secretary-Treasurer as Treasurer, January 15, 1943

Receipts

Balance Dec. 20, 1941		$6,324.98
1941 indiv. dues	$ 347.65	
1941 instit. dues	21.00	
1941 subscriptions	26.08	
1942 indiv. dues	6,633.96	
1942 instit. dues	681.10	
1942 subscriptions	819.69	
Initiation fees	224.00	
Advertising	609.36	
Reprints	219.64	
Sale copies of MONTHLY	158.51	
Sale First Carus Mon.	12.50	
Sale Second Carus Mon.	12.50	
Sale Third Carus Mon.	20.00	
Sale Fourth Carus Mon.	8.75	
Sale Fifth Carus Mon.	47.18	
Sale Sixth Carus Mon.	474.88	
Sale Archibald's *Outline of Hist. of Math.*	123.71	
Sale library books and periodicals	575.16	
Sale Rhind Papyrus	104.60	
Annals subscriptions	7.50	
Duke Journal subscriptions	6.00	
Math. Reviews subscriptions	13.00	
Drury Coll. int. Hardy Fund	120.00	
Refund office expense	7.21	
Refund War Prep. reprints	13.84	
Repaid from Carus Fund	595.60	
Interest	1,189.80	
Payment from restricted Carus Fund	49.70	
Payment from restricted Chace Fund	2.20	
Transfer certificate of deposit	1,120.77	
Restricted funds transferred from Carus Fund	298.20	
Restricted funds transferred from Chace Fund	13.20	14,557.29
Total 1942 receipts to date		$20,882.27

Expenditures

Publisher's bills (Nov. '41–Oct. '42)		$ 5,955.09
Reprints		347.28
President's office		4.48
Editor-in-chief's office 1941		200.00
Editor-in-chief's office 1942		562.41
Expense Exec. and Finance Committees		233.85
Membership maintenance		150.00
War Preparedness Committee		39.96
Secretary-Treasurer's office		
Postage	$ 475.92	
Bond	11.26	
Office expense	155.35	
Express, tel., etc.	103.07	
Clerical work	2,863.08	
Printing	152.01	
Bank charge	28.19	3,788.88
Annals subvention		200.00
Duke Journal subvention		150.00
Math. Reviews subvention		500.00
Expense of sections		182.78
Expense acct. regional governors		169.84
Lehigh meeting		25.15
Poughkeepsie meeting		75.00
New York meeting and annual elec.		111.03
Paid *Annals* subscriptions		10.00
Forwarded *Annals* subscriptions		7.50
Forwarded *Duke Journal* subscriptions		6.00
Forwarded *Math. Reviews* subscriptions		13.00
Sust. memb. in Amer. Math. Society		100.00
Insurance back copies MONTHLY		4.55
Storage back copies MONTHLY		30.00
Paid back copies MONTHLY		177.50
Paid B. F. Finkel int. Hardy Fund		120.00
Refund subscriptions		5.85
Library expense		39.08
Moving Association office		193.37
Expense Ithaca office		44.95
War Policy Committee		113.44
Expense acct. Carus Committee		5.00
Chauvenet Prize Award		100.00
Balance purchase War Bond		200.09
Balance purchase Phelps Dodge Bond and accr. int.		26.15

RECEIPTS (continued)		EXPENDITURES (continued)	
		Transfer Peoples Bank savgs. acct. to Gen. Endowment	1,866.47
		Transfer Defense Bonds to Gen. Endowment	2,000.00
		From certificate of deposit for reinvestment	715.09
Total expenditures	18,473.79	Total expenditures	$18,473.79
Balance to end of 1942 business	$ 2,408.48	Checking account	1,172.47
Received on 1943 business	1,288.85	Oberlin Savgs. Bk. acct. restricted*	811.20
		Cleveland Trust Co. savgs. acct.	1,713.66
Book balance Jan. 15, 1943	$ 3,697.33	Bank balance Jan. 15, 1943	$ 3,697.33

EXHIBIT OF THE FUNDS OF THE ASSOCIATION

CARUS MONOGRAPH FUND

Balance December 20, 1941		$6,392.82
Receipts: Sales	$575.81	
Interest	180.78	756.59
		$7,149.41
Expense acct. Carus Committee		5.00
Balance January 15, 1943		$7,144.41

ARNOLD BUFFUM CHACE FUND

Balance December 20, 1941		$8,163.98
Receipts: Sale Papyrus	$104.60	
Interest	230.46	335.06
Balance January 15, 1943		$8,499.04

JACOB HOUCK MEMORIAL FUND

Balance December 20, 1941	$8,181.17
Interest	224.81
Balance January 15, 1943	$8,405.98

CHAUVENET PRIZE FUND

Balance December 20, 1941	$ 632.94
Interest	14.40
	$ 647.34
Paid on Prize Award December 1941	32.94
Balance January 15, 1943	$ 614.40

LIFE MEMBERSHIP FUND

Liability on earlier life memberships as of January 1, 1942	$ 900.17
To be transferred to current funds, surplus	75.87
Liability on life memberships as of January 1, 1943	$ 824.30

* This is the balance, not yet distributed, of $2,704.00 which was tied up at the time of bank closures in 1933. It is probable that some of the balance, but not all, will be repaid.

PERMANENT INVESTMENTS OF THE ASSOCIATION

	Par Value	Market Value Dec. 31, 1942
U. S. Savings Bonds	$ 1,275.00	. 1,468.00
U. S. Treasury 1% Notes Ser. A 1946	3,000.00	2,979.00
U. S. Treasury $2\frac{3}{4}$% Bond 1947	1,000.00	1,042.50
U. S. Government $1\frac{3}{4}$% Bonds 1948	2,000.00	2,012.00
U. S. Treasury 2% Bonds 1950	3,000.00	3,051.00
HOLC 3% Bonds Ser. A 1944–52	3,000.00	3,096.00
Phelps Dodge Corp. $3\frac{1}{2}$% conv. deb. 1952	1,000.00	1,047.50
U. S. Savings $2\frac{1}{2}$% Bonds Ser. G 1953	3,000.00	3,000.00
U. S. Savings $2\frac{1}{2}$% Bonds Ser. G 1954	8,200.00	8,200.00
Texas Power & Light Co. 5% First Mort. Bond 1956	1,000.00	1,075.00
Amer. Tel. & Tel. Co. 3% Bonds conv. 1956	2,000.00	2,140.00
Commonwealth Edison Co. $3\frac{1}{2}$% Bonds. conv. deb. 1958	2,000.00	2,180.00
N. Y. Steam Corp. $3\frac{1}{2}$% First Mort. Bond 1963	1,000.00	1,061.25
Montana Power Co. $3\frac{3}{4}$% First Ref. Mort. Bonds 1966	3,000.00	3,112.50
Gatineau Power Co. $3\frac{3}{4}$% First Mort. Bond Ser. A 1969	1,000.00	918.75
Penn. R. R. Co. $3\frac{3}{4}$% Genl. Mort. Bonds Ser. C 1970	2,000.00	1,775.00
Cols. & So. Ohio Elec. Co. $3\frac{1}{4}$% First Mort. Bonds 1970	2,000.00	2,160.00
Shawinigan W. & P. Co. $4\frac{1}{2}$% First Mort. Bonds 1970	2,000.00	2,010.00
C. & O. Ry. Co. $3\frac{1}{2}$% Ref. Mort. Bonds Ser. D 1996	3,000.00	3,060.00
Land Trust Certif., Hotel Cleveland Site	700.00	510.00
	$45,175.00	$45,898.50

Of the funds on hand indicated in the first division of this report, $1,279.32 belongs to the Carus Monograph Fund, $574.04 to the Chace Fund, $224.81 to the Houck Fund, $114.40 to the Chauvenet Fund, while $824.30 is held as a Life Membership Fund, representing the liability as of January 1, 1943, on life memberships paid for earlier.

When the accounts were closed by the retiring secretary-treasurer, there remained on the total business for 1942 the following items:

BILLS RECEIVABLE

1942 individual dues	$200.00
Advertising	70.00
Reprints	35.00
	$305.00

BILLS PAYABLE

Publisher's bills Nov., Dec. '42)	$1,250.00
Subsidy *Duke Journal*	50.00
Carus Monograph Fund	1,279.32
Chace Fund	574.04
Houck Fund	224.81
Chauvenet Fund	114.40
Life membership fund	824.30
	$4,316.87

These items are about as usual and the current income of the Association amply provides for their payment as they come due.

The retiring secretary-treasurer holds receipts from the incoming secretary-treasurer for the assets in the general treasury and from the Cleveland Trust Company for the securities listed in "Investments."

W. D. CAIRNS, *Retiring Secretary-Treasurer*

THE SEVENTEENTH ANNUAL MEETING OF THE PHILADELPHIA SECTION

The seventeenth annual meeting of the Philadelphia Section of the Mathematical Association of America was held at the University of Pennsylvania, Philadelphia, Pa., on Saturday, November 28, 1942. Professor C. O. Oakley, chairman of the Section, presided at the morning and afternoon sessions.

The attendance was about thirty-five, including the following twenty-six members of the Association: E. F. Allen, H. W. Brinkmann, P. A. Caris, Mary L. Constable, H. B. Curry, J. E. Davis, Arnold Dresden, V. V. Latshaw, Marguerite Lehr, F. L. Manning, A. E. Meder, Jr., Richard Morris, W. R. Murray, C. A. Nelson, C. O. Oakley, A. E. Pitcher, G. E. Raynor, C. J. Rees, I. J. Schoenberg, J. A. Shohat, L. L. Smail, G. L. Walker, A. D. Wallace, R. M. Walter, Anna Pell Wheeler, P. M. Whitman.

At the business meeting the following officers were elected for next year: Chairman, J. E. Davis, Drexel Institute of Technology; Secretary, P. M. Whitman, University of Pennsylvania. It was voted to hold the next meeting at the University of Pennsylvania, Philadelphia, Pa., on Saturday, November 27, 1943. The Section voted to cooperate in the formation of a local group of technical societies comparable to one existing in Chicago.

The following papers were presented:

1. "On a theorem of Jensen" by Professor I. J. Schoenberg, University of Pennsylvania.
2. "Exterior ballistics" by Professor G. E. Raynor, Lehigh University.
3. "On modern methods in the numerical solution of linear problems" by Dr. Hilda Geiringer, Bryn Mawr College, introduced by Professor Brinkmann.
4. "The Heaviside operational calculus" by Dr. H. B. Curry, Frankford Arsenal.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles.

1. The basis of Prof. Schoenberg's discussion was Jensen's theorem: If circles are described whose diameters are the segments joining pairs of conjugate imaginary roots of a real polynomial $f(z)$, then every non-real root of the derivative $f'(z)$ lies on or within those circles. The first published proof of this theorem is due to Walsh, Annals of Mathematics, 22 (1920) pp. 128–144. This theorem is now extended to a set of points in space, furnishing information about the possible location of points of equilibrium of a particle attracted by the given points according to the inverse distance law of attraction. For the special case of a cubic polynomial $f(z)$ of given roots, a ruler and compasses construction of the roots of $f'(z)$ is given which remains valid if all the roots of $f(z)$ are real (see Walsh, *loc. cit.*, pp. 142–144).

2. In the first part of the paper Professor Raynor gave a description of the force system acting on a spinning projectile. Then a set of simplified equations of motion was discussed and it was explained briefly how ballistic and firing tables are prepared.

3. In order to obtain the effective solution of a system of n nonhomogeneous linear equations $a_{ik}x_k+r_i=0$ with determinant different from zero, Dr. Geiringer outlined a general iteration scheme which contained all particular cases so far known. It reduced respectively to: a) "iteration by simultaneous displacements" and b) "iteration by successive displacements." Well known particular cases of the first procedure are the "ordinary" iteration or iterated transformation of the error-vector $z_i^{(r)}=x_i^{(r)}-x_i$ (x_i=solution), and the "method of steepest descent." In the simplest case of b) one computes successively values $x_i^{(r+1)}$ of the ith unknown by means of the ith equation where the values of the other unknowns are $x_1^{(r+i)}, x_2^{(r+1)}, \cdots, x_{i-1}^{(r+1)}, x_{i+1}^{(r)}, \cdots, x_n^{(r)}$ ($r=0, 1, 2, \cdots$; $x_i^{(0)}$ arbitrary). It is known that this method converges if the matrix (a_{ik}) is positive definite; also in this case any order in iterating the equations is admissible. The procedure is identical with Southwell's Relaxation Method. Besides, necessary and sufficient conditions for the convergence of b) can be indicated, and new simple sufficient conditions are deduced none of which demands symmetry of the matrix. On the other hand, if the matrix is symmetric and $a_{ii}>0$ then positive definiteness is not only sufficient but also necessary for convergence from an arbitrary starting point. (The latter is true for the "ordinary" iteration but there the condition is not sufficient.) In each of these cases the convergence may be accelerated by "group-iteration."

4. Dr. Curry presented a rigorous theory of the Heaviside operational calculus which was algebraic in character and made no use of infinite integrals. Naturally it included only the discrete aspects of the calculus, *i.e.* applications to ordinary differential equations.

P. M. WHITMAN, *Secretary*

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, N. Y.

Associate Professor E. F. Beckenbach of the University of Michigan has been appointed to an associate professorship at the University of Texas.

Associate Professor C. S. Carlson of St. Olaf College has been promoted to a full professorship.

E. W. Holt of Lawrence Academy, Groton, Massachusetts is now a lieutenant, U.S.N.R., and is teaching in the Naval Academy Preparatory School, Naval Operating Base at Norfolk, Virginia.

Associate Professor E. D. Jenkins is on leave from Eastern Kentucky State Teachers College and is serving with the Navy Department at Corpus Christi, Texas.

Sidney Kaplan has been appointed assistant economic statistician of the Industry and Facilities Branch of the War Production Board and is stationed at New York.

Dr. R. H. Moorman of Tennessee Polytechnic Institute has been promoted to an assistant professorship.

J. K. Reckzeh of West Virginia Institute of Technology is now an Ensign teaching Navigation at the Navy Ground School, Corpus Christi, Texas.

Dr. Arthur Rosenthal is now an assistant professor at the University of New Mexico. This is in correction of an item in the November issue of this MONTHLY.

Associate Professor J. B. Rosser and Assistant Professor R. J. Walker have been granted leaves of absence from Cornell University. The former is serving with the National Defense Research Committee in Washington and the latter as a mathematician at Aberdeen Proving Ground.

Dr. Andrew Sobczyk is on leave from Oregon State College and is located at the Radiation Laboratory of Massachusetts Institute of Technology.

Dr. Ellen C. Stokes, instructor in mathematics at the New York State College for Teachers at Albany, has been appointed dean of women.

Professor W. L. Williams has been appointed head of the department of mathematics at the University of South Carolina to succeed Professor J. B. Coleman who recently resigned.

Professor Roscoe Woods has been serving as the acting head of the mathematics department of the University of Iowa since last September. Professor H. L. Rietz, the former head of the department, has relinquished the position on account of ill health.

The following appointments to instructorships are announced:

University of South Carolina: Miss Flora Dinkines

University of Southern California: E. C. Rex, Veryl Throckmorton and Miss F. Marian Clarke (visiting instructors)

Associate Professor Emeritus C. H. Currier of Brown University died on January 5, 1943. He was a charter member of the Mathematical Association.

Professor E. L. Dodd of the University of Texas died on January 9, 1943. He was a charter member of the Mathematical Association.

Professor Emeritus J. H. Scarborough of Central Missouri State Teachers College died on November 25, 1942. He was a charter member of the Mathematical Association.

MEXICAN CONGRESS OF MATHEMATICS

The First National Congress of Mathematics was held in Saltillo, State of Coahuila, Mexico, November 1–7, 1942, in connection with the celebration of the seventy-fifth anniversary of the founding of the Ateneo Fuente, a college located in Saltillo. The Congress was in commemoration of the three hundredth anniversary of the birth of Sir Isaac Newton.

The officers were: President, Dr. Alfonso Nápoles Gándara, head of the Department of Mathematics in the National University and of the Institute of Mathematics, recently organized to stimulate research; Vice-President, Dr. Carlos Graef Fernández, head of the Department of Astrophysics in the National University; Secretary, Ing. Francisco José Alvarez. Officers of the Section of Pure Mathematics were: President, Dr. Carlos Graef Fernández; Secretary, Professor Francisco Zubieta Russi; Reporter, Professor Alberto Barajas Celis. Officers of the Section of Applied Mathematics were: President, Dr. Manuel Sandoval Vallarta; Vice-President, Professor Leopoldo Medina; Reporter, Dr. Nabor Carrillo.

Special guests were Dr. Blas Cabrera, former director of the Institute of Physics of Madrid, Dr. Manuel Sandoval Vallarta of the Massachusetts Institute of Technology, and Dr. P. R. Rider of Washington University, exchange professor at the National University.

One hundred ten persons were registered, but very many more than this were in actual attendance at the sessions. Among the speakers at the general sessions were the Governor of the State of Coahuila, the Secretary of Public Education, the Rector of the National University, and a representative of the United States Ambassador. There were some twelve papers presented at these general sessions, most of them dealing with relations between mathematics and other sciences. About an equal number of expository papers on various mathematical topics were given. In addition, thirty-nine papers of a research nature were presented at the sectional meetings.

There were many pleasant social features, including lunches, dinners, dances, and a one-day trip to Monterrey.

One important outcome of the Congress was the decision to establish a Mexican Mathematical Society, which will sponsor a journal. The Permanent Commission of the Congress, which will have charge of the organization of the Society, is composed of Dr. Alfonso Nápoles Gándara, President; Ing. Francisco José Alvarez, Secretary; Dr. Manuel Sandoval Vallarta; Dr. Carlos Graef Fernández; Professor Alberto Barajas Celis.

P. R. RIDER

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

FROM SELECTIVE SERVICE

Occupational Bulletins No. 10 and No. 23 were amended December 14, 1942. The amendment of particular significance to mathematicians pertains to the occupational classification of graduate students. The statement follows.

"A graduate or postgraduate student undertaking further studies in these scientific and specialized fields following completion of his normal undergraduate course of study may be considered for occupational classification if, in addition to pursuing further studies, he is also acting as a graduate assistant in a recognized university or college. A graduate assistant is a student who in addition to pursuing such further studies is engaged in one of the following:

(a) In scientific research certified by a recognized federal agency as related to the war effort; or

(b) in classroom or laboratory instruction for not less than twelve hours per week."

FROM THE WAR DEPARTMENT

A memorandum entitled, "Call to Active Duty of Students Enlisted in the Enlisted Reserve Corps, Unassigned Group," was issued by the War Department upon January 27, 1943. This memorandum indicates that after the current academic period many students in the Enlisted Reserve Corps and in other categories will be detailed for instruction under the Army Specialized Training Program. Thus, the following statement from the same bulletin will be of interest to mathematicians.

"The following fields of training listed in Selective Service Bulletin No. 10 (amended December 14, 1942) as 'Critical Occupations' are accepted under the Army Specialized Training Program as approved technical engineering courses:

(1) Aeronautical engineers
(2) Automotive engineers
(3) Chemical engineers
(4) Civil engineers
(5) Electrical engineers
(6) Heating, ventilating, refrigerating, and air-conditioning engineers
(7) Mechanical engineers
(8) Radio engineers
(9) Chemists
(10) Mathematicians
(11) Meteorologists
(12) Physicists, including astronomers
(13) Psychologists"

THE PROBLEM OF SECURING TEACHERS OF COLLEGIATE MATHEMATICS FOR WARTIME NEEDS

The eighteen-year-old draft law and the inauguration of the proposed Army and Navy programs for colleges and universities are creating serious personnel problems for departments of mathematics. In an attempt to alleviate this situation, partially at least, the American Mathematical Society and the Mathematical Association of America are establishing in the Philadelphia office of the Society a bureau to provide information regarding available teachers of collegiate mathematics. The cooperation of all mathematicians is needed in order to make this plan a success.

The government has indicated that approximately 250,000 trainees will be sent to a selected group of about 300 colleges and universities. It is estimated that the full-time services of at least 2,500 teachers of mathematics will be required for this program. In addition, institutions will necessarily provide a normal program for men under 18, for women, who are electing mathematics in increasing numbers, and for students rejected, for any reason, by Selective Service. Institutions will also continue special contracts already in existence such as meteorology and aviation pre-flight schools.

To provide for all this instruction there are available about 3,000 persons now functioning as teachers of mathematics in four-year colleges. To ensure the proper handling of this instruction, readjustments of and additions to teaching staffs of many institutions will be necessary. For some institutions, additional personnel might be secured from the following sources:

(1) Members of departments of mathematics in institutions which do not assume government programs;

(2) Persons now teaching in non-critical fields who have had sufficient training in mathematics to make them valuable in this field. These persons should be recruited for the teaching of mathematics rather than being allowed to assume non-academic employment. This should probably be the main source of supply.

It is in order to promote an orderly solution of these problems that the American Mathematical Society and the Mathematical Association of America have established the bureau of information regarding teachers of collegiate mathematics. The purpose is two-fold:

(1) To collect information concerning persons who are trained in mathematics and who are or will be available for teaching during the emergency at institutions other than their own. It is requested that the names and present addresses of persons who are available for employment as teachers of collegiate mathematics be reported to the address given below. Additional information concerning qualifications will be secured directly from the mathematicians so reported.

(2) To furnish to chairmen of departments of mathematics who need additional personnel the information obtained through (1). As the employment offered will, in most cases, be temporary, an attempt will be made to furnish the chairman with an adequate list, mainly of persons living in his vicinity. *The*

purpose of the bureau is to furnish only information and not recommendations. The department chairman will, of course, investigate on his own responsibility the qualifications of the available teachers.

The country's awakened consciousness of the need for mathematics as a preparation for service in the armed forces imposes on the mathematical profession the grave responsibility to meet this need by providing the best possible teaching. The Society and Association aim to make a definite contribution to the war effort by supplying information regarding available teachers of mathematics. To do this, the full cooperation of all mathematicians is needed. It is impossible to predict what portion of the demand will be met in this manner. It is likely that teachers will become available less rapidly than the demands for additional personnel are made, but the cooperation of mathematicians will enable the bureau to function effectively.

All correspondence in this connection should be addressed to:

Committee on Available Teachers of Collegiate Mathematics
110 Bennett Hall
University of Pennsylvania
Philadelphia, Pa.

March, 1943

Committee:
W. D. CAIRNS
ARNOLD DRESDEN
J. R. KLINE

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-sixth Summer Meeting, New Brunswick, N. J., September 11-13, 1943.

Twenty-seventh Annual Meeting, Cleveland, Ohio.

The following is a list of the Sections of the Association, with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., April 17, 1943
ILLINOIS, Notre Dame, Ind., April 9–10, 1943
INDIANA, Notre Dame, April 9–10, 1943
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, Brooklyn, N. Y., May 8, 1943
MICHIGAN, Notre Dame, Ind., April 9–10, 1943
MINNESOTA
MISSOURI, Kansas City
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, Jan. 29, 1944
OHIO, Columbus, April 1, 1943
OKLAHOMA
PHILADELPHIA, Philadelphia, Nov. 27, 1943
ROCKY MOUNTAIN, Denver, April, 1943
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 13, 1943
SOUTHWESTERN
TEXAS, Lubbock, April, 1943
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May 7, 1943

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A GEOMETRICAL BACKGROUND FOR DE SITTER'S WORLD

H. S. M. COXETER, University of Toronto

"He's in prison now, being punished: and the trial doesn't even begin till next Wednesday: and of course the crime comes last of all." The White Queen [1].

1. Introduction. Through a natural misunderstanding, students who have just begun to learn about non-euclidean geometry are apt to say that the elliptic plane can be developed on a sphere, and the hyperbolic plane on a hyperboloid. "Of one sheet, or two sheets?" you ask, and then the statement is admitted to have been confused. But let us see whether there is after all a germ of truth in it.

We must remember that the classification of quadrics as ellipsoids, paraboloids (of two kinds), and hyperboloids (of two kinds) is not peculiar to euclidean geometry, but belongs properly to the wider field of *affine* geometry. All the theorems of affine geometry hold not only in euclidean geometry but also in Minkowskian (or pseudo-euclidean) geometry, where every point is the vertex of a real null cone (or isotropic cone, or cone of minimal lines). A hyperboloid whose asymptotic cone is of this kind has the property that all its points are equidistant from its center; it may thus legitimately be called a sphere. In other words, a euclidean sphere is an ellipsoid, but a Minkowskian sphere is a hyperboloid—of one or two sheets according as the radius is space-like or time-like. The intrinsic geometry of either sheet of the sphere of two sheets is indeed hyperbolic. The sphere of one sheet is quite different, but equally worthy of study; that is where de Sitter's World comes in.

These ideas are not so familiar as they deserve to be, though of course they are well known to experts. It is hoped that the present note will help to clarify the situation by emphasizing the precise relationship of the various geometries. We shall be chiefly concerned with spaces of two or three dimensions; but the extension to more than three dimensions presents no difficulty, and will be seen to have an interesting application in relativistic cosmology.

2. Affine geometry. Affine geometry deals with points, lines, and planes, having the familiar properties of serial order, continuity, and parallelism. (See [2], pp. 161, 162, 186; or [8], pp. 51, 52, 56, 57.) With the aid of parallelograms we can construct the midpoint of any given segment, and describe a *translation*. Since there is a unique line through a given point parallel to a given line, and a unique plane through a given point parallel to a given plane, we can say that every line belongs to a definite *bundle of parallels*, and every plane to a definite *pencil of parallels*. Such bundles and pencils are said to be improper, in contrast to the proper bundle of lines through a point, and the proper pencil of planes through a line. By treating bundles and pencils as points and lines, with the natural interpretation for the relation of incidence, we obtain *real projective geometry*. ([2], pp. 165–174.) Since the *proper* bundles and pencils correspond in an obvious manner to the ordinary points and lines, we can think of the projective space as an extension of the affine space. The extra entities, namely the

bundles and pencils of parallels, are then called *points at infinity* and *lines at infinity*. In the projective geometry these are the points and lines of a plane: *the plane at infinity*.

Conversely, we could begin with real projective geometry, in which all planes are alike, and agree to single out one of them as the plane at infinity, letting any two lines or planes be considered parallel if they intersect on that plane. In this sense, affine space is derived from real projective space by *removing* one plane (with all its points and lines).

Quadrics can be defined projectively, as by von Staudt and Enriques ([2], p. 68) or Seydewitz and Reye ([2], p. 219). We can then distinguish the *ruled* quadrics from the non-ruled or *oval* quadrics. In affine geometry, these can be further classified according to the nature of their intersection with the plane at infinity:

Oval Quadric	*Ruled Quadric*	*Section at Infinity*
Ellipsoid	—	Nothing
Elliptic paraboloid	—	One point
—	Hyperbolic paraboloid	Two lines
Hyperboloid of two sheets	Hyperboloid of one sheet	Conic

The *center* of a quadric is the pole of the plane at infinity. Thus the center of a paraboloid is its point of contact with the plane at infinity. In the other cases the center is an ordinary point, and we can alternatively classify the quadrics according to the manner in which they meet their diameters and diametral planes. Every diameter of an ellipsoid meets the surface twice. But a hyperboloid has some diameters that do not meet it at all. In the case of a hyperboloid of one sheet, any diametral plane meets every generator; so the section of the surface by such a plane is a conic. On the other hand, a hyperboloid of two sheets has some diametral planes that do not meet it at all; the two sheets lie on opposite sides of any such plane.

As an alternative to this synthetic treatment, we can build up a theory of vectors—added, subtracted, or multiplied by scalars—and use any three independent vectors to describe a system of affine *coordinates*. (These are illustrated by oblique Cartesian coordinates with different units of measurement along the three axes.) Then every equation of the first degree represents a plane, and certain equations of the second degree represent quadrics. The extension to projective space is achieved by the familiar device of rendering the coordinates homogeneous, *i.e.*, replacing x_0, x_1, x_2 by x_0/x_3, x_1/x_3, x_2/x_3. Then the plane at infinity is $x_3=0$.

3. The absolute polarity. We have seen that the plane at infinity of affine space is a projective plane. We know that the geometry of a bundle in euclidean space (with its lines and planes interpreted as points and lines) is elliptic, which means that the plane at infinity of euclidean space is an elliptic plane. In the Appendix to [6], Robb shows that the geometry of a bundle in Minkowskian space (or space-time) is hyperbolic, which means that the plane at infinity of

Minkowskian space (of $2+1$ dimensions) is a hyperbolic plane. It is perhaps worth while to outline the converse process, and show how an elliptic or hyperbolic metric in the plane at infinity of affine space induces a euclidean or Minkowskian metric in the whole space. For this purpose, we single out an *absolute polarity* (elliptic or hyperbolic) in the plane at infinity, and consider as *perpendicular* any line and plane whose elements at infinity correspond in this polarity.

Given two points B and C, the *sphere* on BC as diameter may be defined as the locus of the point of intersection of a variable line through B with the perpendicular plane through C (or vice versa). Thus a sphere is a quadric, according to Seydewitz's construction. To see what kind of quadric this is, we investigate the possibility of its containing a point at infinity, say P. In such a case the line BP would be perpendicular to a certain plane through CP, and P would lie on its own polar. This can happen only if the polarity is hyperbolic; the locus of such *self-conjugate* points is then a conic, as defined by von Staudt. Thus the sphere is an ellipsoid or a hyperboloid according as the absolute polarity is elliptic or hyperbolic. (A nice justification for the accepted terminology!)

The properties of congruence may be developed by defining the reflection in a plane α as the harmonic homology with axial plane α and center the point at infinity on lines perpendicular to α. A displacement is then defined as the product of any even number of reflections; in particular, the product of reflections in two parallel planes is seen to be a translation. Given two points O and C, the sphere through C with center O is the locus of the image of C by reflection in all the planes through O. (This clearly agrees with the above construction, if O is the mid point of BC.) It follows that all the radii of a sphere are congruent.

If we now refer to a standard list of axioms of congruence for euclidean geometry (*e.g.* [2], p. 180), we find that they are all valid, *provided the absolute polarity is elliptic.* Consider, for instance, the first axiom:

If A and B are distinct points, then on any ray C/E there is just one point D such that $AB\equiv CD$.

To locate the desired point D, we apply the translation that takes A to C; if this takes B to B', we draw the sphere through B' with center C, to meet the given ray at D. This construction must succeed in the elliptic case, as the sphere then has radii in all directions. But in the hyperbolic case it will fail whenever CE happens to be one of the exterior diameters.

Proceeding similarly with the remaining axioms, we find that the elliptic polarity does indeed give rise to euclidean geometry. (See Veblen [**10**], pp. 287–302.) In order to show that the hyperbolic polarity gives rise to Minkowskian geometry, we could similarly test the postulates of Robb [6].* But the details of the work are rather formidable, so we shall be content to interpret the various entities and appeal to coordinates for the actual identification.

* Since we are at present limiting the number of dimensions to three (*i.e.* $2+1$, instead of $3+1$), the relevant Postulates are I–XVIII, XXI, and the denial of XIX (cf. XX).

4. Minkowskian geometry. In the plane at infinity with a hyperbolic polarity we have a conic, the *Absolute*, which is the locus of self-conjugate points and the envelope of self-conjugate lines. The remaining points and lines at infinity are thereby separated into two classes: interior and exterior points, secants and exterior lines. ([2], p. 56.) In order to examine the induced metric, we classify the lines and planes of the affine space according to their sections by the plane at infinity. (We use the modern names for them, with Robb's in parentheses.)

Line or Plane	*Section at Infinity*
Time-like line (inertia line)	Interior point
Null line (optical line)	Point on the Absolute
Space-like line (separation line)	Exterior point
Minkowskian plane (inertia plane)	Secant
Null plane (optical plane)	Tangent
Euclidean plane (separation plane)	Exterior line

We have seen that in this geometry a sphere is a hyperboloid. Strictly, we should include the case when the hyperboloid degenerates into a cone. In fact, if OC (or BC) is a null line, the locus defined in §3 is the *null cone* which joins O to the Absolute. In the remaining (non degenerate) cases, the radius OC may be either space-like or time-like. The other radii will follow suit, as a line of either kind reflects into a line of the same kind. We proceed to show that a non degenerate sphere is a hyperboloid of *one* or *two* sheets according as its radii are space-like or time-like.

If the radii are space-like, their points at infinity are exterior. Since every line at infinity contains some exterior points, the sphere meets each of its diametral planes. (The curve of intersection is an ordinary circle if the plane is euclidean, two parallel lines if it is a null plane, and in the remaining case a Minkowskian "circle," which is a special hyperbola of the affine geometry.)

If, on the other hand, the radii are time-like, their points at infinity are interior, whereas every point on an exterior line is exterior. Thus a euclidean plane through the center will not meet such a sphere, but will separate the two sheets.

By arbitrarily labelling the two sheets as first and second, or "past" and "future," we assign a definite sense along the time-like diameters. By applying translations, we deduce a consistent sense along *all* time-like lines. Then, for any two points whose join is time-like, we can assert that one is *after* the other.

Two null lines through a point O determine a Minkowskian plane which they decompose into four angular regions, one of which consists of points *after* O. The rays which bound this region determine a definite sense along the two null lines, and so along all null lines. Thus the idea of one point being after another applies to points on a null line as well as to points on a time-like line.

Robb's "conical order" is now established: there is at every point a null cone, whose generators are parallel to respective generators of any other null cone, and a time-like line is interior to the null cone at each of its points. This is essentially the model considered by Robb ([6], pp. 16–22); but by defining it in affine space instead of euclidean, we avoid the "distortion" which he mentions on p. 402.

For the analytical treatment, we choose projective coordinates in the plane at infinity in such a manner that the hyperbolic polarity takes the canonical form

$$X_0 = -x_0, \qquad X_1 = x_1, \qquad X_2 = x_2. \tag{4.1}$$

Then the Absolute has the ordinary equation

$$-x_0^2 + x_1^2 + x_2^2 = 0$$

and the tangential equation $-X_0^2+X_1^2+X_2^2=0$. ([**2**], pp. 86, 87.) The triangle of reference is a self-polar triangle, and may be joined to an arbitrary origin to form a suitable trihedron of reference for affine coordinates. We can use the same symbols for these coordinates by letting $(l_0, l_1, l_2, 0)$ denote the point previously called (l_0, l_1, l_2), which is the point at infinity on all lines parallel to

$$\frac{x_0}{l_0} = \frac{x_1}{l_1} = \frac{x_2}{l_2}\,. \tag{4.2}$$

These lines are time-like, null, or space-like, according as the number $l_0^2-l_1^2-l_2^2$ is positive, zero, or negative; and similarly the plane

$$X_0x_0 + X_1x_1 + X_2x_2 = 0$$

is euclidean, null, or Minkowskian, according to the sign of $X_0^2-X_1^2-X_2^2$. Applying the polarity (4.1), we see that the line (4.2) is perpendicular to the plane

$$-l_0x_0 + l_1x_1 + l_2x_2 = 0$$

(and to any parallel plane, which can be derived by replacing the zero on the right by another constant).

Let the points B and C of §3 have the affine coordinates $(-y_0, -y_1, -y_2)$ and (y_0, y_1, y_2), and let (x_0, x_1, x_2) be the point where the line

$$\frac{x_0 + y_0}{l_0} = \frac{x_1 + y_1}{l_1} = \frac{x_2 + y_2}{l_2}$$

through B meets the perpendicular plane

$$-l_0(x_0 - y_0) + l_1(x_1 - y_1) + l_2(x_2 - y_2) = 0$$

through C. Then the sphere described by the point (x) has the equation

$$-x_0^2 + x_1^2 + x_2^2 = -y_0^2 + y_1^2 + y_2^2,$$

and so is a hyperboloid of one sheet, a null cone, or a hyperboloid of two sheets, according to the sign of the constant on the right.

In pure affine geometry we can compare distances along any one line, or along parallel lines, but the comparison of distances along non-parallel lines simply has no meaning. However, now that we have metricized the space by defining spheres, we can compare distances along any two time-like lines, or along any two space-like lines; but it is still impossible to compare distances along lines of different kinds, or along null lines with different directions, and

we shall not attempt to do so. Accordingly, we define distances (or *intervals*) of two kinds, so as to be able to say that the surface

$$-x_0^2 + x_1^2 + x_2^2 = s^2$$

is a sphere of space-like radius s, while

$$x_0^2 - x_1^2 - x_2^2 = t^2$$

is a sphere of time-like radius t.

It is sometimes convenient to unify the formulae by saying that any point (x_0, x_1, x_2) is distant $\sqrt{x_0^2 - x_1^2 - x_2^2}$ from the origin, in which case a time-like interval appears as a real number, and a space-like interval as a pure-imaginary number. The opposite convention could be used just as well, and the old-fashioned word "imaginary" should not give the subject any air of mystery. The modern view of a complex number, as an ordered pair of real numbers, clarifies the situation and explains the mutual incomparability of the two kinds of interval: no real multiple of $[t, 0]$ can equal $[0, s]$ with $s \neq 0$.

We observe also that in this scheme every distance along a null line is zero. (That is why it is called a null line.) Thus a null cone may be considered as a sphere of zero radius.

As the metric which we have obtained agrees with that defined by Minkowski [5], the desired identification is now complete. On the one hand, the geometry of Minkowski's time-like lines, or of their points at infinity, is hyperbolic; on the other hand, affine space acquires a Minkowskian metric as soon as we specialize a hyperbolic polarity at infinity.

The fact that we have limited the number of dimensions to $1+2$ is unimportant, as the extension to $1+3$ dimensions can be made without difficulty; *e.g.* the interval from $(0, 0, 0, 0)$ to (x_0, x_1, x_2, x_3) is $\sqrt{x_0^2 - x_1^2 - x_2^2 - x_3^2}$. (This x_3 is on a par with the x_1 and x_2, and has nothing to do with the x_3 of §2.) The $(1+2)$-dimensional space is simply a section of the $(1+3)$-dimensional space. Further extensions can be made just as easily.

5. The representation of non-euclidean geometries on spheres. In three-dimensional euclidean space, where the plane at infinity is elliptic, there is a $(1, 1)$ correspondence between points at infinity and diameters of a sphere, and consequently a $(1, 2)$ correspondence between points at infinity and points on a sphere. The elliptic plane cannot be mapped on a sphere without duplication. If we try to map it on a hemisphere, we still have trouble on the peripheral circle. In other words, points at infinity correspond to diameters, whereas points on the sphere correspond to radii; and there is no satisfactory way of associating all the diameters with half the radii.

In $(1+2)$-dimensional Minkowskian space, where the plane at infinity is hyperbolic, there is similarly a $(1, 2)$ correspondence between points at infinity (interior and exterior) and points on a sphere (of time-like or space-like radius, respectively). In the case of exterior points the situation is just as before: the

sphere has a single sheet, and does not provide a (1, 1) mapping of the exterior points at infinity unless we agree to identify each pair of antipodal points of the sphere.

But the case of interior points is quite different. There is still a (1, 2) correspondence between those points at infinity and the points on a two-sheeted sphere; but now any pair of antipodal points consists of one point on each sheet, so there is a (1, 1) correspondence between interior points at infinity and points on either sheet. These interior points are just the "ordinary" points of the hyperbolic plane. Hence *either sheet of a sphere of two sheets provides an undistorted map of the proper hyperbolic plane,* without any duplication. Each line of the hyperbolic plane is represented by a diametral section of the sheet, namely by one branch of a hyperbola. This hyperbola is a Minkowskian circle—a "great circle" of the sphere. Such a curve on the surface is a geodesic in the ordinary sense of "shortest path," as the tangent planes to this kind of sphere are euclidean.

These results are obvious analytically, when we describe hyperbolic geometry in terms of coordinates x_0, x_1, x_2 satisfying

$$x_0^2 - x_1^2 - x_2^2 = 1, \qquad x_0 > 0,$$

with the rule that the distance between points (x) and (y) is

$$\arg\cosh\,(x_0y_0 - x_1y_1 - x_2y_2),$$

so that the line element is given by

$$ds^2 = -\,dx_0^2 + dx_1^2 + dx_2^2.$$

([2], pp. 209, 248.)

Eddington ([4], p. 165) has shown that the sphere of *one* sheet in $(1+2)$-dimensional Minkowskian space provides a model for the $(1+1)$-dimensional de Sitter's world,* wherein coordinates x_0, x_1, x_2 can be found such that

$$-\,x_0^2 + x_1^2 + x_2^2 = R^2,$$

with the line element given by

$$ds^2 = dx_0^2 - dx_1^2 - dx_2^2.$$

His lucid discussion of elliptic space ([4], pp. 157–158) is relevant here, and indicates that, in order to obtain an unduplicated model of the de Sitter plane, we should identify antipodal points of the one-sheeted sphere.

But we have already obtained the same model for the "exterior-hyperbolic" plane, whose points are the absolute poles of the lines of the ordinary hyperbolic plane. Adding two dimensions, *de Sitter's world is four-dimensional exterior-hyperbolic space*—the space whose points are exterior to an oval quadric three-

* Eddington clearly implies that the analogous manifold in $(1+4)$-dimensional Minkowskian space represents the $(1+3)$-dimensional de Sitter's world. Robertson ([7], p. 840) obtains the same result in a different way; but instead of "Minkowskian" he calls the underlying space "euclidean."

fold in projective four-space, with the metric determined by that quadric threefold as Absolute. (This remark was made to me about 1930 by Professor Patrick Du Val, but I have not seen it in print.) The three kinds of line—time-like, null, and space-like—are recognizable in hyperbolic geometry as the ordinary lines, lines at infinity, and ultra-infinite lines, *i.e.* as the secants, tangents, and exterior lines to the Absolute.

6. Projective coordinates. Returning, for simplicity, to the two-dimensional world, we now have *homogeneous* coordinates x_0, x_1, x_2 with

$$-x_0^2 + x_1^2 + x_2^2 > 0.$$

Retaining the convention whereby time-like and space-like intervals are regarded as real and pure-imaginary, respectively, we find (as in [2], p. 209) that the interval from (x) to (y) is

$$R \operatorname{arg\,cosh} \frac{\left| -x_0y_0 + x_1y_1 + x_2y_2 \right|}{\sqrt{-x_0^2 + x_1^2 + x_2^2}\sqrt{-y_0^2 + y_1^2 + y_2^2}}. \tag{6.1}$$

This is time-like or space-like according as

$$-(x_1y_2 - x_2y_1)^2 + (x_2y_0 - x_0y_2)^2 + (x_0y_1 - x_1y_0)^2$$

is positive or negative. In the former case the interval increases to infinity as the point (x) approaches the Absolute,

$$-x_0^2 + x_1^2 + x_2^2 = 0.$$

In the latter case the interval is preferably expressed as

$$iR \arccos \frac{\left| -x_0y_0 + x_1y_1 + x_2y_2 \right|}{\sqrt{-x_0^2 + x_1^2 + x_2^2}\sqrt{-y_0^2 + y_1^2 + y_2^2}}.$$

Thus, as Dirac remarks ([3], p. 658), all space-like lines are finite.

To describe the manner in which Minkowski's metric arises as a limiting case of de Sitter's, we make use of the homogeneity of our coordinates by arbitrarily putting $x_1 = R$. We are then left with non-homogeneous coordinates x_0, x_2 (or x_0, x_2, x_3, x_4), such that the interval from (x_0, x_2) to (y_0, y_2) is

$$\begin{aligned} &R \operatorname{arg\,cosh} \frac{R^2 - x_0y_0 + x_2y_2}{\sqrt{R^2 - x_0^2 + x_2^2}\sqrt{R^2 - y_0^2 + y_2^2}} \\ &\quad = R \operatorname{arg\,sinh} \sqrt{\frac{R^2(x_0 - y_0)^2 - R^2(x_2 - y_2)^2 + (x_2y_0 - x_0y_2)^2}{(R^2 - x_0^2 + x_2^2)(R^2 - y_0^2 + y_2^2)}}. \end{aligned} \tag{6.2}$$

It follows (as in [2], p. 255) that the line element is given by

$$ds^2 = \frac{dx_0^2 - dx_2^2 + (x_2dx_0 - x_0dx_2)^2/R^2}{\{1 - (x_0^2 - x_2^2)/R^2\}^2}.$$

When R tends to infinity, the interval becomes

$$\sqrt{(x_0 - y_0)^2 - (x_2 - y_2)^2},$$

and the line element takes its usual form for the Minkowskian plane, with time x_0 and distance x_2.

This result suggests the possibility of giving x_0 and x_2 the same interpretation *before* making R tend to infinity. What the cosmologists actually do, however, is to define an "angular" distance χ and time t, which are connected with our x_0, x_1, x_2 by the relations

$$x_0 = R \cos \chi \sinh t, \qquad x_1 = R \cos \chi \cosh t, \qquad x_2 = R \sin \chi$$

(or $x_0 : x_1 : x_2 :: \sinh t : \cosh t : \tan \chi$). Then the interval from (χ, t) to (χ', t') is

$$R \operatorname{arg} \cosh \{\cos \chi \cos \chi' \cosh (t - t') + \sin \chi \sin \chi'\}, \tag{6.3}$$

which reduces to $R|t-t'|$ when $\chi = \chi' = 0$, and to $iR|\chi - \chi'|$ when $t = t'$.

The world line of an undisturbed particle, initially at the origin ($x_0 = x_2 = 0$) is a straight line of the form $x_2 = ux_0$, or

$$\tan \chi = u \sinh t.$$

Thus the particle recedes from the origin as t increases. This is the phenomenon which gives de Sitter's world its chief experimental support, in view of the recession of nebulae. ([4], p. 162.)

7. Triangles in de Sitter's world. Exterior-hyperbolic geometry has been described by E. Study [9]. In particular, he discusses the possibility of one side of a triangle being greater than the sum of the other two. For such a comparison to have any meaning, the three sides must be all time-like or all space-like. If, further, we restrict consideration to triangles whose interior points are all exterior to the Absolute, we find that Study's results are expressible in the following form: there is *always* one side greater than the sum of the other two, say

$$AC > AB + BC.$$

This may be proved very simply in terms of the coordinates χ and t, by taking the line AC and the perpendicular line through B to be $\chi = 0$ and $t = 0$, or *vice versa*. The three sides, when time-like, determine three chords of the Absolute; the chord determined by AC is, in an obvious sense, *between* the other two. (See Fig. 1.) In the other case the Absolute lies entirely within one of the three "colunar" triangles, namely the one that shares the side AC of the triangle under consideration. (See Fig. 2.)

Thus a triangle with time-like sides can be expressed in the form

$$A(0, t), \qquad B(\chi, 0), \qquad C(0, -t'),$$

where χ, t, t' are positive. Then the sides are

$$AB = \text{arg cosh } (\cos \chi \cosh t) < t,$$
$$BC = \text{arg cosh } (\cos \chi \cosh t') < t',$$

and

$$AC = t + t'.$$

Eddington has given the following practical illustration of this result: Twin brothers, one of whom stays at home while the other travels very fast in various directions, will find when they meet again that the sedentary one has aged more than the traveller.

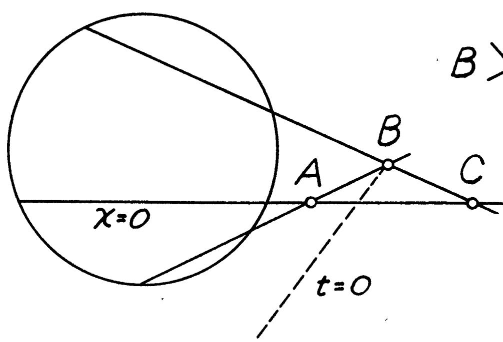

FIG. 1

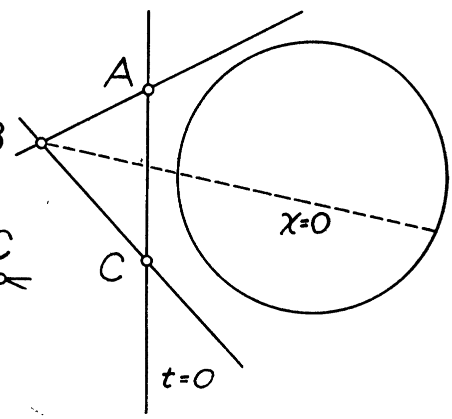

FIG. 2

Similarly, a triangle with space-like sides can be expressed as

$$A(\chi, 0), \qquad B(0, t), \qquad C(-\chi', 0),$$

with χ, χ', t positive. Dropping the i from the expression

$$iR \text{ arc cos } \{\cos \chi \cos \chi' \cosh (t - t') + \sin \chi \sin \chi'\}$$

for a space-like interval, we now have

$$AB = \text{arc cos } (\cos \chi \cosh t) < \chi,$$
$$BC = \text{arc cos } (\cos \chi' \cosh t) < \chi',$$

and

$$AC = \chi + \chi'.$$

Thus the straight path from A to C is longer than any other (everywhere space-like) path from A to C. In fact, the crooked distance can be made as small as we please by taking the origin to be the mid-point of AC (so that $\chi' = \chi$), and choosing B so that $\cosh t$ is nearly as great as $\sec \chi$.

When we add the extra dimensions for de Sitter's four-dimensional world, we find that a space-like line is neither minimum nor maximum. ([9], p. 115;

cf. [6], p. 3.) In fact, such a line, besides lying in de Sitter planes (*i.e.* secant planes to the Absolute) lies also in elliptic planes (exterior to the Absolute).

8. Through the looking-glass. In describing the two-dimensional de Sitter's world, Eddington writes of the totality of an observer's experience as forming a "lune." ([4], p. 165. His "90°" seems to have crept in through a false analogy with euclidean space.) We now recognize this lune as the angular region between two null lines (tangents to the Absolute). The corresponding figure in the whole four-dimensional world is the interior of the observer's null cone (which is an enveloping cone to the Absolute). The "horizon" which worried de Sitter is thus seen to be the section of the Absolute by the polar hyperplane of the observer. This evidently changes with the observer, like a rainbow. Its illusory character is further revealed by the fact that the world is strictly *exterior* to the Absolute, so that the horizon is not really a part of the world at all.

On the other hand, a rather disturbing paradox remains. If A and B are two observers on a space-like line (so that B is neither before nor after A), there are events (shaded in Fig. 3) which are common to A's future and B's past, and

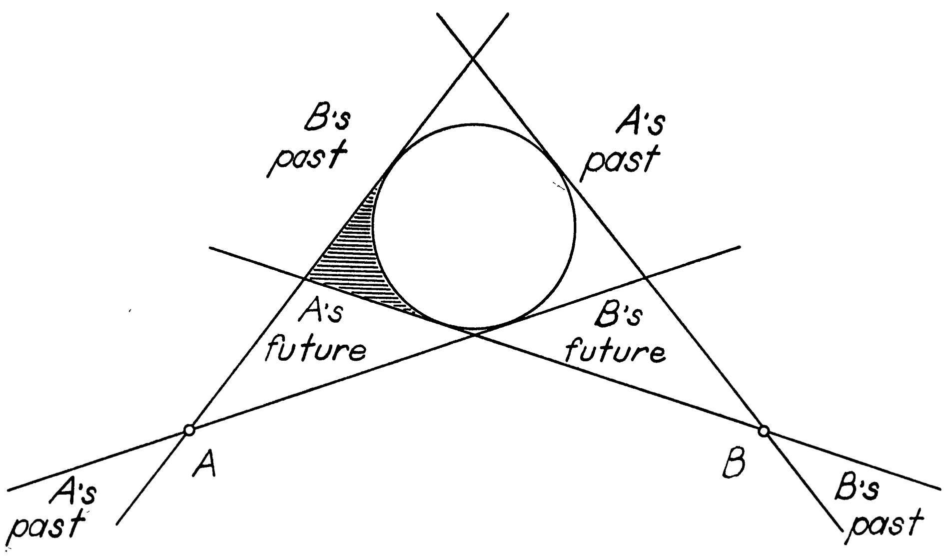

FIG. 3

other events common to A's past and B's future. This overlapping of the null cones becomes more pronounced as we increase the distance AB, until when $AB=\pi$ it is complete: A's future *is* B's past, and *vice versa*. But π (or πR) is the *whole* length of a space-like line, so A and B then coincide. In other words, if you could travel all along a space-like line, you would return to the starting-point with your past and future interchanged! There are two possible solutions for this paradox. You may argue that such a journey is essentially impossible:

the line joining you—"here and now"— to any event that you could possibly experience, is time-like. On the other hand, if you insist that the distinction between past and future is absolute, you can replace the exterior-hyperbolic space by its orientable covering manifold (which is representable on the whole one-sheeted hyper-sphere in Minkowskian space of $1+4$ dimensions). But do you not find it disturbing to envisage an exact replica of yourself at the "antipodes," living backwards?

9. Conclusion. Apart from all these difficulties, we have ignored the embarrassing subject of the essential emptiness of de Sitter's world, which has led cosmologists to propose a modification in the direction of an earlier hypothesis: Einstein's "cylindrical" world. ([4], p. 160.) But the fact remains that de Sitter's is the theoretical world of greatest interest to pure geometers, as it alone has an interesting group. ([3], p. 657.) Since the points of exterior-hyperbolic space are the absolute poles of the hyperplanes of ordinary hyperbolic space (of four dimensions), the above remarks show that this group is the hyperbolic metric group: its elements are the collineations that preserve an oval quadric in real projective four-space.

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FOLDING THE CONICS

R. C. YATES, U. S. Military Academy

The idea of applying the process of paper folding to the construction of the conics originated with Row.* The methods he gives, however, are quite involved in a structural sense. Improvements (at least for the central conics) are offered by Lotka† but the usefulness and charm of his methods are somewhat obscured by an accompanying analytical justification.

The purpose of this note is to offer the best of the methods of Lotka and Row and to present simple proofs to establish the processes involved. It is hoped

* T. Sundara Row, Geometric Exercises in Paper Folding (trans. by Beman and Smith) Chicago, 1901.

† A. J. Lotka, School Science and Mathematics, VII, 1907, 595–597; Scientific American Supplement, 1912, 112.

that this formation of the conics by paper folding will somehow find its way into the clasroom where it will undoubtedly be received with enthusiasm.

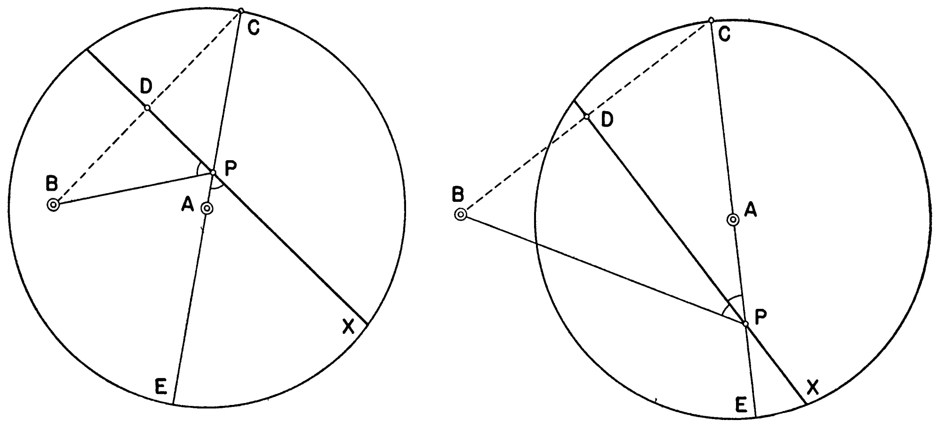

FIG. 1

A fixed point B is selected within a circle of radius r and center A. The point B is folded over upon the circle, as at C, forming the crease DX (the perpendicular bisector of BC) which meets the diameter EC in the point P. As B takes positions along the circle, the path of P is the *ellipse* having foci at A and B, major axis r, and the creases as tangents. For, since P lies on the perpendicular bisector of BC,

$$AP + BP = AP + PC = r$$

and

$$\angle BPD = \angle DPC = \angle APX.$$

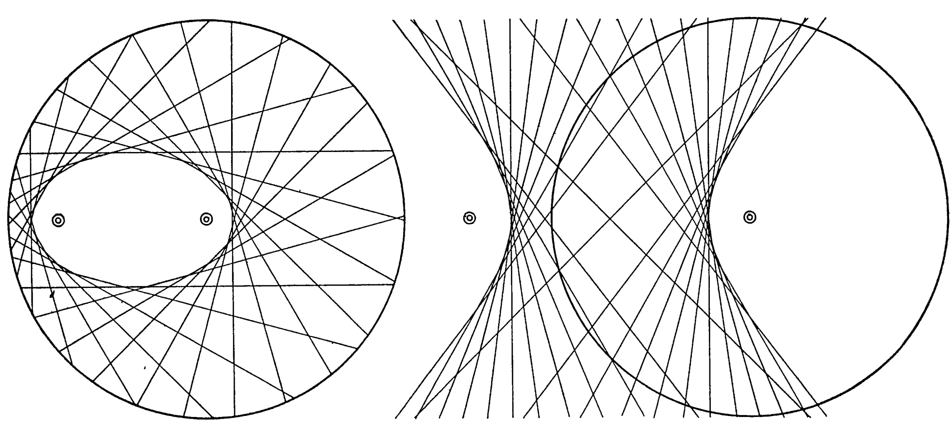

FIG. 2

(If B is taken at A, the locus of P is the circle with center at A and radius $r/2$.)

If B is selected outside of the circle, the locus of P (the intersection of crease and corresponding diameter) is the *hyperbola* having A and B as foci, real axis r, and the creases as tangents. For, since P lies on the perpendicular bisector of BC,

$$BP - AP = CP - AP = r$$

and

$$\angle BPD = \angle DPA.$$

The rectangular hyperbola is formed if $AB = r\sqrt{2}$. If B is taken on the circle, the locus of P is the point A.

The *parabola* (see Figure 3) presents a special case in which a line L replaces the circle of the central conics. The point B is folded over upon L, as at C, producing the crease DX. The perpendicular to L at C meets the crease in P. Points P form the parabola having B as focus, L as directrix, and the creases as tangents. For, since P lies on the perpendicular bisector of BC,

$$BP = PC$$

and

$$\angle BPD = \angle DPC.$$

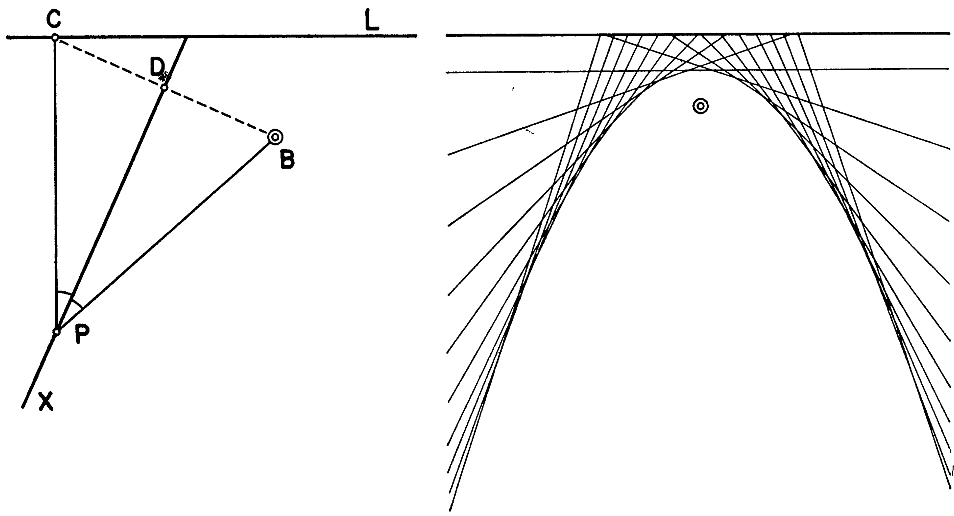

FIG. 3

This fascinating art of paper folding, which seems unfortunately relegated to the limbo, may be brought to full expression through the medium of ordinary wax paper found in every kitchen cabinet.

THE LIBRATION POINTS IN AN n-BODY PROBLEM*

J. J. L. HINRICHSEN, Iowa State College

1. Statement of problem. Let us consider the restricted problem of n bodies in which $n-1$ of the bodies have equal mass and are assumed fixed in the vertices of a regular $(n-1)$-sided polygon while the nth body is an infinitesimal mass, lying in the same plane, whose motion is influenced by the other $n-1$ bodies but which is unable to exert any influence upon them. The $n-1$ bodies are assumed to attract one another according to the Newtonian inverse square law and to be rotating in a circle about their common center of mass.

We shall choose the unit of mass as the sum of the masses of the $n-1$ bodies, the unit of distance as the distance from one of these bodies to the center of mass of the other $n-2$ bodies, and the unit of time such that the angular velocity of the system is unity.

We shall introduce a system of rectangular coordinates moving with the bodies such that the origin is at the center of the regular $(n-1)$-sided polygon. The x-axis shall be so chosen that for n even, one vertex of the polygon lies on the negative half of the x-axis and if n is odd, such vertices fall on both positive and negative sides of the axis. Then if the vertices of the regular polygon are denoted by P_i, their coordinates by (x_i, y_i), $(i=1, 2, \cdots, n-1)$, and the infinitesimal mass by P with coordinates (x, y), the potential function may be written

$$\Omega = \frac{1}{2(n-1)} \sum_{i=1}^{n-1} \left[r_i^2 + \frac{2}{r_i} \right],$$

where

$$r_i^2 = (x - x_i)^2 + (y - y_i)^2 .$$

The equations of motion of the infinitesimal mass become

$$\frac{d^2x}{dt^2} - 2\frac{dy}{dt} = \Omega_x$$

$$\frac{d^2y}{dt^2} + 2\frac{dx}{dt} = \Omega_y,$$

where Ω_x and Ω_y denote the partial derivatives of Ω with respect to x and y respectively. The equations admit the well-known integral of Jacobi,

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2\Omega - C,$$

where C is a constant of integration and v denotes the velocity of P.

2. The libration points. This paper is concerned with the libration points of the system, namely those points whose coordinates satisfy

* Presented to the American Mathematical Society, November 21, 1941.

$$\Omega_x = \Omega_y = 0.$$

They are points about which infinitesimal periodic orbits may exist. They are also critical points of the system of curves

$$2\Omega - C = 0,$$

first discussed for the restricted problem of three bodies by G. W. Hill.*

We shall proceed to write the equations. If n is even,

$$x_i = r \cos \frac{2i - 1}{n - 1} \pi, \qquad y_i = r \sin \frac{2i - 1}{n - 1} \pi, \qquad (i = 1, 2, \cdots, n - 1),$$

and if n is odd,

$$x_i = r \cos \frac{2i}{n - 1} \pi, \qquad y_i = r \sin \frac{2i}{n - 1} \pi, \qquad (i = 1, 2, \cdots, n - 1),$$

where the distance from one body to the center of mass of the remaining $n-2$ bodies is set equal to unity. Then

$$r = \frac{n - 2}{n - 2 + 2 \sum_{i=1}^{(n-2)/2} \cos \frac{2i - 1}{n - 1} \pi}, \quad \text{if } n \text{ is even, } (n > 2),$$

and

$$r = \frac{n - 2}{n - 1 + 2 \sum_{i=1}^{(n-3)/2} \cos \frac{2i}{n - 1} \pi}, \quad \text{if } n \text{ is odd, } (n > 3).$$

Now

$$\Omega_x = \sum_{i=1}^{n-1} \frac{\partial \Omega}{\partial r_i} \frac{\partial r_i}{\partial x} = \frac{1}{n - 1} \sum_{i=1}^{n-1} \left(1 - \frac{1}{r_i^3}\right)(x - x_i),$$

$$\Omega_y = \sum_{i=1}^{n-1} \frac{\partial \Omega}{\partial r_i} \frac{\partial r_i}{\partial y} = \frac{1}{n - 1} \sum_{i=1}^{n-1} \left(1 - \frac{1}{r_i^3}\right)(y - y_i),$$

and the equations whose solutions give the coordinates of the libration points may be written

$$\sum_{i=1}^{n-1} \left(1 - \frac{1}{r_i^3}\right)(x - x_i) = 0,$$

$$\sum_{i=1}^{n-1} \left(1 - \frac{1}{r_i^3}\right)(y - y_i) = 0.$$

* G. W. Hill, American Journal of Mathematics, vol. 1, 1878, pp. 5–29, 129–147, 245–260. Also see G. H. Darwin, Acta Mathematica, vol. 21, 1897, pp. 99–242.

If we multiply the first of these equations by $(y-y_j)$, the second by $(x-x_j)$ and subtract, we eliminate $(1-1/r_j^3)$ and obtain

$$\sum_{i=1}^{n-1} \left[(y_i - y_j)x - (x_i - x_j)y + (x_iy_j - x_jy_i)\right]\left(1 - \frac{1}{r_i^3}\right) = 0,$$
$$(j = 1, 2, \cdots, n-1).$$

These equations are not linearly independent. The term obtained for $i=j$ vanishes, and it is to be noted that

$$(y_i - y_j)x - (x_i - x_j)y - (x_iy_j - x_jy_i) = 0$$

is the equation of the straight line determined by the points (x_i, y_i), (x_j, y_j), while

$$1 - \frac{1}{r_i^3} = 0,$$

is the equation of the circle of radius one with (x_i, y_i) as center. These lines and circles divide the plane into regions. By investigating the signs of the terms of the above sums in a certain region, we are able to conclude the possibility or impossibility of existence of libration points in the region and are thereby able to establish certain bounds for those points. For $n=3$, 4 and 5, approximate locations of the libration points have been found.

3. The case $n=3$. This is the classical restricted problem of three bodies but with two equal masses. Here

$$x_1 = -\tfrac{1}{2}, \quad y_1 = 0; \qquad x_2 = +\tfrac{1}{2}, \quad y_2 = 0,$$

and the equations satisfied by the coordinates of the libration points are

$$y\left(1 - \frac{1}{r_1^3}\right) = 0$$
$$y\left(1 - \frac{1}{r_2^3}\right) = 0.$$

It follows either $y=0$ or $r_1 = r_2 = 1$. In the first case, we may locate the libration points along the x-axis by approximating the roots of the equation $\Omega_x = 0$, or

$$2x = \frac{x + \frac{1}{2}}{r_1^3} + \frac{x - \frac{1}{2}}{r_2^3}.$$

There exist five libration points of which three are collinear with the bodies and have coordinates $(\mu, 0)$, $(0, 0)$, $(-\mu, 0)$, where $1.20 < \mu < 1.21$ while the remaining two points $(0, \pm\sqrt{3}/2)$ lie at the vertices of equilateral triangles formed with the bodies as other vertices. The system of curves of zero velocity for a given energy constant C, studied by Hill, have these five points as critical points. Each of the libration points is within a distance $3/2$ from the center of mass of the system.

4. The case $n=4$. Our formulas yield

$$x_1 = \tfrac{1}{3}, \quad y_1 = \frac{1}{\sqrt{3}}; \qquad x_2 = -\tfrac{2}{3}, \quad y_2 = 0; \qquad x_3 = \tfrac{1}{3}, \quad y_3 = -\frac{1}{\sqrt{3}},$$

and the equations satisfied by the coordinates of the libration points are

$$\left(\frac{1}{\sqrt{3}}x - y + \frac{2}{3\sqrt{3}}\right)\left(1 - \frac{1}{r_2^3}\right) + \left(\frac{2}{\sqrt{3}}x - \frac{2}{3\sqrt{3}}\right)\left(1 - \frac{1}{r_3^3}\right) = 0,$$

$$\left(\frac{1}{\sqrt{3}}x - y + \frac{2}{3\sqrt{3}}\right)\left(1 - \frac{1}{r_1^3}\right) + \left(-\frac{1}{\sqrt{3}}x - y - \frac{2}{3\sqrt{3}}\right)\left(1 - \frac{1}{r_3^3}\right) = 0,$$

$$\left(\frac{2}{\sqrt{3}}x - \frac{2}{3\sqrt{3}}\right)\left(1 - \frac{1}{r_1^3}\right) + \left(\frac{1}{\sqrt{3}}x + y + \frac{2}{3\sqrt{3}}\right)\left(1 - \frac{1}{r_2^3}\right) = 0.$$

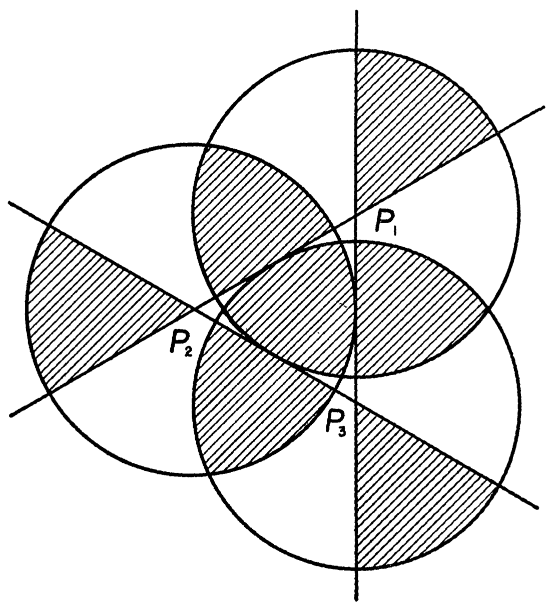

FIG. 1. Areas of the plane (shaded) in which Libration Points may lie for $n=4$.

The plane is divided into regions by the lines

$$\frac{1}{\sqrt{3}}x - y + \frac{2}{3\sqrt{3}} = 0,$$

$$\frac{2}{\sqrt{3}}x - \frac{2}{3\sqrt{3}} = 0,$$

$$-\frac{1}{\sqrt{3}}x - y - \frac{2}{3\sqrt{3}} = 0,$$

and the circles $r_i=1$, $(i=1, 2, 3)$. To be solutions of these equations, the coordinates (x, y) cannot make both terms of the first members either positive or negative. Such points can therefore only lie in the various shaded regions of the plane indicated in Figure 1.

Because of symmetry considerations, the libration points must lie along one of three axes of which $y=0$ is typical. The position of the points along this line may be found by solving the equation $\Omega_x=0$, or

$$3x = \frac{x-\frac{1}{3}}{r_1^3} + \frac{x+\frac{2}{3}}{r_2^3} + \frac{x-\frac{1}{3}}{r_3^3}.$$

There exist four roots which may be written $x=0$, $\frac{1}{3}$, μ_1, and $-\mu_2$, where $.87<\mu_1<.88$ and $1.20<\mu_2<1.21$. Hence there exists $1+3+3+3=10$ libration points in this restricted problem of four bodies.

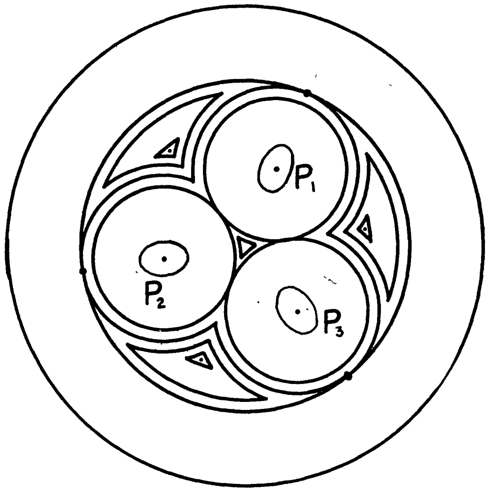

FIG. 2. Zero Velocity Curves for $n=4$. (approximate)

5. Hill curves for $n=4$. We shall next investigate the Hill curves for this problem and seek those points at which this system of curves has singularities. Consider the equation

$$2\Omega - C = 0,$$

or

$$r_1^2 + r_2^2 + r_3^2 + 2\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) = 3C.$$

For C large, either r_1, r_2 or r_3 must be large, in which case all three of these distances are large, or else one of the distances r_1, r_2 or r_3 must be very small. The motion must therefore be such that P is outside of a large closed curve enclosing the three bodies or else P is inside of a small closed curve about either P_1, P_2

or P_3. As C decreases the large curve diminishes in size while the small inner curves increase. The shapes of the curves also vary, but vary continuously with C. The first singularity arises when the three small curves simultaneously meet in three points on the sides of the equilateral triangle. This gives rise to three libration points namely those above corresponding to $x=\frac{1}{3}$. As C decreases further, the large outer curve continues to decrease while the three inner curves have now merged to form an outer and an inner closed curve. The outer one of these curves expands and meets the diminishing exterior curve simultaneously in three points namely those corresponding to $x=-\mu_2$. The innermost curve meanwhile decreases and shrinks to a point, disappearing in the origin $x=0$. The two outer curves, after meeting, break up into three separate closed curves each one of which shrinks to a point as C continues to decrease. These points correspond to $x=\mu_1$. Thus each one of the ten libration points is a critical point of the system of zero velocity curves for the problem under consideration. Each of the points is furthermore within a distance 5/3 from the center of mass of the system.

6. The case $n=5$. The formulas yield

$$x_1 = 0, \quad y_1 = \tfrac{3}{4}; \qquad x_2 = -\tfrac{3}{4}, \quad y_2 = 0; \qquad x_3 = 0, \quad y_3 = -\tfrac{3}{4}, \quad x_4 = \tfrac{3}{4}, \quad y_4 = 0,$$

and the equations satisfied by the coordinates of the libration points are

$$(x - y + \tfrac{3}{4})\left(1 - \frac{1}{r_2^3}\right) + 2x\left(1 - \frac{1}{r_3^3}\right) + (x + y - \tfrac{3}{4})\left(1 - \frac{1}{r_4^3}\right) = 0,$$

$$-(x - y + \tfrac{3}{4})\left(1 - \frac{1}{r_1^3}\right) + (x + y + \tfrac{3}{4})\left(1 - \frac{1}{r_3^3}\right) + 2y\left(1 - \frac{1}{r_4^3}\right) = 0,$$

$$2x\left(1 - \frac{1}{r_1^3}\right) + (x + y + \tfrac{3}{4})\left(1 - \frac{1}{r_2^3}\right) + (x - y - \tfrac{3}{4})\left(1 - \frac{1}{r_4^3}\right) = 0,$$

$$(x + y - \tfrac{3}{4})\left(1 - \frac{1}{r_1^3}\right) + 2y\left(1 - \frac{1}{r_2^3}\right) - (x - y - \tfrac{3}{4})\left(1 - \frac{1}{r_3^3}\right) = 0.$$

The plane is now divided into regions by the lines

$$\begin{aligned} x - y + \tfrac{3}{4} &= 0 \\ x + y + \tfrac{3}{4} &= 0 \\ x - y - \tfrac{3}{4} &= 0 \\ x + y - \tfrac{3}{4} &= 0, \\ x &= 0 \\ y &= 0, \end{aligned}$$

and the circles $r_i=1$, $(i=1, 2, 3, 4)$. Each equation contains three terms in its first member and for their sum to vanish, these terms cannot all be either posi-

tive or negative. We are thus able to conclude that the libration points can lie only in certain restricted regions of the plane.

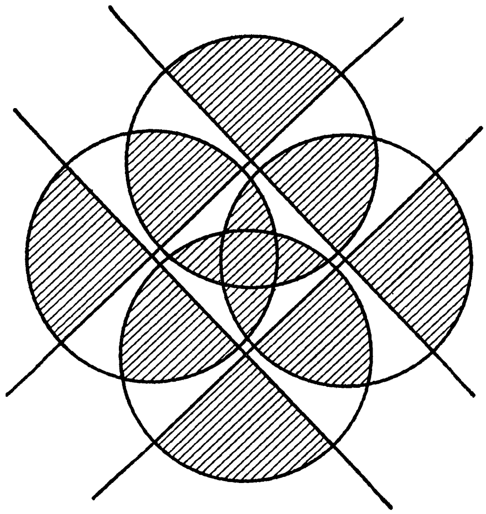

FIG. 3. Areas of the plane (shaded) in which Libration Points may lie for $n=5$.

Because of symmetry, the libration points must lie along one of four lines. Consider the lines $y=0$ and $y=x$ which are typical. The position of the points along $y=0$ may be found by solving the equation $\Omega_x=0$, or

$$4x = \frac{2x}{r_1^3} + \frac{x+\frac{3}{4}}{r_2^3} + \frac{x-\frac{3}{4}}{r_4^3}.$$

This equation possesses the roots $x=0$ and $x=\mu$, where $1.25<\mu<1.26$. For $y=x$, Ω_x becomes identical with Ω_y and we must solve

$$4x = \frac{2x-\frac{3}{4}}{r_1^3} + \frac{2x+\frac{3}{4}}{r_2^3},$$

which is found to have, in addition to $x=0$, two roots μ_1 and μ_2, where $.46<\mu_1<.47$ and $.62<\mu_2<.63$. With the symmetric points included, there exist $1+4+4+4=13$ libration points. Each is within a distance $7/4$ from the center of the system.

7. General remarks. In conclusion, we shall make several general remarks. There will exist $3n-2$ libration points each of which must lie within a distance $(2n-3)/(n-1)$ from the center of the system. As n increases, the regions within this distance excluded to libration points become smaller and smaller and have almost disappeared for $n=8$. The libration points must always lie on axes of symmetry. One will lie at the center of mass while $n-1$ will be equally distributed along each of three concentric circles about the center. The discussion of the system of curves of zero velocity may be extended to values of $n>3$.

MATHEMATICS AND MECHANICS IN THE POSTGRADUATE SCHOOL AT ANNAPOLIS*

R. E. ROOT, U. S. Naval Academy

1. The early years. Postgraduate education at Annapolis had its beginning in 1909, when a small group of young Naval Officers were ordered there to engage in studies in engineering science. For a few years this work proceeded without organized curricula, with no regular schedule of classes. A library was provided, and the facilities of the academic departments, and of the Engineering Experiment Station were available. The student officers were directed by a single officer, designated as the Head of the Postgraduate Department, who had the assistance of a civilian engineer and a small office force. The Department was housed in the loft of Isherwood Hall, the building occupied by the Department of Marine Engineering.

At this early period, as at all periods in the history of the school, the program benefited by generous cooperation of civilian institutions. The Head of the Department arranged for frequent lectures on engineering topics, some series of lectures constituting virtually a course of instruction. Professors from Columbia, Harvard, Johns Hopkins and perhaps other universities, and experts from the bureaus in Washington, participated in this program. Short courses, provided by part time assistance of members of academic departments, were increasingly relied upon to overcome deficiencies in the fundamental training of student officers. The deficiencies most evident were in mathematics, mechanics and physics. Mathematicians will be interested to know that Professor W. W. Johnson, then well past seventy, closed his illustrious career as a teacher by giving some of this part time instruction.

The outline of courses for 1914–15 contained courses in mathematics, mechanics, applied mechanics, physics, electrical engineering, machine design and thermodynamics. At this time the student body included only those taking mechanical and electrical engineering, and the year's work at Annapolis was preparatory to a year at Columbia University. The students were regular line officers of the Navy, with about five years of experience since graduation from the Naval Academy. The course in mathematics covered reviews in certain essentials of algebra, trigonometry and analytic geometry, becoming more thorough with the advanced topics in calculus, and including brief treatments of Fourier series, probability and precision, and ordinary differential equations. The course in mechanics covered statics and dynamics, and the applied mechanics covered strength of materials and the motion and balance of reciprocating engines. These three sequences were taught by one professor.

The work was interrupted by the declaration of war in April, 1917. Commander and student officers were at once ordered to other duty and the teaching force was assigned to the appropriate departments of the Naval Academy.

* Presented at the Annual meeting of the Mathematical Association of America, at Vassar College, September 7, 1942.

2. After the war. The school opened in its own building in June 1919, with Admiral E. J. King, then a Captain, as head of the department. Of the former teaching force, only two were on hand, the Professor of electrical engineering and physics and the Professor of mathematics and mechanics. A Professor of mechanical engineering and an assistant professor in each of the three divisions were provided at once. During the two years of Captain King's administration the organization was greatly improved. A professor in chemistry and metals was appointed and officer assistants were detailed to the school for administrative duties. The institution became officially the Postgraduate School. Curricula were established for mechanical, electrical, radio and aeronautical engineers, and in ordnance, naval construction and civil engineering. The first four curricula were identical for the year spent at Annapolis, specialization beginning the next year at other institutions. The other three groups could never quite be fitted into this system, but they did spend most of a year at Annapolis, and then go to other institutions.

The mathematics of this period was about the same as before the war, with the addition of a "mathematical laboratory" course of ten afternoons in the construction of charts and nomograms and in mechanical integration. The mechanics was expanded to include hydraulics and a smattering of aeronautics, and a course in modern ballistics was developed for the ordnance group.

During the next ten years, under the successors of Captain King, certain trends were evident and changes in fundamental policy were made. Emphasis on unification was limited to summer, fall and winter terms, and the spring term became a period of specialization, when each group was given courses especially planned to prepare for the next year's work at another institution. We were able to readjust our work in mechanics by having the hydraulics and strength of materials assigned to other divisions, reducing the general course to the conventional statics and dynamics, with a special course in engine balance and vibrations for mechanical and aeronautical engineers and a special course in aeronautics for aeronautical engineers. Ordnance students were assigned to sub-specialties, as torpedoes, explosives, metals, fire control, design and ballistics. All took much the same work at Annapolis, but went to different schools for one or two years of specialization. But after a few years it became evident that this degree of specialization was not desirable. Then the fire control, design and ballistics specialties were discontinued and replaced by a larger group in "general ordnance." This group took one full year at Annapolis, one summer at the Naval Gun Factory, then two more terms (eleven weeks per term) at Annapolis, followed by 14 months at various naval, army and industrial establishments. This second year ordnance curriculum called for advanced courses in differential equations, probability and statistical methods, dynamics of rigid bodies, exterior ballistics, and a course in applied elasticity emphasizing energy methods, dynamic stresses and vibrations.

During these years a General Line curriculum was introduced, for officers not specializing in any technical branch. In 1931–32 there were 63 officers in

this work, while 70 officers were in the first year technical curricula and 8 in second year ordnance. The general line courses included tactics, gunnery, communications, economics, government, international relations, administration and organization, psychology, navigation and electricity. Much of the teaching of these subjects was done by commissioned officers. In 1931 the Postgraduate School staff, under the Head of the School and the Executive officer, included 8 line officers and 15 civilian teachers, besides 2 visiting professors conducting certain general line courses. The civilians were divided, four in mathematics and mechanics, three in mechanical engineering, three in electrical engineering, two in metallurgy and chemistry, one in physics, one in radio, and one in modern languages.

3. The new plan. The year 1932–33 was a transition period, during which a new plan and revised curricula became effective. Under this plan all students were to enter the "School of the Line" for one year, selected groups were to follow specializing curricula the second year at Annapolis, after which qualified officers would take the advanced training at other institutions. The new class was to be divided at the beginning into groups corresponding to the various specialties, selection for certain sub-specialties to follow later.

For the pre-technical groups five hours per week were allotted to mathematics and mechanics, so that during the first year the mathematics could be covered up to differential equations, and the work in the regular statics and dynamics was finished the second summer. This permitted development of specialized courses in mathematics and mechanics for the various groups for the second year.

Under the new arrangement the naval constructors and civil engineers were not sent to Annapolis, but went directly to the Massachusetts Institute of Technology and to Rensselaer Polytechnic Institute. It should be understood that, ordinarily, these groups were selected within two years after graduation, usually from among those having very high scholastic standing, for transfer from the Line to the other Corps, while other groups were selected at least six years after graduation. The younger men could fit directly into the programs of civilian schools more easily than officers coming from six or seven years of sea duty. Moreover, the Construction Corps and the Civil Engineer Corps candidates would not require general line courses.

During the next few years the number of student-officers at Annapolis was sometimes as high as 275 men, over 100 of them in the second year work. The trend was away from unification, with a growing tendency to plan each curriculum to meet its particular objective, independent of other curricula. In most cases the students in a curriculum constituted at least one section suitable for separate class room instruction, and in few cases could the teaching load be effectively reduced by combining groups in different major curricula. Moreover, economy of teaching load is not a prime consideration when the students draw salaries comparable to those of the professors. Even if courses in different cur-

ricula were of similar content, differences in time schedules might make separate treatment necessary. This diversity of curricula was nearly complete for the second year students.

The general policy of the school, the character and objectives of the curricula and the assignment of student officers to these curricula, all are determined by the Head of School, in cooperation with the various bureaus of the Navy, and under the Bureau of Personnel, formerly the Bureau of Navigation. Each curriculum is administratively the charge of a commissioned officer who is a member of the staff. His responsibility extends to the welfare of the students in this specialty, and pertains to the work at other institutions as well as at Annapolis. Each of the courses designed to accomplish the general objective has a specific objective and a time allotment. It is the task of a professor of the subject to work out the details of a course, and we consider it well worth while to regard the content, arrangement and treatment of each course as a separate problem, to be considered in relation to the specific objectives and in coordination with other courses in the same curriculum.

Certain factors which affect the layout of courses are worthy of mention. First, the student body is very homogeneous, nearly all with a common Naval Academy experience followed by active naval duties. Occasionally, officers of the Marine Corps and Coast Guard take our courses, and a few officers from the navies of our South American neighbors are admitted, but in general we know what to expect. In fact, selections are so carefully made, usually after application of the candidate for the course, that failures in the courses are, indeed, very rare.

The second factor is the cooperation of the institutions to which our students are to go. Usually relations with these schools continue over many years, and mutual understanding eliminates unnecessary duplications and avoids serious omissions.

The third factor, particularly affecting revisions, is the comment of the students themselves after completing the courses, and sometimes after later naval experience. Often the officers in charge of curricula, and other officers of the staff, are former student officers.

With the increased divergence of curricula there was a growing tendency for the professors individually to center their interest in particular fields. Indeed this tendency had been encouraged from the start in relation to the specialized courses, by assigning each special course to the same teacher year after year. And now the attention of one professor in the mathematics and mechanics group is given almost entirely to aeronautical subjects, and that of another to ordnance. The remaining three of us have not been so fortunate as to confine our efforts to a single field, but each sequence of courses in a major curriculum has been made the particular responsibility of some one professor.

4. Courses in mathematics and mechanics. To this point I have spoken of conditions before the impact of the present national emergency on the work of

the school. I will now briefly characterize the courses given by the mathematics and mechanics professors in several major curricula just before this impact, say in 1939–40.

Naval Engineering. The objective here is to prepare officers to direct the design, inspection, operation and maintenance of naval machinery, and to direct related research.

The mathematics of the first year was followed by a course of 40 classroom hours in ordinary and partial differential equations, and by 40 hours divided among: (a) Numerical analysis, covering interpolation formulas, numerical integration and differentiation, numerical solution of algebraic and differential equations; (b) Solution by series, Bessel's equation and Bessel functions, and (c) introduction to vector analysis.

In mechanics the statics and dynamics were followed by 30 recitations and 10 problem periods, in advanced stress analysis, with emphasis on energy methods, and 40 recitations and 10 problem periods in hydromechanics. The problem periods are about two hours in length.

Radio Engineering. The objective is to prepare the officers to supervise the work relative to specification, design and research problems of radio and sound engineering, and to supervise the operation, maintenance and testing of radio and sound apparatus.

The first year mathematics was followed by a sequence of four courses in the four terms of the second year. (a) A course of 30 periods in ordinary differential equations, (b) 50 hours treating hyperbolic functions, harmonic analysis, numerical and series methods in differential equations, gamma and Bessel functions; (c) 50 hours in the application of differential equations to electric circuits, including Heaviside methods; and (d) 50 hours continuing Heaviside methods and including vector analysis with applications to electrical theory.

For this group the essential elements of mechanics were covered in physics courses.

Ordnance Engineering. The objective is to fit the officer to inspect ordnance material, to deal with problems of design and development in the Bureau of Ordnance, the Naval Gun Factory and the Naval Proving Ground, and to serve as experts in the operation of ordnance material in the fleet.

The mathematics of the second year was presented in four courses: (a) Twenty hours in ordinary differential equations; (b) 30 hours in statistical analysis including least squares, interpolation and frequency distributions; (c) 30 hours in exterior and interior ballistics, covering modern methods of trajectory computation and the derivation and application of the Le Duc formulas; and (d) 30 hours giving statistical methods covering correlation, theory of sampling, and control of quality by tests of samples.

The mechanics of the second year was in three courses: (a) Thirty class room hours and 10 problem periods in advanced stress analysis emphasizing energy methods; (b) 30 hours covering the theory of the thick cylinder with applications to built-up guns, also vector analysis with applications to dynamics of rigid

bodies; and (c) 30 hours covering the theory of the gyroscope and the stability of a rotating projectile in flight, also corrections of trajectories for non-standard conditions and for effects due to the earth's rotation, and the ballistics of bombing and of armor penetration.

Aeronautical Engineering. The objective is to fit the officer to cope with any problem arising in the naval aeronautical organization.

Three courses in mathematics in the second year were: (a) Thirty hours in ordinary differential equations; (b) 40 hours treating partial differential equations and numerical and series methods for ordinary differential equations, also vector analysis and complex variable theory applied to fluid mechanics; and (c) 30 recitations and 10 problem periods in aerodynamic theory of the airfoil.

The five courses in mechanics were: (a) Thirty class room periods and 10 problem periods in advanced stress analysis, including graphical and energy methods; (b) 30 hours and 10 problem periods in airplane structures; (c) 30 hours and 10 laboratory periods in aerodynamics of the airplane, including stability and performance and wind tunnel methods and equipment; (d) 40 hours and 10 problem periods, conducted in part as a seminar and dealing with problems of aircraft design; and (e) 30 hours and 10 problem periods in the dynamics of engine and shaft, analyzing periodic forces, balance of multicylinder engines, elastic vibrations and critical speeds.

Aerological Engineering. The objective is to prepare officers to become weather forecasters, to improve methods of forecasting at sea, and to participate in the solution of problems involving atmospheric conditions, as visibility, ballistic winds, etc.

Second year courses were: (a) Thirty hours in ordinary differential equations; (b) 40 hours in partial differential equations and vector analysis with applications in fluid mechanics; (c) 50 hours in mathematical processes in meteorology, deriving the general equations of motion on the rotating earth, and covering the kinematical methods of Petterssen for predicting changes in the pressure field; and (d) 40 hours in statistical analysis, covering interpolation, numerical differentiation and integration, numerical and series solutions of differential equations, least squares methods, correlation, Fourier analysis and the investigation of periodicity of data.

5. The present emergency. As I have said, when the war broke in 1917, the Postgraduate work was suspended. As the present national emergency approached, the school girded itself for greater burdens. Many months before Pearl Harbor the movement toward a two ocean navy had brought an urgent demand for many more naval officers with specialized technical training. The present Head of the Postgraduate School, Captain James A. Logan, realizing the important role that the school should play in the emergency, and foreseeing that the very urgent need for officers would make it difficult to have officers at school for long periods, directed revision of the curricula to reduce the time of each to the shortest minimum consistent with its objective.

General line subjects have been eliminated and some curricula of small tech-

nical content have been suspended, while the principal technical sequences have been shortened by months, and in some cases by years. Some emergency curricula have been added. With this adaptation of the school to the requirements of the emergency, the school is operating at full load, and only the limited capacity of the building prevents the ordering of larger groups to its courses.

I would mention two marked departures from previous conditions. First, the aerological course, which formerly took two years at Annapolis, and one year at another school, now is completed in five successive terms of ten weeks each, all at Annapolis. Officers who are competent forecasters conduct the practical part of the course, and the present class numbers 40, all but 5 of them reserve officers, graduates of universities or engineering schools, several with graduate training. Second, a curriculum for naval constructors has been established, running through three ten week terms, and when one class has finished another is ready to start. Each class, about 80 in number, is composed of young engineering graduates with commissions as reserve officers. They take a refresher course in mathematics, running five hours per week for ten weeks. The school calendar is now virtually five terms of ten weeks each.

Of the civilian teaching staff, most are reserve officers, and now only five of us are not in uniform. Of the force in mathematics and mechanics, formerly five in number, one has been ordered elsewhere, where his special abilities are required, and another divides the week between the Postgraduate School and the Naval Proving Ground at Dahlgren; but we have the assistance of four additional reserve officers, teachers from the departments of other institutions.

It will now be realized that the development of the Postgraduate School has not been hampered by any restricting conventions. As a school we have had no fixed standards to which we must conform other than those growing out of the needs of the navy. In our efforts to meet these needs we have made our own traditions.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Allegheny College, Meadville, Pa.

NUMERICAL INTEGRATION

J. S. FRAME

1. Introduction. The subject of numerical integration is one which could be of practical importance to students of engineering and other sciences if they but knew of it. However, it is usually not taught when the student begins integration, but is reserved for a course in advanced calculus which the majority of engineers and other scientists never take. The mathematics club can perform

a distinct service by bringing this topic to the attention of a wider group than the advanced mathematics majors.

The object of numerical integration is to obtain an approximate numerical value for a definite integral $\int_{x_0}^{x_0+l} f(x)dx$ in terms of certain known values of the integrand $f(x)$, and perhaps also in terms of additional data such as the values of the derivative $f'(x)$ at the ends of the interval. Assuming that the interval of integration (denoted by Δ), has finite length l, we denote the value of the integral by Al, so that A is by definition the *average value* of $f(x)$ on the interval Δ. Four different methods, or rules, for approximating this average A will be derived in this article. In each case an estimate will be given for the order of magnitude of the error. The comparative excellence of the approximations will then be illustrated by numerical examples. Finally, a brief mention of some applications will show instances when one of these methods gives exact results.

The function $f(x)$ will be assumed to be single-valued, continuous and integrable on the given interval Δ; and, in the last method of approximation, the derivative $f'(x)$ will also be assumed continuous

2. Notation. We subdivide the given interval $\Delta:[x_0 \leqq x \leqq x_0+l]$ into n equal subintervals Δ_i: $[x_{i-1} \leqq x \leqq x_i]$, whose equal lengths l/n are denoted by $2h$ and whose respective midpoints are denoted by a_i. We denote by b_i the ordinate at the midpoint a_i of Δ_i:

$$b_i = f(a_i) = f\left(\frac{x_{i-1} + x_i}{2}\right). \tag{2.1}$$

We further denote the ordinate at the right-hand endpoint x_i of Δ_i by y_i, the derivative $f'(x_i)$ by y_i', the average of the ordinates y_{i-1} and y_i by c_i, and the average of the extreme ordinates y_0 and y_n by C_1:

$$c_i = \tfrac{1}{2}(y_{i-1} + y_i) = \tfrac{1}{2}[f(x_{i-1}) + f(x_i)]. \tag{2.2}$$

$$C_1 = \tfrac{1}{2}(y_0 + y_n). \tag{2.3}$$

The symbol s_i will be used to denote that portion of the graph of the function $y=f(x)$ which corresponds to values of x in the interval Δ_i.

The average value of $f(x)$ on the interval Δ_i will be denoted by f_i. By our definitions this may be written in the two forms

$$f_i = \frac{n}{l}\int_{x_{i-1}}^{x_i} f(x)dx = \frac{1}{2h}\int_{a_i-h}^{a_i+h} f(x)dx. \tag{2.4}$$

The important quantity A, with whose value we are concerned, is easily seen to be the exact arithmetic mean of the n quantities f_i defined by (2.4).

$$A = (f_1 + f_2 + \cdots + f_n)/n. \tag{2.5}$$

Each of our approximations to A will be the arithmetic mean of n quantities which approximate f_i, so we shall have need of the two averages B_n and C_n, defined as follows:

$$B_n = (b_1 + b_2 + \cdots + b_n)/n, \tag{2.6}$$

$$C_n = (c_1 + c_2 + \cdots + c_n)/n. \tag{2.7}$$

By using (2.2) and (2.3) we may express C_n in the more useful form

$$C_n = (y_1 + y_2 + \cdots + y_{n-1} + C_1)/n. \tag{2.8}$$

3. Linear approximations: the tangent and trapezoid rules. Either b_i or c_i may be taken as an approximation to the average value f_i of $f(x)$ on Δ_i. In the first case the actual curve s_i is approximated by the tangent drawn at the extremity (a_i, b_i) of its mid-ordinate, and the area under the curve $y=f(x)$ is approximated by the "*tangent area*" under these segments of tangent lines, one of them drawn in each interval. Since the area under the ith tangent segment is $2hb_i$, the average ordinate under this segment is b_i. The arithmetic mean B_n of these mid-ordinates b_i is an approximation to the true average A.

$$\begin{aligned} \textbf{Tangent rule: } \mathrm{A} \doteq \mathrm{B}_n &= \frac{\text{Area under tangents}}{\text{Length of interval}} \\ &= \text{Average of mid-ordinates of } n \text{ equal subintervals} \end{aligned} \tag{3.1}$$

(The symbol $\doteq$ means "is approximately equal to").

In the second case, the area under the curve $y=f(x)$ is approximated in each interval Δ_i by the "*trapezoid area*" or "*chord area*" $2hc_i$ under the chord joining the end points of the arc s_i. We thus obtain as an approximation to A the arithmetic mean C_n of the n values c_i.

$$\begin{aligned} \textbf{Trapezoid rule: } \mathrm{A} \doteq \mathrm{C}_n &= \frac{\text{Area under chords}}{\text{Length of interval}} \\ &= \frac{y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}(y_0 + y_n)}{n}. \end{aligned} \tag{3.2}$$

4. Parabolic approximation: Simpson's rule. Instead of approximating the curve s_i by a line segment, we may approximate it by a parabolic arc through the three points (x_{i-1}, y_{i-1}), (a_i, b_i), (x_i, y_i). The computation is simplified if we introduce a new independent variable t:

$$t = (x - a_i)/h, \qquad -1 \leqq t \leqq 1, \tag{4.1}$$

in terms of which the equation of the parabola may be written

$$p_i(t) = b_i + \tfrac{1}{2}(y_i - y_{i-1})t + (c_i - b_i)t^2. \tag{4.2}$$

We verify by direct substitution, using (2.2), that this function takes on the values y_{i-1}, b_i, y_i at the points $t = -1, 0, 1$, respectively. We find by direct integration that the average ordinate p_i under this curve is as follows:

$$p_i = \frac{1}{2}\int_{-1}^{1} p_i(t)dt = b_i + 0 + \frac{1}{3}(c_i - b_i). \tag{4.3}$$

The arithmetic mean of the n quantities p_i, which we denote by P_n, is usually a much better approximation to A than either B_n or C_n. It is in fact a weighted average of B_n and C_n, which *gives the exact value of A whenever the function $f(x)$ is a polynomial of degree less than or equal to three.* The value of P_n may be expressed in several ways, using (4.3), (2.6), and (2.7).

$$\text{(4.4)} \qquad \textbf{Parabolic rule: } \mathbf{A} \doteq \mathbf{P}_n = \frac{\text{Area under parabolic arcs}}{\text{Length of interval}}$$

$$= \mathbf{B}_n + \tfrac{1}{3}(\mathbf{C}_n - \mathbf{B}_n) = \tfrac{1}{3}(2\mathbf{B}_n + \mathbf{C}_n).$$

This formula may also be written in a form known as Simpson's rule:

$$\text{(4.5)} \qquad Area \doteq l(y_0 + 4b_1 + 2y_1 + 4b_2 + \cdots + 2y_{n-1} + 4b_n + y_n)/6n.$$

In a numerical example, however, it may save time to compute P_n (for $n>1$) from B_n and C_n as in (4.4), instead of using (4.5).

5. A quartic approximation. If the end values of the derivatives $f'(x)$ are known to the desired degree of accuracy (and if $f'(x)$ is assumed continuous), a still better approximation to A can be obtained with very little extra computation. We approximate the arc s_i by a quartic curve, passing through the three points (x_{i-1}, y_{i-1}), (a_i, b_i), (x_i, y_i) and also tangent to s_i at the first and third points. In terms of the variable t this polynomial $q_i(t)$ has the equation

$$\text{(5.1)} \qquad q_i(t) = p_i(t) + (t^3 - t)k_i + (t^4 - t^2)(b_i - c_i + d_i),$$

where

$$\text{(5.2)} \qquad d_i = 2h(y_i' - y_{i-1}')/8, \qquad k_i = (y_{i-1}'h + y_i'h + y_{i-1} - y_i)/4.$$

The function $q_i(t)$ is seen to coincide with $p_i(t)$ for $t=-1, 0, 1$. Furthermore, its derivative with respect to x, which is $q_i'(t)/h$, is seen to have the values y_{i-1}' and y_i' for $t=-1$ and 1, respectively. The average value q_i on the interval Δ_i is easily obtained by the integration of equation (5.1). It can be expressed as follows

$$\text{(5.3)} \qquad \begin{aligned} q_i &= p_i + 0 + (\tfrac{1}{5} - \tfrac{1}{3})(b_i - c_i + d_i), \\ &= b_i - \tfrac{1}{3}(b_i - c_i) - \tfrac{2}{15}(b_i - c_i + d_i). \end{aligned}$$

Fortunately, in forming the arithmetic mean D_n of the n quantities d_i, all the derivative values y_i' cancel out except the two end values $y_n' = f'(x_0+l)$ and $y_0' = f'(x_0)$. For our new approximation to A the only new quantity to be computed in addition to B_n and C_n is D_n:

$$\text{(5.4)} \qquad D_n = [f'(x_0 + l) - f'(x_0)]l/8n^2 = (d_1 + d_2 + \cdots + d_n)/n.$$

We then obtain as a quartic approximation to A the arithmetic mean Q_n of the n quantities q_i, which is in many cases an even better estimate of A then is P_n.

In fact, it gives A *exactly whenever* $f(x)$ *is a polynomial of degree less than or equal to five.*

$$\text{(5.5)}\quad \textbf{Quartic rule: } \mathbf{A} \doteq \mathbf{Q}_n = \mathbf{P}_n - \tfrac{2}{15}(\mathbf{B}_n - \mathbf{C}_n + \mathbf{D}_n) \\ = \mathbf{B}_n - \tfrac{1}{3}(\mathbf{B}_n - \mathbf{C}_n) - \tfrac{2}{15}(\mathbf{B}_n - \mathbf{C}_n + \mathbf{D}_n).$$

6. Estimate of the error. In each of these methods of numerical integration, the error can be estimated if the derivatives of $f(x)$ are continuous and are known throughout the interval. Let M_ν be the maximum value of $|f^\nu(x)|$ in Δ. Then the following inequalities can be shown to hold. We state them here without proof.

$$\text{(6.1)}\quad |A - B_n| < \frac{M_2}{24}\left(\frac{l}{n}\right)^2, \qquad \text{(6.2)}\quad |A - C_n| < \frac{M_2}{12}\left(\frac{l}{n}\right)^2,$$

$$\text{(6.3)}\quad |A - P_n| < \frac{M_4}{180}\left(\frac{l}{n}\right)^4, \qquad \text{(6.4)}\quad |A - Q_n| < \frac{M_6}{7!5!}\left(\frac{l}{n}\right)^6.$$

The errors obtained by use of the last two methods are seen to be small quantities of higher order in l/n than those of the first two. For a fixed value of n, however, it is possible to exhibit functions for which any chosen one of the four estimates B_n, C_n, P_n, Q_n is exactly equal to A, but the other three are different from A by as large an error as might be named. For instance, the quartic approximation is useless for a function whose derivative becomes infinite at one of the end points of the interval Δ, since D_n would then be infinite. In the examples which follow, however, it will be seen that Q_n gives the best approximation to A, and P_n the next best approximation.

7. Numerical examples. To illustrate the theory we shall compare the four approximations with the exact value of A for the two functions $1/(1+x^2)$ and $\sin x$ which can also be integrated by exact methods.

EXAMPLE 1. The integral $\int_0^1(1+x^2)^{-1}dx$ has the value $\pi/4$, $=0.7853981634$ correct to ten decimal places. Since $l=1$, we have $A=\pi/4$. Let us first choose $n=1$, and then choose $n=5$, and estimate the integral by each of the four methods. For $n=1$ we have $D_1=-1/16=-.0625$. Furthermore,

$$\text{(7.1)}\quad \begin{aligned} B_1 &= b_1 = f(\tfrac{1}{2}) = 0.8000, & C_1 &= \tfrac{1}{2}(1 + 0.5) = 0.7500, \\ P_1 &= B_1 - \tfrac{1}{3}(B_1 - C_1) = 0.7833, & Q_1 &= P_1 - \tfrac{2}{15}(-\ .0125) = 0.7850. \end{aligned}$$

Hence the errors in the four approximations are as follows:

$$\text{(7.2)}\quad A - B_1 = -0.0146, A - C_1 = 0.0354, A - P_1 = 0.0021, A - Q_1 = 0.0004.$$

The last method of approximation is obviously the most accurate in this case.

If we now take $n=5$ in the same example, we find that Q_5 is an approximation to $\pi/4$ which is accurate to eight decimal places. The computation can be arranged as follows.

$$f(x) = 1/(1+x^2), \qquad 0 \leqq x \leqq 1, \quad n = 5, \quad h = 0.1.$$

$$\begin{array}{ll}
f(.1) = .99009\ 90099 & f(.2) = .96153\ 84615 \\
f(.3) = .91743\ 11927 & f(.4) = .86206\ 89655 \\
f(.5) = .80000\ 00000 & f(.6) = .73529\ 41176 \\
f(.7) = .67114\ 09396 & f(.8) = .60975\ 60976 \\
f(.9) = .55248\ 61878 & C_1 = .75000\ 00000 \\
\hline
5B_5 = 3.93115\ 73300 & 5C_5 = 3.91865\ 76422
\end{array}$$

$$\begin{array}{llll}
(7.3) & B_5 = .78623\ 14660 & D_5 = -\ .00250\ 00000 & \\
 & C_5 = .78373\ 15284 & B_5 - C_5 = .00249\ 99376 & -(B_5 - C_5 + D_5) = .00000\ 00624 \\
 & \textit{Divide by } 3: & B_5 - P_5 = .00083\ 33125 & .00000\ 00208 \\
 & P_5 = .78539\ 81535 & & Sum/10 = Q_5 - P_5 = .00000\ 00083 \\
 & Q_5 = .78539\ 81618 & A = .78539\ 81634 \text{ (by integration).} &
\end{array}$$

The errors in the four approximations are as follows:

$$\begin{array}{lll}
(7.4) & A - B_5 = -\ .00083\ 33026, & A - C_5 = .00166\ 66350, \\
 & A - P_5 \doteq .00000\ 00099, & A - Q_5 = .00000\ 00016.
\end{array}$$

As a check in the computation, it should be noticed that D_n is nearly equal to $C_n - B_n$ whenever the problem is such that the quartic approximation is a good one. If this condition is not satisfied, the computation should be checked to see if mistakes have been made.

EXAMPLE 2. Let us consider next the integral $\int_0^\pi \sin x\, dx$, whose exact value is 2. Taking $n=3$, we can easily compute the required values of $f(x) = \sin x$ in terms of radicals. We have $l=\pi$, $A=2/\pi$, $D_3 = -\pi/36$. Hence

$$3B_3 = (\tfrac{1}{2} + 1 + \tfrac{1}{2}) = 2, \qquad 3C_3 = (\tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\sqrt{3} + 0) = \sqrt{3}.$$

$$\begin{array}{lll}
B_3 = .66666\ 667 & D_3 = -\ .08726\ 646 & \\
C_3 = .57735\ 027 & B_3 - C_3 = .08931\ 640 & B_3 - C_3 + D_3 = .00204\ 993 \\
\textit{Divide by } 3: & B_3 - P_3 = .02977\ 213 & .00068\ 331 \\
P_3 = .63689\ 453 & & Sum/10 = P_3 - Q_3 = .00027\ 332 \\
Q_3 = .63662\ 121 & A = .63661\ 977 \text{ (by integration).} &
\end{array}$$

The errors in the four approximations are as follows:

$$\begin{array}{lll}
(7.6) & A - B_3 = -\ .03004\ 690 & A - C_3 = 0.5926\ 950 \\
 & A - P_3 = -\ .00027\ 476 & A - Q_3 = -\ .00000\ 144.
\end{array}$$

In this particular example, it is easy to compare these errors with the estimates given in §6, since $M_2 = M_4 = M_6 = 1$, $l/n = \pi/3 = .10472$. These estimates are as follows:

$$\begin{array}{lll}
(7.7) & |A - B_3| < (\pi/3)^2/\ 24 = .045693, & |A - C_3| < .091386, \\
 & |A - P_3| < (\pi/3)^4/180 = .006681, & |A - Q_3| < .00000218.
\end{array}$$

It is obvious that in this example the quartic approximation is decidedly better than the other three.

8. Some exact averages. Problems in which the parabolic approximation P_1 gives the exact average A occur frequently in applications of integration. In the **prismoidal volume formula,** the volume of a solid is expressed as the product of the height by a certain average area of cross section:

$$\text{(8.1)} \qquad \textit{Volume} = \textit{Height } (\textit{Lower base} + 4 \textit{ Mid-section} + \textit{Upper base})/6.$$

The average is precisely the average P_1, and the formula is exact for segments of such solids as cylinders, cones, spheres, wedges, prisms, and pyramids, for which the area of cross section is a polynomial of degree less than or equal to three in the distance from the lower base.

The problem of finding the moment of inertia of a straight rod or pipe of uniform density about an arbitrary axis in space is just that of integrating a squared distance r^2, which may be fairly complicated to express algebraically, but which is just a quadratic function of the arc length. The average value of r^2, which is called the square (k^2) of the radius of gyration, is given exactly by the parabolic approximation P_1, namely

$$\text{(8.2)} \qquad k^2 = (r_0^2 + 4r_m^2 + r_1^2)/6,$$

where r_0, r_1, and r_m are the respective distances to the axis from the end points and midpoint.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The department of Discussions and Notes is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A SIMPLE CRITERION FOR RATIONAL ROOTS

R. S. UNDERWOOD, Texas Technological College

This note calls attention to some tests for rational roots of algebraic equations which are simple and sweeping enough to be worthy of inclusion in any elementary algebra which has a chapter on the theory of equations.

Suppose c/d, $(c, d) = 1$, is indicated by the theorem on rational roots as a possible root of the algebraic equation with integral coefficients,

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0.$$

With this notation, two theorems and a corollary may be stated briefly.

THEOREM 1. *We may reject c/d as a root of $f(x) = 0$ if $d - c$ is not a divisor of $f(1)$.*

This is a corollary of the general theorem given by L. E. Dickson in his *New First Course in the Theory of Equations*, page 29. It may be stated explicitly as follows: If c/d, $(c, d) = 1$, is a rational root of the algebraic equation $f(x) = 0$ with

integral coefficients, then $dm-c$ divides $f(m)$, m an integer.

COROLLARY. *We may reject c/d as a root of $f(x)=0$ if $f(1)$ is odd and $d-c$ is even.*

THEOREM 2. *If a_0, a_n, and $f(1)$ are odd, $f(x)=0$ has no rational roots.*

For, by the premises and the theorem on rational roots, c and d are odd, so that $d-c$ is even and the above Corollary applies.

In practice when Theorem 2 does not apply at once (and it evidently will apply only about one time in eight with a random choice of coefficients), the student should express $f(1)$ as the product of its prime factors. He can then use the Corollary and Theorem 1 in that order to reject by inspection and in wholesale lots many of the root-possibilities which he often tests laboriously via synthetic division.

A SUGGESTION FOR A SIMPLIFIED TRIGONOMETRY

A. A. ALBERT, University of Chicago

The Law of Cosines is not well adapted to the solution of an oblique triangle in which two sides and the included angle are given, and so the solution is usually accomplished by the use of the Law of Tangents. This latter procedure involves several auxiliary computations and is particularly undesirable in the case where the unknown angles are very nearly equal. There is, however, a little used third method which has some distinct advantages over the other two. It may be presented as follows.

We use the diagrams

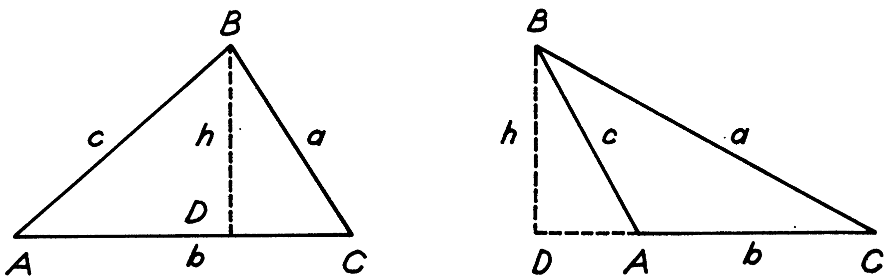

FIG. 1

In either figure

$$\cot A = \frac{AD}{h} = \frac{b - DC}{h} = \frac{b - a\cos C}{a \sin C}.$$

But then we have

$$\cot A = \frac{b}{a \sin C} - \cot C, \tag{1}$$

and the principal step in the solution has been accomplished. The other angle $B=180°-(A+C)$, but we may check the results by the use of the symmetrical formula

$$\cot B = \frac{a}{b \sin C} - \cot C. \tag{2}$$

The determination of A by the use of (1) requires the logarithmic computation of the quotient $b/(a \sin C)$ followed by two applications of a table of natural functions. The check formula requires relatively little additional tabular work. Formula (1) is comparable in derivation to the Law of Sines, and the corresponding computations begin with logarithmic work of precisely the same nature as in the Law of Sines.

Besides the application of (1) to triangle solution it may be combined with the Law of Sines to yield the formula for $\sin(A+C)$. It is customary to derive the latter formula by a complicated figure and only in the case where A, C, $A+C$ lie between 0° and 180°. Then A and C are the angles of a triangle in which $B=180°-(A+C)$, $\sin B=\sin(A+C)$. We use the Law of Sines to replace b/a by $\sin B/\sin A$ and have

$$\cot C + \cot A = \frac{\sin B}{\sin A \sin C}$$

Solving for $\sin B$ we have

$$\sin(A+C) = \sin A \sin C \left(\frac{\cos C}{\sin C} + \frac{\cos A}{\sin A}\right) = \sin A \cos C + \cos A \sin C,$$

as desired. To derive the formula for $\cos(A+C) = -\cos B$ we use the symmetrical formula

$$\sin(A+B) = \sin C = \sin A \cos B + \cos A \sin B.$$

Then

$$\begin{aligned}\sin A \cos(A+C) &= \cos A \sin B - \sin C \\ &= \cos A(\sin A \cos C + \cos A \sin C) - \sin C(\sin^2 A + \cos^2 A) \\ &= \sin A(\cos A \cos C - \sin C \sin A)\end{aligned}$$

and we have the desired formula. The extension to general angles as usual is carried out analytically.

As a result of the procedure outlined above a simplified trigonometry may be constructed. The first four chapters on angular measurements, functions of a general angle, logarithms, functions of large angles might have the usual content. Chapter five would have as subject matter the solution of *all* triangles. It would contain formula (1) and the Law of Sines which would suffice for all cases of the oblique triangle save the relatively rare three sides case. The Law of Cosines

is well adapted to machine computation in this case and would be used.* Our final chapter would include all aspects of trigonometric analysis and would begin with the non-geometric derivation of the addition formulae presented above. This plan would seem to provide a more closely knit theory than any heretofore presented.

AN INSTRUMENT FOR DRAWING CONFOCAL ELLIPSES AND HYPERBOLAS†

R. M. SUTTON, Haverford College

In the physical problem of sound-ranging to find the location of a distant gun, the usual procedure consists in determining the time of arrival of sound

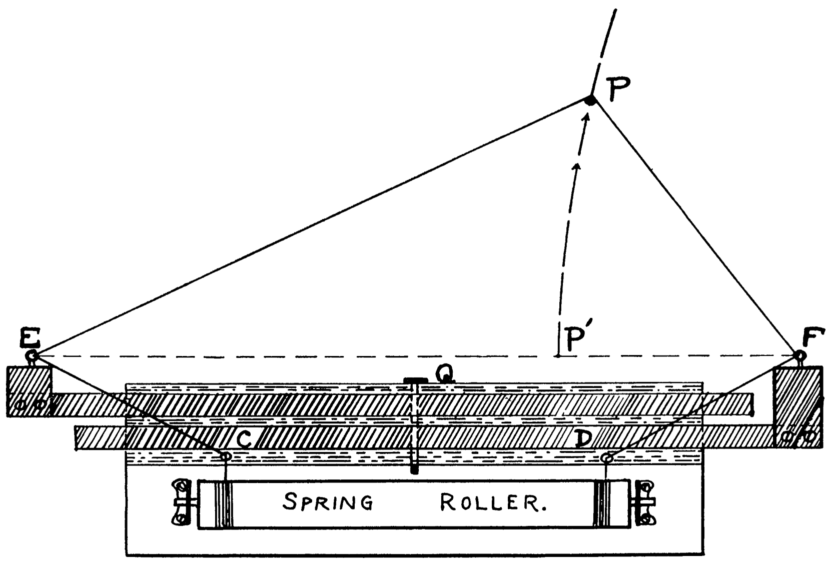

FIG. 1

from the gun at each of three or more stations. From the *difference* in time at any two stations, and from the known velocity of sound, it is possible to determine a hyperbolic curve with the two stations as foci, on which curve the source must lie; and if two or more such curves are determined, their intersection must coincide with the source of sound. To illustrate the manner of solving this problem graphically, the writer has made a simple instrument by which hyperbolas and ellipses may be drawn rapidly. As the idea of drawing such curves by string and chalk is very old, the merit of this arrangement must be sought, not in new principles, but in its convenience and flexibility.

On a plywood board $12'' \times 3''$ is mounted a window-shade spring roller with

* For use with tables of logarithms and natural functions it would be written in the form $\cos A + 1 = (a+b+c)(b+c-a)/2bc$.

† Shown at the Washington Meeting of the American Mathematical Society, May 3, 1941.

rachet removed (see figure) to which is tied an eight-foot length of fishline in two strands. If, now, the chalk is fixed to the line at any point P' and the line is pulled out, it pays out string at the same rate from the two sides of the roller through the small eyelets C, D, E, and F; the latter two eyelets are the foci of the hyperbola PP' described by P whose path has a constant *difference* of distance $(EP-PF)$ from the focal points E and F. It is a simple matter, with a little practice, to hold the chalk firmly against the cord and to move the point P out along its path of mechanical constraint under the tension of the roller. The eccentricity of the curve can be selected at will, and a whole family of confocal hyperbolas can be drawn in rapid succession by simply shifting the chalk on the string. Only one-half of each curve can be drawn at a time and the instrument must be rotated 180° to draw the second half.

For the drawing of ellipses, the roller can be held by the thumb of the left hand while the chalk, held in the right hand, traverses half of an ellipse as it slides along the cord. In this case, of course, the loop of cord is of fixed length during the drawing of any one ellipse, but the eccentricity of the ellipse may be changed readily by altering the length of loop within which the chalk is free to move. It is thus possible to draw in a few minutes a whole family of orthogonal confocal hyperbolas and ellipses. The distance between foci EF, may be changed quickly by removing a pin at Q so as to slide the two arms which bear the focal eyelets E and F in or out until the desired spacing is obtained. In the apparatus illustrated this distance can be altered from 6″ to 30″ in steps of one inch.

In physics, the rectangular hyperbola, $xy=k$, is of particular interest because many physical laws are reducible to equations of such a form. If one desires to graph the equation $xy=k$ on horizontal X- and vertical Y-axis, the instrument must be set at 45° to the X-axis and a family of curves all of the *same eccentricity* $(\sqrt{2})$ may then be generated by changing the focal distance each time a new member of the family is drawn. This is, however, a little more time-consuming than the process of drawing the confocal set previously described.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

Mathematics for Technical Training, Calculus. By P. L. Evans. Boston, Ginn and Company, 1942. 7+126 pages. \$1.25.

This book is the third in a series. It comprises both differential and integral calculus and their ordinary applications. As to pages it is indeed a brief course, but it is not abbreviated when it comes to the topics usually considered in a

standard course. There is no lack of problems, whether they be problems for guidance, worked out and explained, or the type of problems suitable for the home study of the reader, or for drill in the classroom. The interested student will be pleased to find that the answers are given for a sufficient number of the problems.

The text begins with a table of contents and closes with an index. Furthermore, the first chapter is a summary of formulas from the field of the prerequisite mathematics, and the last thirteen pages are given to a rather complete table of integrals. Principles are stated, but attempts at proofs, or derivations of the basic formulas are omitted. The book may also be considered as a suitable source for extra problems for the mastery of techniques. It is pleasing in its appearance and its arrangement.

H. T. R. AUDE

Mathematics of Modern Engineering, Vol. II. By E. G. Keller. New York, John Wiley and Sons, 1942. 12+309 pages. $4.00.

This book is one of a series appearing under the imprimatur of the General Electric Company. It is the second of two volumes bearing the same title and, according to its author, its aim is not so much to teach mathematics as to present to the practicing engineer various mathematical disciplines that are effective in the solution of certain engineering problems. Those of us who had to defend or explain the "practicality" of mathematics to engineering faculties and students cannot but feel grateful for the service being rendered by the author. The present book is divided into three broad chapters that may be labeled (the labeling is the reviewer's): Mathematico-Physical Principles; Matrices and Tensors; Differential Equations. The first chapter deals with the elements of the Calculus of Variations, Hamilton's principle, Lagrange's equations for holonomic and non-holonomic systems, Rayleigh's principle and classical vector analysis. The applications in this chapter are to dynamical systems and such problems as forced vibrations and oscillatory systems are treated.

The second chapter is devoted to matrices and tensors primarily as they are used in electrical networks. The definitions may seem unnecessarily restricted, but this may be justified by the objectives of the book.

The third chapter is concerned with the solution of systems of total differential equations, especially series solutions, their convergence and the various methods of approximate and numerical solutions. There is also a very brief account of elliptic and hyperelliptic functions and of Lalesco's non-linear integral equation. The applications are to electric circuits and rotating machines.

The reviewer feels that he would be derelict in his duty if he failed to point out many faults which the book possesses. The eternal mystery of the differential of a function is still unresolved in this book (p. 9) and even the definition of such a fundamental concept as potential energy is misstated (p. 19). One can hardly indorse "guessing" as one method suggested for solving an algebraic equation (p. 76) or "observing" that $x_2=4x_1/3$ minimizes w_a^2 and remarking that "This is a good estimate, since the exact value of x_2 is $4x_1/3$ (p. 87). It may be a matter

of taste that a great deal of space is devoted to dominant series and dismissing Picard's method by a trivial example without even mentioning the name, but one should object to equating (p. 233) the fourth approximation (a polynomial) to the solution (an exponential function).

Some of the "explanations" and definitions are either very much confused or are incorrect. There is no room—in the reviewer's opinion—in any book for such a statement as (p. 222). "We may think of a_i as constants, since they are the n arbitrary constants of the solution, or as variables which in turn have constant values for some specified values."

Many theorems or expressions which are valid only under certain restrictions are used without even mentioning what these restrictions are; nowhere in the book was the reviewer able to find the restriction necessary for the existence of the inverse of a matrix. Stating and using Cramer's rule without saying a word about the non-vanishing of the determinant of the coefficients is neither mathematics nor good engineering.

The reviewer found over forty misprints, and numerous statements, which, if not meaningless, are at least controversial. Many of the misprints are doubtless slips, but a number of them cannot be so explained. The whole section 2.25 is incorrect from its title to its emphasized statement (p. 127). The impedance tensor Z' on p. 151 is not the sum of the two tensors Z_1' and Z_2' as the sum is defined on p. 122, but the composition or direct sum of the two. The statement (p. 196) "shortest lines $\cdots$ in a curved affine space" is either nonsense or a misprint.

In spite of these blemishes, this book, as well as some of the others of this series, deserve wide circulation not only in industry but also in the engineering schools, and the reviewer hopes that books of this type—more carefully written—will become more frequently used.

M. S. KNEBELMAN

Calculus. By G. E. F. Sherwood and A. E. Taylor. New York, Prentice-Hall, Inc., 1942. 14+503 pages. $3.75.

This book is somewhat unusual in presenting a practical motivating "Foreword to the Student" which even precedes the Greek alphabet and the usual elementary formulas for reference.

The purpose of this text is "to set forth in systematic and thorough manner the fundamental principles, methods, and uses of the calculus."

If the engineering student can postpone one semester the use of integrals in his other courses he will find this an excellent text. The definite integral is started on p. 183 and precedes the formal treatment of indefinite integrals beginning in Chapter X on p. 198.

This text appeals to the reviewer as a most successful attempt to inject modern rigor into a practical first course for the sound training of the prospective engineer, mathematician, or scientist.

Limiting processes are emphasized from the start. Use is made of Cauchy's Principle of Convergence, uniform continuity, and Duhamel's Principle as enunciated by Osgood without mention of the word infinitesimal.

The authors are careful to state when assumptions are made and to state a theorem, even when the proof must be omitted, in order to secure accuracy of theory. For instance a careful distinction is made between double and iterated integrals. The Jacobian condition is stated in connection with implicit functions. Term by term integration is said not always to be valid for series which are not power series. Likewise term by term differentiation is not taken for granted even for power series.

Although the better students will appreciate the proofs of the theorems on limits of a sum, product and quotient, they may be omitted in a brief course, or if otherwise desired. The same holds true for several other sections such as those on evolutes, on roulettes, on the existence of an integral, on the osculating plane, and on envelopes.

As mentioned in the preface the chapters on hyperbolic functions, further methods of integration, infinite series, and Taylor's series, which occur rather late, might be taken up much earlier, just after Chapter X, or Chapter XI on geometrical applications. In fact part of Chapter XVI on hyperbolic functions could be studied just after Chapter V on differentiation of transcendental functions.

One valuable feature of the book is a thirty-three page chapter on solid analytic geometry. Another is the chapter summary outlining the important new ideas in each chapter and followed by a list of problems involving them. These lists are given in addition to the usual exercises under the separate topics. A majority of the answers are given but a sufficient number are omitted to invite the students to test their independence.

The format of the book is very pleasing. The paper is smooth, the print clear, and the figures are unusually good. There are over 138 numbered figures only a few of which might be improved, *e.g.*, 86a, 86b, 87a, 87b, on pp. 297, 298. A few defects in printing were noticed. On page 146 in the answer to problem 12 "A" is omitted; on p. 194 the fifth line from the bottom should end with -1; on p. 341, near the middle, the partial derivative of v lacks an "x"; on p. 500 under "indeterminate forms" 1^0 should be replaced by 1^∞. On p. 205, footnote, the statement that "log sin x has no meaning" is not quite accurate.

The book abounds in applications to Physics. Among these is an unusually full treatment of gravitational attraction. Another leads to the elliptic integral, which is handled in a practical way by infinite series.

One might have expected some discussion of the error in Simpson's rule. On the other hand one would not expect to find the infinite product for π derived from Wallis' integral formulas as given on page 382.

Preceding a good index of seven pages is a table of over a hundred integrals. The treatment of the limit e is properly made by considering the graph of

$(1+t)^{1/t}$ near $t=0$ and relegating an interesting existence proof to the back of the book as a four page appendix.

C. C. CAMP

Laboratory Geometry. By Elizabeth Roudebush. New York. Prentice-Hall, Inc., 1942. 192 pages. $1.12.

Laboratory Geometry is a combination text and work book designed for students at high school level. The book is a spiral note book, and therefore many of the exercises and constructions may easily be worked out in the spaces provided in the text. There are a great many illustrative figures; the print is large and very easy to read.

In Unit I the author has most successfully introduced the subject by the consideration of geometric figures: lines, angles, circles and triangles. The student's attention will be held by his active participation in supplying missing words and in performing simple constructions. The drawings are very instructive.

The idea of geometric proof, the axioms and postulates as well as the fundamental theorems on straight lines are included in Unit II. The proofs of the theorems are so well motivated that the student will be able to grasp the whole point of geometry more easily than he would, had the approach been purely axiomatic. Congruent triangles, polygons, and locus problems are dealt with in Units III, IV, and V respectively.

Excellent summaries at the end of each Unit list the definitions, axioms, postulates and theorems contained in the unit. A review of the first five units includes many practical problems. Circles, ratio and proportion, similar polygons, areas and regular polygons are treated in the Second Half of the book. In the unit on similar polygons a table of sines and tangents for degree intervals is included to enable the student to solve simple right triangles.

In my opinion the book is excellent both from the stand-point of teaching and material covered. Throughout, the ideas that are most difficult for the student to grasp are very carefully introduced in a conversational style by using worth while every day illustrations. The author is to be congratulated for the writing of Laboratory Geometry.

ELMER TOLSTED

NEW BOOKS RECEIVED

Mathematical Recreations. By M. Kraitchik. New York, W. W. Norton and Co., Inc., 1942. 328 pages. $3.75.

Seven place values of Trigonometric Functions for every thousandth of a degree. By J. Peters. New York, D. Van Nostrand Co., Inc., 1942. 376 pages. $7.50.

A Primer of Formal Logic. By J. C. Cooley. New York, Macmillan Co., 1942. 11+378 pages. $3.00.

Tables of arc tan *x*. New York, Work Projects Administration, 1942. 25+173 pages. $2.00.

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PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 566. *Proposed by Michael Wilensky, Cincinnati, Ohio*

Suppose there is a principle according to which a claimant gets a share of the contestable thing, proportionate to his claim; so that when one of two claimants claims the whole, while the other claims only half of it, the former gets three quarters (*viz.*, the incontestable half, and half the contestable half) and the latter gets one quarter (which is half his claim). Find a formula for the share of each of n claimants, when the kth claimant claims $1/k$ of the entity $(k=1, 2, \cdots, n)$.

E 567. *Proposed by V. Thébault, San Sebastián, Spain*

Using compasses alone, construct a regular polygon of thirty sides.

E 568. *Proposed by P. D. Thomas, U. S. Coast and Geodetic Survey, Lucedale, Miss.*

In a given triangle show that the radical axes of the circumcircle with the respective circles whose diameters are any three concurrent Cevians meet the corresponding sides in three collinear points. (Cf. E 467 [1942, 63].)

E 569. *Proposed by David Matlack, Grinnell College*

Through a fixed point A on a circle (O), a line is drawn, parallel to a variable radius OP, meeting the circle again at Q. Find the envelope of the chord PQ.

E 570. *Proposed by L. M. Kelly, U. S. Coast Guard Academy*

If the six conics determined by each five of a set of six points are congruent, must they coincide?

SOLUTIONS

Circles through the Gergonne Point

E 527 [1942, 404]. *Proposed by V. Thébault, San Sebastián, Spain*

Show that the sum of the radii of the circles C_1, C_2, C_3 of E 457 is equal to the diameter of the incircle, and that the sum of the radii of the three analogous circles whose centers are exterior to the segments A_iI is three times as great.

Solution by Howard Eves, Syracuse University. Let us use the notation of E 457 [1941, 637], denoting the radius of C_i by r_i and the radius of the incircle by r. We shall use bars to designate the corresponding elements for the analogous set of circles.

Now we have $r_i/r = A_iP/A_iP_i$. But, from an elementary theorem on concurrency, $\sum A_iP/A_iP_i = 2$. Therefore $\sum r_i = 2r$.

Since C_iP and IP_i are corresponding lines of homothetic figures it follows that they are parallel, whence C_iP is perpendicular to A_jA_k. This property applied to C_j and C_k is sufficient to guarantee that A_{ji}, P, A_{ki} are collinear, whence $A_{ji}A_{ki}$ is the radical axis of C_i and $\overline{C}_i$. Therefore $A_{ji}A_{ij} = A_{ji}\overline{A}_{ij}$, or, since $A_{ji}P_k = P_kA_{ij}$, we have $P_k\overline{A}_{ij} = 3P_kA_{ij}$. From this it readily follows that $\bar{r}_i = 4r - 3r_i$, and finally $\sum \bar{r}_i = 6r$.

Note. If Q_i is the other point of intersection of A_iP_i with the incircle, we have incidentally shown that $\sum A_iP/A_iQ_i = 6$, and that the chords cut off on A_iP_i by C_i and $\overline{C}_i$ are quadrisected by Q_i and P_i, respectively.

Also solved by the proposer, using trigonometry.

An Orthocentric Group of Lines

E 533 [1942, 475]. *Proposed by N. A. Court, University of Oklahoma*

Prove that, if an orthocentric group of points occurs as a section of an orthocentric group of lines, then the plane of section is perpendicular to one of the lines.

Solution by the Proposer. Let A, B, C, H be the traces of the orthocentric group of lines DA, DB, DC, DH in a plane ABC. By assumption, the points A, B, C, H form an orthocentric group.

If the plane ABC is perpendicular to one of DA, DB, DC, the proposition is proved. We assume, therefore, that this is not the case, and shall show that the plane ABC is then necessarily perpendicular to DH.

If AA' is the perpendicular from A upon the plane DBC, and AA'' the perpendicular from A upon the line BC, then BC is perpendicular to the plane $AA'A''$. Now, since the four lines DA, DB, DC, DH are orthocentric, the plane DAH, perpendicular to DBC, contains AA'. Again, since the four points A, B, C, H are orthocentric, the perpendicular AA'' passes through H, and so lies in the plane DAH. Consequently the plane $AA'A''$ is identical with DAH. This plane is perpendicular to the line BC, and therefore to the plane ABC. In a like manner it may be shown that the planes DBH and DCH are each perpendicular to ABC. This proves, superabundantly, that DH is perpendicular to ABC.

Note. This is the converse of a known proposition. See the proposer's *Modern Pure Solid Geometry*, 1935, p. 28, art. 69.

An Unsolved Problem

E 534 [1942, 475]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Show that 4, 5, 7 are the only values of n for which $n!+1$ is a perfect square.

Remark by Paul Erdös, University of Pennsylvania. It would be very hard to prove even that $n!+1$ cannot be a perfect fourth power. In this direction, Obláth and I have proved the following results:

(1) The Diophantine equation $n!\pm m!=x^k$ has only a finite number of solutions if $n\geqq m>1$.

(2) The equation $n!=x^4\pm y^4$ has no solutions with n greater than a certain definite number.

(3) If x and y are relatively prime, $n!=x^k\pm y^k$ has no solutions for odd values of k, nor for $k=2^l$ where $l>2$.

Concurrent Simson Lines

E 535 [1942, 475]. *Proposed by A. H. Stone, Institute for Advanced Study*

Let A', B', C' be three points on the circumcircle of a triangle ABC, whose Simson lines with respect to ABC all meet in a point, O. Prove that the Simson lines of A, B, C, with respect to the triangle $A'B'C'$, concur at the same point O.

Solution by Howard Eves, Syracuse University. Let us designate the Simson lines with respect to ABC of A', B', C' by α', β', γ', and the Simson lines with respect to $A'B'C'$ of A, B, C by α, β, γ. We first establish a condition for α, β, γ to concur. The numbers in parentheses refer to articles in R. A. Johnson's *Modern Geometry*, 1929, where a treatment of this problem is outlined (338).

LEMMA. *A necessary and sufficient condition for α, β, γ to concur is that α be perpendicular to BC, β to CA, and γ to AB.*

(i) Suppose α, β, γ concur at a point O. Let H' be the orthocenter of $A'B'C'$, and produce $H'O$ its own length to H. Now, since α bisects both AH' (327) and HH', it follows that α is parallel to AH. Similarly β is parallel to BH, and γ to CH. Therefore

$$\measuredangle\, BHC = \measuredangle\, (\beta\gamma) = \measuredangle\, CAB$$

(326, second corollary). (We are here using Johnson's *directed angles* (16). In the ordinary notation $\angle BHC = 180° - A$, with a suitable modification if B or C happens to be obtuse.) Similarly we have

$$\measuredangle\, CHA = \measuredangle\, ABC \quad \text{and} \quad \measuredangle\, AHB = \measuredangle\, BCA.$$

Hence H is the orthocenter of ABC. Since α is parallel to AH, it then follows that α is perpendicular to BC, with similar remarks for β and γ.

(ii) Suppose α is perpendicular to BC, β to CA, γ to AB. Let H be the orthocenter of ABC, and let O be the midpoint of HH'. Now α is parallel to AH and bisects AH'. Therefore α passes through O. Similarly β and γ pass through O. Thus α, β, γ concur.

We use the above lemma to show that the concurrence of α', β', γ' implies the concurrence of α, β, γ (at the same point O). Let the isogonal of $A'A$ with respect to angle A' be $A'A''$ (with A'' on the circumcircle). Then α is perpendicular to $A'A''$ (326, Theorem). It is readily seen that AA'' is parallel to $B'C'$, whence by the lemma, AA'' is perpendicular to α'. This guarantees that AA'' and AA' are isogonals with respect to angle A, so that $A'A''$ is parallel to BC, whence α is perpendicular to BC. Similarly β is perpendicular to CA, and γ to AB. This proves that α, β, γ concur. As in (ii), then, it follows that the point of concurrence is the midpoint of HH', where H and H' are the orthocenters of ABC and $A'B'C'$.

A Needle in a Bowl

E 536 [1942, 546]. *Proposed by Norman Miller, Queen's University*

In a smooth hemispherical bowl of radius a, a smooth needle of length $2l$ ($l < a$) is placed with one end projecting over the rim and is then released. Show that the needle will come to rest in a horizontal position if l is less than $a\sqrt{\frac{2}{3}}$.

Solution by Howard Eves, Syracuse University. Let A and B be the ends of the needle, M its midpoint, and O the center of the bowl. Three situations are possible when the needle comes to rest: (1) A and B may both be within the bowl; (2) A, say, may be within the bowl while B projects outside the rim; (3) A may be within the bowl and B just on the rim.

In (1) the needle is acted on by three forces: those at A and B, each acting normal to the bowl (and hence through O), and one acting vertically at M. Since the needle is in equilibrium, these three forces are concurrent. But for the vertical force at M to pass through O we must have the needle horizontal.

In (2) the needle is again acted on by three forces: one at A, normal to the bowl, a second at C (where the needle touches the rim) normal to the needle, and a third acting vertically at M. Again the three forces must be concurrent, and from the geometry of the situation the point of concurrence is D, diametrically opposite to A on the sphere determined by the bowl. Here, of course, the needle is inclined.

In (3) the needle is in unstable equilibrium.

To find the condition on l leading to the three cases, set $MC=m<l$, $DC=n$, $\angle BAD=\angle CDM=\theta$. Then

$$n/(l+m) = \tan\theta = m/n,$$

whence $n^2=(l+mm)$. Now, by Pythagoras' theorem, we have

$$4a^2 = (l+m)^2 + n^2 = (l+m)(l+2m) < 6l^2,$$

whence $l>a\sqrt{\frac{2}{3}}$. This is the condition for situation (2). If $l=a\sqrt{\frac{2}{3}}$ we have (3), and if $l<a\sqrt{\frac{2}{3}}$ we have (1).

For an alternative method of approach see Bowser's *Analytic Mechanics*, p. 82, Ex. 9.

Remark by W. B. Carver, Cornell University. Suppose l lies between $a\sqrt{\frac{2}{3}}$ and a. If the needle is released from a sufficiently inclined position, it will slide past the position of equilibrium with sufficient momentum to slip entirely into the bowl, and will come to rest in the horizontal position, as in case (1).

Also solved by J. H. Butchart, L. M. Kelly, W. O. Pennell, K. Yamakawa, and the proposer.

Editorial Note. In view of Professor Carver's remark, the wording of the problem has been slightly changed since its first appearance.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4078. *Proposed by Raphael Robinson, Univ. of California at Berkeley*

Show that the continued fraction

$$\frac{1}{2^{1!}+}\,\frac{1}{2^{2!}+}\,\frac{1}{2^{3!}+}\,\frac{1}{2^{4!}+}\cdots,$$

represents a transcendental number by using the following theorem of Liouville, *Journal de Mathématiques pures et appliquées*, 1851.

The denominator of a convergent of a continued fraction representing the root x of an nth degree algebraic equation with rational coefficients never exceed the product of a certain constant by the $(n-2)$nd power of the denominator of the preceding convergent.

4079. *Proposed by S. Beatty, Univ. of Toronto*

If two complexes, in projective n-space, are reciprocal with respect to a quadric, so that each vertex of one is the pole of a bounding hyperplane of the other, then the $n+1$ lines joining corresponding vertices are associated,* in the sense that every $(n-2)$-space which meets n of them meets the remaining one also. Reciprocally, the $n+1$ of the $(n-2)$-spaces of intersection of corresponding hyperplanes are such that every line which meets n of them meets the remaining one also.

4080. *Proposed by V. Thébault, San Sebastián, Spain*

Given five points in space, if five parallel forces applied at these points are in equilibrium, then equilibrium results also when each force is translated to the center of the circumsphere of the tetrahedron which has for vertices the points of application of the four other forces.

Note. The similar problem for four points in a plane was proposed by J. Neuberg, *Educational Times*, 1891, vol. 55, p. 82.

4081. *Proposed by Otto Dunkel, Washington Univ.*

Through the vertices of the triangle ABC parallels $A\alpha$, $B\beta$, $C\gamma$ of arbitrary direction are drawn meeting the transversal Δ in the points α, β, γ; and through the latter points straight lines are drawn parallel respectively to BC, CA, AB rotated through the angle θ, thus forming a triangle $A_1B_1C_1$ similar to ABC. Prove that: (1) As the direction of the parallels varies the vertices A_1, B_1, C_1 describe straight lines concurrent in a point $\phi(\theta)$. (2) For the particular set of parallels $A\alpha$, $B\beta$, $C\gamma$ which have the direction of Δ rotated through the angle $-\theta$, the triangle $A_1B_1C_1$ reduces to the point $\phi(\theta)$. (3) The locus of $\phi(\theta)$ is a unicursal cubic passing through the circular points at infinity, the point at infinity of the Newton line (ABC, Δ), the orthopole of Δ with respect to ABC, and A_0, B_0, C_0, the points of intersection of the sides of ABC with Δ.

SOLUTIONS

Miquel Circle and Point

3890 [1938, 554]. *Proposed by V. Thébault, San Sebastián, Spain*

Given four straight lines Δ_1, Δ_2, Δ_3, Δ_4, in a plane; through the orthogonal projections of the vertices of the triangles $T_1\equiv(\Delta_2, \Delta_3, \Delta_4)$, $T_2\equiv(\Delta_1, \Delta_3, \Delta_4)$, $T_3\equiv(\Delta_1, \Delta_2, \Delta_4)$, $T_4\equiv(\Delta_1, \Delta_2, \Delta_3)$ on Δ_1, Δ_2, Δ_3, Δ_4, respectively, parallels are

* For example, when $n=3$, the four lines are generators of a regulus. The case where $n=2$ is Hesse's theorem.

drawn to the sides opposite the corresponding vertices of the triangle considered: these parallels determine four other triangles T_1', T_2', T_3', T_4' symmetrically equal to the first. (1) Show that the Miquel circles of the quadrilaterals (T_1', Δ_1), (T_2', Δ_2), (T_3', Δ_3), (T_4', Δ_4) are equal to the nine-point circles of T_1, T_2, T_3, T_4 and tangent to the circumcircles of T_1', T_2', T_3', T_4'. (2) Show that the Miquel points of the above quadrilaterals are collinear.

Solution by the Proposer. (1) The first part of this problem is a direct application of the following:

THEOREM. *The parallels to the sides BC, CA, AB of a triangle ABC drawn through the orthogonal projections α, β, γ of the vertices A, B, C on any given straight line Δ of the same plane determine a triangle $A_1B_1C_1$ symmetrically equal to ABC. The Miquel circle of the quadrilateral $[A_1B_1C_1, \Delta]$ is equal to the nine point circle of ABC and it is tangent to the circumcircle of $A_1B_1C_1$ at the orthopole ϕ of Δ with respect to ABC. Also the respective symmetrics of the vertices A_1, B_1, C_1 with respect to the circumcenters of triangles $A_1\beta\gamma$, $B_1\gamma\alpha$, $C_1\alpha\beta$ coincide with the point ϕ.* (V. Thébault, *Mathesis*, vol. 49, 1935, p. 298.)

Proof. Suppose first that Δ' is a diameter of the circumcircle (O) of triangle ABC parallel to Δ. The parallels to the sides BC, CA, AB drawn through the orthogonal projections α', β', γ' of the vertices A, B, C on Δ' determine a triangle $a_1b_1c_1$ symmetrically equal to triangle ABC.* We have also shown that the orthopole ϕ' of the diameter Δ' for triangle ABC is the common symmetric of the points a_1, b_1, c_1 with respect to the midpoints A_m, B_m, C_m of the sides of triangle ABC, and that it is the point of contact of the circumcircle (O_1) of $a_1b_1c_1$ with the nine point circle (O_9) of ABC.† From this we infer that the circle (O_9) is the Miquel circle for the quadrilateral $[a_1b_1c_1, \Delta']$. If we now translate Δ' with the attached triangle $a_1b_1c_1$ by the vector δ perpendicular to it to the parallel position Δ, the triangle $a_1b_1c_1$ assuming the position $A_1B_1C_1$, the latter triangle has its sides parallel to the corresponding sides of ABC and passing through the translated points α, β, γ which are the orthogonal projections of A, B, C on Δ, and $A_1B_1C_1$ is symmetrically equal to ABC.‡

The Simson line with respect to $A_1B_1C_1$ of the orthopole ϕ of the line Δ for the triangle ABC coincides with Δ; if σ is the Miquel circle for the quadrilateral $[A_1B_1C_1, \Delta]$ the figure $\{A_1B_1C_1, \sigma\}$ is the translation of $\{a_1b_1c_1, (O_9)\}$ by the translation δ, and the theorem is proved. It follows that the point ϕ on the circumcircle of $A_1B_1C_1$§ is the focus of a parabola inscribed in this triangle with Δ as the tangent at the parabola vertex. Thus the point ϕ is the Miquel point of the complete quadrilateral $[A_1B_1C_1, \Delta]$. The proof of the first part of the problem is now complete.

(2) The second part follows from the theorem that the orthopoles ϕ_1, ϕ_2, ϕ_3, ϕ_4

* J. Neuberg, *Wiskundig Tydschrift*, vol. 10, p. 80.

† V. Thébault, *Mathesis*, vol. 49, 1935, p. 270.

‡ J. Neuberg, *loc. cit.*

§ V. Thébault, *Nouvelles Annales de Mathématiques*, 1914, p. 223.

of the straight lines $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ with respect to the triangles T_1, T_2, T_3, T_4 are collinear* and coincide as shown above with the Miquel points of the quadrilaterals $[T_1', \Delta_1]$, $[T_2', \Delta_2]$, $[T_3', \Delta_3]$, $[T_4', \Delta_4]$.

Note. The parallels to $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ drawn through the orthocenters H_1, H_2, H_3, H_4 of triangles T_1, T_2, T_3, T_4 form a complete quadrilateral such that the four triangles $T_1'', T_2'', T_3'', T_4''$ are symmetrically equal to the triangles T_1, T_2, T_3, T_4. The orthopole ψ_1 of the straight line $\Delta \equiv (H_1, H_2, H_3, H_4)$ for the triangle T_1 lies on a parallel Δ'' to Δ symmetric with respect to Δ to the tangent Δ' at the vertex of the parabola with focus F inscribed in the quadrilateral $(Q) \equiv [\Delta_1, \Delta_2, \Delta_3, \Delta_4]$. The orthopole ψ_1'' of Δ with respect to triangle T_1'' is therefore on the straight line symmetric to Δ'' with respect to Δ, that is on Δ', for the center of symmetry of triangles T_1 and T_1'' is the intersection of Δ with the Newton line of the quadrilateral (Q),† and moreover we know that the orthopoles of a line Δ with respect to two homothetic triangles T_1 and T_1'' are collinear with the projection on the considered straight line of the homothetic center σ of the two triangles, and their distances from σ are in the ratio of the homothetic ratio of the two triangles.‡ Similarly, the orthopoles $\psi_2'', \psi_3'', \psi_4''$ of Δ in the triangles T_2'', T_3'', T_4'' are also on Δ'. But we know also that $\psi_1'', \psi_2'', \psi_3'', \psi_4''$ lie on the orthocentric lines of the quadrilaterals $[T_1'', \Delta]$, $[T_2'', \Delta]$, $[T_3'', \Delta]$, $[T_4'', \Delta]$, that is on $\Delta_1, \Delta_2, \Delta_3, \Delta_4$.§ From this follows:

THEOREM. *The orthopoles $\psi_1'', \psi_2'', \psi_3'', \psi_4''$ of the orthocentric line Δ of the quadrilateral (Q) with respect to triangles $T_1'', T_2'', T_3'', T_4''$ coincide with the orthogonal projections of the Miquel point F on the sides of the quadrilateral (Q).*

This property is also true for the orthopoles $\psi_1', \psi_2', \psi_3', \psi_4'$ of the straight line $\Delta \equiv (H_1, H_2, H_3, H_4)$ with respect to the triangles T_1', T_2', T_3', T_4', and in general for the figure where we consider the variable parallels projecting the vertices of the triangles T_1, T_2, T_3, T_4 on $\Delta_1, \Delta_2, \Delta_3, \Delta_4$. For the triangles T_1', T_2', T_3', T_4', and in general $T_1^i, T_2^i, T_3^i, T_4^i$, are obtained from $T_1'', T_2'', T_3'', T_4''$ by translations perpendicular to Δ. The orthopoles $\psi_1'', \psi_2'', \psi_3'', \psi_4''$ remain fixed, coinciding therefore with those for the line Δ for all of these quadruples of triangles.

Editorial Note. It is assumed here that no two of the lines Δ_i are parallel and that their six intersections A_{jk} are distinct. The theorem in the first part of the above solution results from the following considerations. Let parallel straight lines of arbitrary direction through A and B of triangle ABC meet the transversal Δ in α and β, and let parallels to BC and CA through α and β respectively intersect in C_1. Consider the six straight lines CA, AB, BC, $C_1\alpha$, $\Delta = \alpha\beta$, βC_1, and for simplicity suppose $\alpha \neq \beta$. The intersections of the pairs of parallels CA, βC_1 and BC, $C_1\alpha$ determine the line at infinity, and $A\alpha$ and $B\beta$ meet on this line.

* R. Goormaghtigh, *Nouvelles Annales de Mathématiques*, 1919, p. 39.

† V. Thébault, *Mathesis*, 1937, pp. 187–242.

‡ V. Thébault, *Mathesis*, 1933 (supplement, p. 31).

§ R. Goormaghtigh, *Nouvelles Annales*, 1939, p. 39, and V. Thébault *Mathesis*, 1937, *loc. cit.*

From this it follows that a conic $\{S\}$ is tangent to these six lines, and its center S is the midpoint of CC_1. Hence the symmetric of AB with respect to S is also a tangent forming with the two lines through C_1 the triangle $A_1B_1C_1$ symmetrically equal to ABC with respect to S. Let A_1B_1 meet Δ in γ and consider the six tangents AB, BC, CA, $A_1\beta$, $\beta\gamma$, γA_1. Here the parallel pairs CA, $A_1\beta$ and γA_1, AB determine the line at infinity, and hence $B\beta$, $C\gamma$ are parallel.

Let Δ meet BC, CA, AB in A_0, B_0, C_0, then taking first the six tangents C_1A_1, A_1B_1, B_1C_1, CA_0, A_0B_0, B_0C, and next the six A_1B_1, B_1C_1, C_1A_1, AB_0, B_0C_0, C_0A, we show that A_0A_1, B_0B_1, C_0C_1 are parallel. The three triangles AA_0A_1, BB_0B_1, CC_0C_1 are such that the common midpoint of AA_1, BB_1, CC_1 is S, and the straight lines from S to the midpoints of AA_0, BB_0, CC_0 are parallel respectively to the parallels A_0A_1, B_0B_1, C_0C_1. Hence S and the midpoints of AA_0, BB_0, CC_0 are collinear on (ABC, Δ), and by similar reasoning we show that S lies also on $(A_1B_1C_1, \Delta)$. From this it follows that the parabola tangent to Δ and to the sides of ABC has its axis parallel to (ABC, Δ); see the solutions of 3817 [1939, 177] and 3818 [1939, 178].

Suppose now that the parallels $A\alpha$, $B\beta$, $C\gamma$ are perpendicular to Δ, then it is known that the perpendiculars from α, β, γ to BC, or B_1C_1, *etc.*, meet in a point ϕ, the orthopole of Δ with respect to ABC. From the fact that the projections α, β, γ of ϕ on the sides of $A_1B_1C_1$ lie on the straight line Δ, it is also known that ϕ lies on the circumcircle (O_4) of $A_1B_1C_1$. Since $A_1\gamma\phi$ and $A_1\beta\phi$ are right triangles the circumcircle (O_1) of triangle $A_1\gamma\beta$ has its center O_1 at the midpoint of ϕA_1; and since ϕA_1 is the common chord of (O_1) and (O_4) the angle ϕO_1O_4 is a right angle. If O_2 and O_3 are the circumcenters of $B_1\alpha\gamma$ and $C_1\beta\alpha$, we see similarly that ϕO_2O_4 and ϕO_3O_4 are also right angles. Hence O_1, O_2, O_3, O_4, ϕ are on a circle (M) with its center M at the midpoint of ϕO_4; and (O_4) and (M) are tangent at ϕ. In this special case $(A_1B_1C_1, \Delta)$ is also perpendicular to Δ. It follows from the above that ϕ is the focus of a parabola tangent to the sides of $A_1B_1C_1$ with Δ as the vertex tangent. It also follows that, if parallel straight lines of arbitrary direction through A_1, B_1, C_1 meet Δ in α_i, β_i, γ_i, the parallels to B_1C_1, C_1A_1, A_1B_1 through the latter points form a triangle $A_iB_iC_i$ symmetrically equal to $A_1B_1C_1$ with the center of symmetry S_i on $(A_1B_1C_1, \Delta)$, and $A_i\alpha$, $B_i\beta$, $C_i\gamma$ are parallel to $(A_1B_1C_1, \Delta)$ and consequently perpendicular to Δ. Thus ϕ is the orthopole of Δ for each $A_iB_iC_i$ and ABC is one of these triangles.

The proof of the second part can be stated briefly by using the formulas of the solution of 3839 [1939, 604] where the sides of the quadrilateral $[ABC, \Delta]$ are denoted by Δ_1, Δ_2, Δ_3, Δ_4. If ϕ_i is the orthopole of Δ_i with respect to triangle $T_i = [\Delta_j, \Delta_k, \Delta_l]$, then we find for the coordinates of ϕ_i

$$\phi_i; \quad -a, \ a(\sigma_1^{(i)} + \sigma_2^{(i)} m_i + \sigma_3^{(i)} m_i^2 + m_i^3)/(1 + m_i^2).$$

Hence the four orthopoles ϕ_i lie on the directrix of the parabola tangent to the four lines, and as is known the four orthocenters H_i of T_i lie on the same line.

Proofs of the theorem of the proposer's Note are given in the solutions of 3991 [1942, 550].

Pedals of Regular Polygons

3970 [1940, 574]. *Proposed by V. Thébault, San Sebastián, Spain*

Let $(H)\equiv A_1A_2A_3A_4A_5A_6$ and $(D)\equiv A_1\alpha_1A_2\alpha_2 \cdots A_6\alpha_6$ be a regular hexagon and a regular dodecagon inscribed in a circle (O). Show that: (1) The Simson lines Δ_1, Δ_2 of any point M of (O) with respect to the triangles $A_1A_3A_5$, $A_2A_4A_6$ are perpendicular and intersect at the midpoint of MO. (2) The consecutive sides of the pedal (H') of M with respect to (H) are parallel to Δ_1, Δ_2. (3) The opposite sides of the pedal (D') of M with respect to (D) are parallel to the bisectors of the angles between Δ_1 and Δ_2. (4) Two sides of (D'), separated by a side, are perpendicular. (5) If we denote by S_6, S_{12}, Σ_{12} the areas of (H), (D), (D') then $\Sigma_{12}=S_6+S_{12}/2$. (6) Extend (3) and (4) to pedal polygons of M with respect to a regular polygon of $6k$ sides, k being any integer.

Solution by J. W. Clawson, Ursinus College. We place the polygons (H) and (D) in the circle with unit radius and center at the point O in the complex plane, so that $A_j=\text{cis }(j-1)\pi/3$, $j=1, 2, 3, 4, 5, 6$, where $\text{cis }\theta=\cos\theta+i\cdot\sin\theta$. Then $\alpha_j=\text{cis }(2j-1)\pi/6$.

(1) Taking M to be the turn t, the Simson line of $A_1A_3A_5$ is $t^2z-t\bar{z}=(t^3-1)/2$, where $\bar{z}$ is the reflection of z in the axis of reals, *i.e.*, if $z=r\cdot\text{cis }\theta$, $\bar{z}=r/\text{cis }\theta$.

Similarly the Simson line of $A_2A_4A_6$ is $t^2z+t\bar{z}=(t^3+1)/2$.

These lines are mutually perpendicular and intersect at $t/2$, which is the midpoint of MO.

(2) Let the foot of the perpendicular from M to A_jA_{j+1} be B_j. Then B_j is

$$\frac{1}{2}\left[\text{cis }(2j-1)\frac{\pi}{6}+t-\text{cis }(2j-1)\frac{\pi}{3}\cdot 1/t\right].$$

The equation of B_jB_{j+1} turns out to be $tz-\text{cis }\pi j\cdot\bar{z}=tB_j-\text{cis }\pi j\cdot\bar{B}_j$. If $j=1$, 3 or 5, the left-hand side of this equation becomes $tz+\bar{z}$; if $j=2$, 4 or 6, it becomes $tz-\bar{z}$.

(3) The bisectors of the angles between Δ_1 and Δ_2 are $2t^2z+2it\bar{z}=t^3+i$ and $2t^2z-2it\bar{z}=t^3-i$.

The equations of $A_j\alpha_j$ and of α_jA_{j+1} are

$$z+\text{cis }(4j-3)\frac{\pi}{6}\cdot\bar{z}=\frac{\sqrt{6}+\sqrt{2}}{2}\cdot\text{cis }(4j-3)\frac{\pi}{12},$$

$$z+\text{cis }(4j-1)\frac{\pi}{6}\cdot\bar{z}=\frac{\sqrt{6}+\sqrt{2}}{2}\cdot\text{cis }(4j-1)\frac{\pi}{12}\cdot$$

Call the foot of the perpendiculars from M to $A_j\alpha_j$ and to α_jA_{j+1}, C_j and C_j' Then C_j is

$$\frac{1}{2}\left[\frac{\sqrt{6}+\sqrt{2}}{2}\text{ cis }(4j-3)\frac{\pi}{12}+t-\text{cis }(4j-3)\frac{\pi}{6}\cdot 1/t\right]$$

and the equation of C_jC_j' turns out to be

$$tz - \text{cis}\,(2j-1)\frac{\pi}{2}\cdot\bar{z} = tC_j - \text{cis}\,(2j-1)\frac{\pi}{2}\cdot\overline{C}_j.$$

If $j=1$, 3 or 5 the left hand side of this equation is $tz-i\bar{z}$; if $j=2$, 4 or 6 it is $tz+i\bar{z}$.

The equation of $C_j'C_{j+1}$ is $tz-\text{cis}\,j\pi\cdot\bar{z}=tC_j'-\text{cis}\,j\pi\cdot\overline{C}_j'$. If $j=1$, 3 or 5, the left hand side of this equation is $tz+\bar{z}$; if $j=2$, 4 or 6, it is $tz-\bar{z}$. Thus three sets of opposite sides of the pedal (D') appear to be parallel to the bisectors of the angles between Δ_1 and Δ_2; while the other three sets of opposite sides appear to be parallel to the lines Δ_1 and Δ_2 themselves.

(4) This follows immediately.

(5) S_6 is $3\sqrt{3}/2$ and S_{12} is 3. To find $\sum_{12}$ we set up the value of

$$\tfrac{1}{2}i[C_1C_2' - C_1'C_2 + \cdots + C_6C_1' - C_6'C_1].$$

This is laborious but finally yields $6(\sqrt{3}+1)/4$.

(6) Let us call the vertices of the polygon of $6k$ sides, beginning at A_j.

$$A_j,\ \beta_j,\ \beta_j',\ \beta_j'',\ \cdots,\ \beta_j^{(k-2)},\ A_{j+1},\ \cdots.$$

Then β_j is cis $(jk-k+1)\pi/3k$, β_j' is cis $(jk-k+2)\pi/3k$, and so on. Working as in (3), we find that the equations of the sides of the pedal polygon D_jD_j' and $D_j'D_j''$ are

$$tz - \bar{z}\cdot\text{cis}\,(jk-k+1)\pi/k = tD_j - \overline{D}_j\cdot\text{cis}\,(jk-k+1)\pi/k,$$
$$tz - \bar{z}\cdot\text{cis}\,(jk-k+2)\pi/k = tD_j' - \overline{D}_j'\cdot\text{cis}\,(jk-k+2)\pi/k.$$

Hence, opposite sides of the polygon are mutually perpendicular; sides separated by k sides are mutually perpendicular; sides ending at the original vertices $A_1, A_2, \cdots$ are parallel to the lines Δ_1 and Δ_2, while the other sides are parallel to the $2k$ lines which divide the angles between Δ_1 and Δ_2 into k equal parts.

Editorial Note. Let P_n be a regular polygon inscribed in circle (O) with the vertices $1, 2, 3, \cdots, n$, and let M be a point on (O) say on the arc $\widehat{12}$, not at an end point. The projection of M on the chord ij is denoted by (i, j); then the pedal P_n' of M with respect to P_n has the vertices $(1, 2), (2, 3), \cdots, (i, i+1), (i+1, i+2), (i+2, i+3), \cdots, (n, 1)$. The area of triangle $(i, i+1)M(i+1, i+2)$ is

$$(R^2\sin 2\phi)/2\left[\cos^2\phi + \tfrac{1}{2}\cos 2\phi - 2\cos^2\phi\cos(2\theta_i+2\phi) + \tfrac{1}{2}\cos(4\theta_i+4\phi)\right],$$

where R is the radius of (O), 2ϕ is the angle subtended at O by each side of P_n, $2\theta_i$ is the angle subtended by the chord Mi at O. The area of P_n' is the sum of the areas of the n triangles with $i=2, 3, 4, \cdots, n, 1$, where $\theta_i=(i-2)\phi+\alpha$, angle $MO2=2\alpha$, $n\phi=\pi$. The sum of each of the two last terms is zero, so that denoting by P_n' its area, we have

$$P_n' = (\tfrac{3}{2} - 2\sin^2\phi)P_n \tag{1}$$

where n may be even or odd. If $n=2m$, where m may be even or odd,

$$P'_{2m} = \tfrac{1}{2}P_{2m} + \cos 2\phi P_{2m} = \tfrac{1}{2}P_{2m} + P_m. \tag{2}$$

We now consider the relation between the polygons P_{2m}' and P_m'. The points $(i, i+1)$, $(i+1, i+2)$, $(i+2, i)$ are collinear, being the projections of M on the sides of triangle $(i, i+1, i+2)$; the chord $M\, i+2$ of (O) subtends right angles at $(i, i+2)$, $(i+1, i+2)$, $(i+2, i+3)$, $(i+2, i+4)$, where the points $(i+2, i+3)$, $(i+3, i+4)$, $(i+4, i+2)$ are collinear for the same reason as above. From this it results that the external angles of P_{2m}' at its vertices $(i+1, i+2)$ and $(i+2, i+3)$ are each 3ϕ, also side $(i, i+2)$, $(i+2, i+4)$ of P_m' is parallel to side $(i+1, i+2)$, $(i+2, i+3)$ of P_{2m}', forming with it an isosceles trapezoid. In a similar manner we find that the external angles of P_m' at its vertices $(i, i+2)$ and $(i+2, i+4)$ are each 6ϕ. There is a second polygon P_m'' inscribed in P_{2m}' with the two consecutive sides $(i-1, i+1)$, $(i+1, i+3)$ and $(i+1, i+3)$, $(i+3, i+5)$ which are parallel respectively to the sides of P_{2m}' $(i, i+1)$, $(i+1, i+2)$ and $(i+2, i+3)$, $(i+3, i+4)$. Thus the angle between corresponding sides of P_m' and P_m'', say $(i, i+2)$, $(i+2, i+4)$ and $(i+1, i+3)$, $(i+3, i+5)$ is 3ϕ in the positive sense from the first to the second.

We consider now the case $m=6$. For the equilateral triangle with the vertices 1, 5, 9 inscribed in (O) the point M is the focus of a parabola tangent to its sides. Here $3\phi=\pi$, and the pedal P_3' of M with respect to this triangle is the straight line of (1, 5), (5, 9), (9, 1) which is the vertex tangent of the parabola. Since its directrix passes through the orthocenter O of the triangle, the radius OM is bisected by P_3'. Similarly, the pedal P_3'' of M with respect to triangle 3, 7, 11, that is the straight line of (3, 7), (7, 11), (11, 3) bisects OM. The two triangles determine a hexagon P_6 with the pedal P_6' whose exterior angles are $\pi/2$. Hence the angle between P_3' and P_3'' is $\pi/2$. One vertex of P_n', $n>3$, requires special consideration; it is vertex (1, 2) if M lies on arc $\overparen{1\,2}$ of (O) but not at a vertex. This vertex lies inside (O) while $(n, 1)$ and (2, 3) lie outside. The chord M 2 of (O) subtends right angles at (1, 2), $(2, n)$, (2, 3) where (2, 3), (3, 1), (1, 2) are collinear. The angle $(2, n)$ 2 $(2, 3)=3\phi$, and hence angle $(n, 1)$, (1, 2), $(2, 3)=3\phi$. Thus M lies inside the triangle with vertices $(n, 1)$, (1, 2), (2, 3) and angle $(n, 1)$, (1, 2) $M=2\phi-\alpha$.

The regular polygon P_{6k} leads to the pedal P_{6k}' which has $\pi/2k$ for the external angle at its vertices, and we consider the $k+1$ consecutive sides $(2k, 2k+1)$, $(2k+1, 2k+2)$, $(2k+1, 2k+2)$, $(2k+2, 2k+3)$, $\cdots$, $(3k, 3k+1)$, $(3k+1, 3k+2)$. The triangle with vertices 1, $2k+1$, $4k+1$ is equilateral and its pedal P_3' is the line of collinearity of $(1, 2k+1)$, $(2k+1, 4k+1)$, $(4k+1, 1)$ which we call Δ_1. Similarly, the equilateral triangle $k+1$, $3k+1$, $5k+1$ gives the pedal P_3'', or Δ_2. From the above Δ_1 and Δ_2 are perpendicular and intersect at the midpoint of OM. The chord M $2k+1$ of (O) subtends right angles at $(2k, 2k+1)$, $(2k+1, 2k+2)$, $(2k+1, 4k+1)$, and we have angle $(1, 2k+1)$, $(2k+1, 4k+1)$, $(2k, 2k+1)=$angle $(1, 2k+1)$, $(2k+1)$, $(2k, 2k+1)=(2k-1)\phi$,

where $\phi=\pi/6k$. Similarly, angle $(2k+1,\ 2k+2)$, $(1,\ 2k+1)$, $(2k+1,\ 4k+1)$ $=$angle $(2k+2)$, $(2k+1)$, $(2k+1,\ 4k+1)=(2k-1)\phi$; and hence $(1,\ 2k+1)$, $(2k,\ 2k+1)$, $(2k+1,\ 2k+2)$, $(2k+1,\ 4k+1)$ is an isosceles trapezoid, and the first of the above $k+1$ consecutive sides of P_{6k}' is parallel to Δ_1; the next side is parallel to Δ_1 turned through the angle $\pi/2k$; the third is parallel to Δ_1 turned through the angle π/k; and so on, the last is parallel to Δ_1 turned through the angle $\pi/2$, or it is parallel to Δ_2. This may be continued and we come to $(4k,\ k+1)$, $(4k+1,\ 4k+2)$ which is parallel to Δ_2 turned through the angle $\pi/2$, or is parallel to Δ_1, and this side is opposite to $(2k,\ 2k+1)$, $(2k+1,\ 2k+2)$. Here we have k pairs of perpendiculars similar to Δ_1, Δ_2 each pair intersecting at the midpoint of OM.

For related theorems see the solutions of 3861 [1940, 118] and of 3886 [1940, 405]. For completeness we give another area formula which is similarly obtained. Here M is any point at the constant distance r from O and we denote by P_n and $\overline{P}_n$ the areas of the regular polygons of n sides inscribed respectively in the circles of radii R and r, and by P_n' the area of the pedal of M with respect to P_n.

$$P_n' = \tfrac{1}{2}P_n + \tfrac{1}{2}(P_n + \overline{P}_n)\cos 2\phi.$$

Special Circulants

3994 [1942, 341] (corrected). *Proposed by C. E. Springer, University of Oklahoma*

If

$$a_{11} = a_{22} = a_{33} = \sum_j \binom{K}{j}(n-1)^{K-j}, \qquad j \equiv 0 \pmod 3;$$

$$a_{12} = a_{23} = a_{31} = \sum_j \binom{K}{j}(n-1)^{K-j}, \qquad j \equiv 1 \pmod 3;$$

$$a_{13} = a_{32} = a_{21} = \sum_j \binom{K}{j}(n-1)^{K-j}, \qquad j \equiv 2 \pmod 3;$$

show that the determinant

$$|a_{ij}| = [(n-1)^3 + 1]^K.$$

Solution by John Williamson, Queens College. The determinant $|a_{ij}|$ is a circulant and as such has the value

$$\prod_{i=1}^{3} (a_{11} + \omega_i a_{12} + \omega_i^2 a_{13}) \tag{1}$$

where ω_1, ω_2, ω_3 are the cube roots of unity. Moreover

$$
\begin{aligned}
(x^3+1)^K &= \prod_{i=1}^{3}(x+\omega_i)^K \\
&= \prod_{i=1}^{3}\left\{\sum_j \omega_i^j\binom{K}{j}x^{K-j}\right\} \\
&= \prod_{i=1}^{3}(a_{11}+\omega_i a_{12}+\omega_i^2 a_{13}), \quad \text{if} \quad x=n-1, \\
&= |a_{ij}| \quad \text{by} \quad (1).
\end{aligned}
$$

The proof of the following generalization is exactly similar to the above. *Let* $|a_{rs}|$ *be a circulant of odd order* n, *where*

$$a_{rs} = \sum_j \binom{K}{j} x^{K-j}, \qquad j \equiv s - r \bmod n. \tag{2}$$

Then

$$|a_{rs}| = (x^n + 1)^K.$$

The corresponding theorem for n even or odd is the following. *Let* $|b_{rs}|$ *be a circulant of order* n, *where*

$$b_{rs} = \sum_j (-1)^j \binom{K}{j} y^{K-j}, \qquad j \equiv s - r \bmod n. \tag{3}$$

Then

$$|b_{rs}| = (y^n - 1)^K.$$

If $y=-x$ and n is odd it follows from (2) and (3) that $b_{rs}=(-1)^K a_{rs}$ and therefore that $|b_{rs}|=(-1)^K|a_{rs}|$. Since $(y^n-1)^K=(-1)^K(x^n+1)^K$ the theorem for n odd may be deduced from this last one.

Solved also in a different manner by the proposer with the remark that n and K of the problem are integers greater than unity and that a generalization of the result is possible in which r replaces 1 and the odd integer p replaces 3.

Editorial Note. The generalization mentioned by the proposer may be obtained by setting

$$a_u = \sum (-1)^j c^j \binom{K}{j} y^{K-j}, \qquad j \equiv u - 1 \bmod n,$$

where K is a positive integer and c is not zero. Then

$$\sum_{u=1}^{n} \omega_i^{u-1} a_u = (y - \omega_i c)^K,$$

where $\omega_1, \omega_2, \cdots, \omega_n$ are the distinct nth roots of unity. The circulant C_n whose first row is $a_1, a_2, \cdots, a_n$ is such that, if we replace its first column

c_1 by $c_1+\omega_i c_2+\omega_i^2 c_3+\cdots+\omega_i^{n-1}c_n$, it will be seen that it is divisible by $(y-\omega_i c)^K$, and hence by the product of the n such factors $(y^n-c^n)^K$. Since the term of C_n of the highest degree in y is y^{nK}, we have

$$C_n = (y^n - c^n)^K.$$

There are several proofs of the factorization theorem for the general circulant C_n. One is similar to the above where the elements a_u are regarded as n independent variables instead of polynomials in the single variable y, see Scott's *The Theory of Determinants*, 1904, pp. 102, 103; on page 104 is an example somewhat similar to the one of this problem.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, N. Y.

Professor W. D. Cairns has been appointed visiting professor of mathematics at the University of New Mexico.

Dr. Nancy Cole of Sweet Briar College is now a visiting assistant professor at Kenyon College.

W. B. Coleman of the Agricultural and Mechanical College of Texas is now a second lieutenant in the Army Air Corps stationed at the San Antonio Aviation Cadet Center.

Assistant Professor J. H. Curtiss and Dr. Joseph Lehner have left Cornell University. The former has been commissioned lieutenant (j.g.) in the Naval Reserve and the latter is in the employ of the Kellex Corporation of New York City.

Dr. H. H. Goldstine of the University of Michigan is now a first lieutenant in the Army Air Force stationed in the ordnance department at Philadelphia.

Dr. Mary C. Graustein of Radcliffe College has been appointed to an assistant professorship at Oberlin College.

Dr. D. W. Hall of Brown University has been appointed to an assistant professorship at the University of Maryland.

Assistant Professor R. N. Johanson of Bradley Polytechnic Institute is now an assistant professor at Hamilton College.

Associate Professor Fritz John of the University of Kentucky is now in the Ballistic Research Laboratory at Aberdeen Proving Ground.

Assistant Professor E. M. Justin has a leave of absence from Case School of Applied Science in order to serve as an instructor of airplane drawing and mathematics at the University of California.

E. C. Molina of the Bell Telephone Laboratories has retired.

Dr. J. M. H. Olmsted of the University of Minnesota has been promoted to an assistant professorship.

Dr. Grace S. Quinn has been appointed associate in mathematics at George Washington University.

Professor L. W. Sheridan is on leave of absence from the College of Mount St. Vincent and is doing research in meteorology at the central office of the United States Weather Bureau in Washington, D. C.

Professor A. J. Smith of Susquehanna College has been appointed to an assistant professorship at the College of William and Mary.

Assistant Professor F. H. Steen of Georgia School of Technology has been appointed to an assistant professorship at Allegheny College.

The following appointments to instructorships have been announced:

Army Air Forces Technical Training School at Scott Field, Illinois: K. F. McLaughlin (assistant instructor)
Brown University: D. A. Jonah, M. E. Munroe, P. C. Rosenbloom
Cornell University: F. W. Dittman (part-time)
Grays Harbor Junior College (Aberdeen, Washington): Miss Ruth E. Porter
Hamilton College: Dr. V. O. McBrien
University of Minnesota: W. D. Munro
Queens College: Dr. Deborah M. Maria

Professor W. M. Carruth of Hamilton College died January 23, 1943. He was a charter member of the Association.

Dr. W. A. Granville, vice-president of the Washington National Insurance Company and author of one of the most widely used calculus texts of our generation, died February 4, 1943 at the age of seventy-nine. He was a charter member of the Association.

Professor Emeritus L. M. Passano of Massachusetts Institute of Technology died January 30, 1943.

Assistant Professor N. S. Risley of Fenn College died December 30, 1942.

A copy of G. N. Watson's *Theory of Bessel's Functions* is urgently needed in connection with important war research. Anyone who has a copy he would be willing to sell, or to lend for the duration, is requested to communicate with Dr. Warren Weaver, Chief of the Applied Mathematics Panel, National Defense Research Committee, Room 5500, 49 West 49th Street, New York, N. Y.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

FROM SELECTIVE SERVICE

Occupational Bulletin Number 11 was amended March 1, 1943. The recommendation of the new bulletin in regard to the deferment of graduate students is essentially the same as that of December 14, 1942 except for a significant change in wording; the latest bulletin states that graduate students in some twenty technical fields, including mathematics, "*should* be considered for occupational classification" whereas the previous wording has been "*may* be considered for occupational classification."

The new bulletin announces a liberalization of draft deferment policies in regard to undergraduate students. It specifies that "a student in undergraduate work in any of the scientific and specialized fields listed *should* be considered for occupational classification if he is a full-time student in good standing in a recognized college or university and if it is certified by the institution as follows: (1) That he is competent and gives promise of successful completion of such course of study, and (2) That if he continues his progress he will graduate from such course of study on or before July 1, 1945."

RETRAINING PROGRAMS FOR TEACHERS

The magnitude of the Army-Navy Educational Program emphasizes the shortage of mathematics and physics teachers available for instructional purposes upon the college level. An acute situation has existed in physics for many months, and now it is generally acknowledged that the supply of mathematicians is quite inadequate. Consequently, many departments of mathematics are following the advice issued to departments of physics by Dr. Homer L. Dodge, Director of the Office of Scientific Personnel, National Research Council. He states that "many departments of physics have available some graduate and senior students who can immediately be called upon to serve as teachers. Most departments, however, will have to meet most or all of the demands for staff expansion by recruitment from the staffs of other departments and outside sources. The sooner this is realized the better, for it is essential that such persons enjoy a period of thorough training."

One of the first programs for the retraining of teachers was inaugurated at New York University under Mr. M. C. Giannini, Director of the ESMWT Program in the College of Engineering, University Heights, New York. The courses have been open to all faculty members in New York University and to faculty members of neighboring institutions, including high schools. The first retraining course in mathematics comprised the subject matter of college algebra

and trigonometry. This was a sixty-hour program, meeting in the late afternoon for three, two-hour sessions every week. Twenty-seven trainees enrolled in the course, and twenty-two successfully completed the program. The trainees included several persons with the Ph.D. degree and two chairmen of departments; they came from such departments as modern language, psychology, and philosophy. At the completion of the first course, twenty members of the class petitioned for a continuation of the training, and a sixty-hour course in analytical geometry has been started. It is expected that many of these trainees, all of whom did exceptional work, will be used in the mathematics department.

THE CURTISS-WRIGHT ENGINEERING CADETTE PROGRAM

The Curtiss-Wright Corporation recently introduced a ten-months engineering training course for women. Women to be trained have been assigned to the following institutions: Cornell University, Iowa State College, University of Minnesota, Pennsylvania State College, Purdue University, and the University of Texas. In addition to mathematics, the courses to be studied are job terminology and specifications, aircraft drawing and design, elementary mechanics, and aircraft materials.

The outline of the course in engineering mathematics follows:

1. *Logarithms* (12 hours): Definition of logarithms; use of common logarithms in multiplication, division, powers, and roots; natural logarithms and conversions.
2. *Slide Rule* (12 hours): Addition and subtraction with simple scales; relationship of the slide rule to logarithms; multiplication, division, powers, and roots by slide rule; rules for decimal points.
3. *Trigonometry* (60 hours): (a) Definitions; functions; solution of right triangles (42 hours). (b) Solution of general triangles; functions of double angles (18 hours).
4. *Areas and Volumes* (18 hours): Areas of plane figures; volumes and areas of surfaces of prisms, cones, pyramids, frustums; volumes of solids of revolution (including Simpson's rule and the rule of Pappus); centroids of simple areas and volumes; use of planimeter.
5. *Review of Algebra* (30 hours): Topics through quadratic equations.
6. *Analytical Geometry* (33 hours): Rectangular coordinate systems; properties of conic sections; brief treatment of polar coordinates.
7. *Aircraft Problems* (33 hours): Graph paper, rectangular and polar coordinates; logarithmic papers; construction of charts; graphical solution of equations; curve plotting; plotting of normals and tangents to irregular curves; construction of graphs as computing aid.

ENGINEERING, SCIENCE, AND MANAGEMENT WAR TRAINING

Many of the war training courses in mathematics have been under the sponsorship of ESMWT. This has been especially true in various industrial regions. Thus the following information in regard to ESMWT furnished by the

Federal Security Agency, U. S. Office of Education, will be of interest to mathematicians.

Origin and Purpose. The Engineering, Science, and Management War Training program is the successor of the Engineering, Science, and Management Defense Training program of 1941–42, and the Engineering Defense Training program of 1940–41. ESMDT was established to provide short courses of college grade, designed to meet the shortage of engineers, chemists, physicists, and production supervisors in activities essential to the national defense. Under the program, courses were conducted at 196 institutions at a cost of approximately $18,000,000. About 438,000 trainees were enrolled. The EDT program of 1940–41 was authorized to provide short courses of college grade in engineering only. These were given in 144 engineering institutions with enrollments of about 120,000. The cost of EDT was approximately $6,140,000.

Present Status. On June 30, 1942, in Public Law 647—77th Congress, Second Session, a sum of $30,000,000 was appropriated to continue the training of engineers, chemists, physicists, and production supervisors for war service throughout the fiscal year ending June 30, 1943.

Organization of Courses. Tax-exempt colleges and universities offering recognized degrees with majors in engineering, chemistry, physics, or production supervision, are eligible to participate in Engineering, Science, and Management War Training, usually called ESMWT. Before a course is organized, the sponsoring college or university determines the need for the contemplated training through consultation with nearby industries and the ESMWT regional adviser for the area concerned, and prepares an estimate of the probable number and qualifications of those available for the training. If conditions are favorable, one or more short courses are designed to prepare available trainees for the jobs in which a shortage was found. Pertinent information, including estimates of costs, is sent to the U. S. Office of Education in a formal proposal to give each course that is planned. Those proposals that meet all legal, educational, and practical standards are approved, and instruction may begin as soon thereafter as qualified trainees can be enrolled. Additional proposals may be submitted as other training needs are established.

Types of Instruction. Regional differences in facilities and in war training needs dictate wide variations in the courses offered. Some are designed to prepare persons for new fields of work; others to fit those already employed in war activities for more difficult and responsible assignments. Classes may meet on the college campus or elsewhere; many institutions are conducting courses simultaneously in several cities. All classes receive personal instruction from qualified teachers, except for a few specialized correspondence courses in the subject matter of mathematics and physics which are offered to prepare high-school teachers to conduct courses in those subjects. Some courses require the full time and attention of those enrolled; others are given after working hours for the benefit of employed persons. The time required to complete a course varies from a few weeks to several months, depending upon the extent and

nature of the training. Subjects range from basic courses, such as engineering drawing and precision inspection, to refined specialties, such as geometrical optics and the X-ray diffraction analysis of metals. In general, the war training needs of an area determine the courses offered there, but some courses are conducted to meet Nation-wide needs of the armed forces and government war production activities.

Although all instruction under ESMWT is of college grade, it is not a substitute for regular courses of study leading to degrees. Regularly enrolled college students may not be admitted to any ESMWT course unless they intend, upon its completion, to enter war employment or the armed forces and do not intend to reenroll in college within the next academic year.

Costs. The Federal Government pays the cost of instruction. Therefore, no tuition or fees are required of ESMWT enrollees. Trainees must pay for textbooks and minor supplies, however, and they must defray their own living and transportation expenses.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-sixth Summer Meeting, New Brunswick, N. J., September 11–13, 1943.

Twenty-seventh Annual Meeting, Cleveland, Ohio.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, April 17, 1943
ILLINOIS, Notre Dame, Ind., April 9–10, 1943
INDIANA, Notre Dame, April 9–10, 1943
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, BROOKLYN, N. Y., May 8, 1943
MICHIGAN, Notre Dame, Ind., April 9–10, 1943
MINNESOTA
MISSOURI, Kansas City
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, Jan. 29, 1944
OHIO, Columbus, April 1, 1943
OKLAHOMA
PHILADELPHIA, Philadelphia, Nov. 27, 1943
ROCKY MOUNTAIN, Denver, April 16–17, 1943
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 13, 1943
SOUTHWESTERN
TEXAS, Lubbock, April, 1943
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May 7, 1943

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WHAT ARE EIGEN-WERTE?

R. E. LANGER, University of Wisconsin

1. Introduction. Of the many stages upon which the dramas of mathematical analysis are enacted, a notable one is that upon which the *Eigen-wert* plays a major role. The inception of the plot there centers in the main around the interrelations of mathematics and the physical sciences, and its development shows well how these sister disciplines influence each other—how the association has yielded inspiration and direction to the one, and to the other precision, power and form.

The differential equation is the most powerful mathematical tool for the formulation of physical laws and problems. Hence, when formulated mathematically, a vast category of physical problems call for a solution of some differential equation—in the great majority of cases for such a solution as fulfills on the side certain prescribed "initial conditions" insofar as the time is concerned, and certain "boundary conditions" relative to its space coordinates. It is pertinent to consider in outline, a few such formulations. Though the configurations to be referred to are specialized, and the remarks but brief and fragmentary, they nevertheless serve as vehicles for the ideas we wish to set forth. Throughout the discussion the symbol $\nabla^2\phi$ will be used to signify the expression

$$\frac{\partial^2\phi}{\partial x^2}, \quad \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}, \quad \text{or} \quad \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2},$$

according as the portion of space that is in question, and which will be denoted by V, is one, two, or three dimensional. The boundary (points, curve or surface) of V will be designated by S, and $\partial\phi/\partial n$ will be interpreted as the normal derivative if S is a curve or surface, and as $\partial\phi/\partial x$ if x is the sole space dimension of V.

2. Eigen-werte in problems of vibration. If a homogeneous and isotropic material body occupies a space V, its elastic vibrations are governable by a function ϕ which satisfies a pair of equations

$$(1)\qquad \begin{aligned} \nabla^2\phi &= a^2\frac{\partial^2\phi}{\partial t^2}, \quad \text{within } V,\\ \alpha\phi + \beta\frac{\partial\phi}{\partial n} &= 0, \quad \text{upon } S. \end{aligned}$$

In these a^2 is a non-vanishing constant, and α and β are constants, or more generally functions of the space coordinates. At the initial instant which may be taken without loss of generality as $t=0$, the function ϕ fulfills furthermore a pair of relations

$$(2)\qquad \begin{aligned} \phi\Big|_{t=0} &= f_1,\\ \frac{\partial\phi}{\phi t}\Big|_{t=0} &= f_2, \end{aligned}$$

in which f_1 and f_2 are prescribed functions.

A classical method for an analysis of such a function ϕ proceeds along the lines of the following considerations. If any product of a factor v, which depends only on the time, by a factor u which is independent of that variable, is to solve the equations (1), that condition is found at once to imply the relation

$$\frac{\nabla^2 u}{a^2 u} = \frac{\dfrac{d^2 v}{dt^2}}{v}.$$

Inasmuch as neither member of the equality depends upon any variable which the other appears to involve, they must both be constant. With the designation of this constant common value by $-\lambda$ (the minus sign is a mere formal convenience), it follows that v and u must respectively be solutions of the équations.

$$\frac{d^2 v}{dt^2} + \lambda v = 0, \tag{3}$$

and of the "differential boundary problem"

$$\begin{aligned} \nabla^2 u + \lambda a^2 u &= 0, \quad \text{within } V, \\ \alpha u + \beta \frac{\partial u}{\partial n} &= 0, \quad \text{upon } S. \end{aligned} \tag{4}$$

The equation (3) is solvable irrespective of the value of λ. Not so the boundary problem (4). This, on the contrary, can be shown by suitable analysis to be solvable by a non-trivial function (*i.e.*, by one that does not vanish identically) only if λ has one of a specifically singled-out set of values. These values, which depend upon the coefficients of the problem and upon the boundary S, are known as "characteristic values," or, in German terms, as *Eigen-werte.* The non-trivial solutions which exist for these values are known as "characteristic solutions," or *Eigen-funktionen.* If u_k is such a solution and λ_k the corresponding characteristic value, the respective solution of the equation (3) is

$$v_k(t) \equiv c_k \cos \rho_k t + \gamma_k \sin \rho_k t, \tag{5}$$

with $\rho_k^2 = \lambda_k$, and, however the constants c_k and γ_k may be chosen, the product $v_k u_k$ satisfies the equations (1).

This bears in the following way upon the envisaged determination of the function ϕ. Because of the linearity of the equations (1), they are satisfied by any finite sum of products $v_k u_k$, and beyond that by any infinite series

$$\sum_{k=1}^{\infty} v_k(t) u_k, \tag{6}$$

which has suitable properties of convergence. These properties may be attainable through suitable choices of the coefficients c_k and γ_k which appear in (5).

If a series (6) is to represent the function ϕ, however, it is furthermore requisite, by virtue of the conditions (2), that the series

$$\sum_{k=1}^{\infty} v_k(0)u_k, \qquad \sum_{k=1}^{\infty} \frac{dv_k}{dt}\bigg|_{t=0} u_k, \tag{7}$$

represent respectively the prescribed functions f_1 and f_2. Whether, and under what conditions, given functions are so representable in infinite series of characteristic solutions, are problems in the so-called "expansion theory" which is associated with any boundary problem (4). This expansion theory is quite inextricably interwoven with elements from the theory of characteristic values.

3. The case of the vibrating string. Concrete and familiar applications of these considerations are called forth by the instances of the vibrating string or the vibrating membrane. The function ϕ in these cases represents the displacement from the position of equilibrium, and the functions f_1 and f_2 describe the initial displacements and velocities. In the case of the string, under tension, fastened at the points $x=0$ and $x=l$, and vibrating in a plane, the equations (1) are explicitly

$$\begin{gathered} \frac{\partial^2\phi}{\partial x^2} = a^2\frac{\partial^2\phi}{\partial t^2}, \quad \text{for} \quad 0 \leqq x \leqq l, \\ \phi\,\big|_{x=0} = 0, \qquad \phi\,\big|_{x=l} = 0. \end{gathered} \tag{1*}$$

The boundary problem (4) is accordingly

$$\begin{gathered} \frac{d^2u}{dx^2} + \lambda a^2 u = 0, \\ u(0) = 0, \qquad u(l) = 0, \end{gathered} \tag{4*}$$

and the simplest considerations suffice to show that any solution of the differential equation in (4*) that satisfies one of the boundary conditions, can satisfy the other also if and only if λ is one of the values $\lambda_k = (k\pi/al)^2$, with $k=1, 2, 3, \cdots$. The associated characteristic solutions are the functions $\sin k\pi x/l$, and the series (7), which are to represent the functions $f_1(x)$ and $f_2(x)$, are respectively

$$\begin{gathered} \sum_{k=1}^{\infty} c_k \sin\frac{k\pi x}{l}, \\ \sum_{k=1}^{\infty} \gamma_k \frac{k\pi}{al} \sin\frac{k\pi x}{l}. \end{gathered} \tag{7*}$$

In this case the expansion theory is evidently the theory of Fourier's series.

4. Eigen-werte in problems of flow and potential. A category of physical problems, of which the flow of heat in a material body may be taken as presenting a typical one, admits of formulation through the medium of a function ϕ

(the temperature in the case of heat flow) which is characterized as the solution of the differential system

$$
\begin{aligned}
\nabla^2\phi &= a^2\frac{\partial\phi}{\partial t}, \quad \text{within } V, \\
\alpha\phi + \beta\frac{\partial\phi}{\partial n} &= 0, \quad \text{upon } S, \qquad (8) \\
\phi\,\big|_{t=0} &= f_1.
\end{aligned}
$$

Although of a physical character quite distinct from those of vibration, these problems admit of analysis differing only in minor details from that above. In particular, the appearance of characteristic values and their rôle, is precisely such as has been indicated.

Many problems, such as those of steady flow, say of electricity through a conductor, are mathematically subsumed in the theory of potential. The governing differential system in such cases is of the form

$$
\begin{aligned}
\nabla^2\phi &= 0, \quad \text{within } V, \\
\phi &= f, \quad \text{upon } S,
\end{aligned} \qquad (9)
$$

the function f being again prescribed. In the instance of certain configurations V and functions f such problems yield to analysis through the medium of characteristic values. Thus if the volume V is a sphere, and with its center as pole the spherical coordinates in which ρ is radial distance and θ, ψ are latitude and longitude are introduced into the equations (9) to replace the Cartesian (x, y, z), it is found that if v is a function of ρ alone and u is independent of ρ, then

$$\frac{\rho^2\nabla^2(vu)}{vu} = \frac{1}{v}\left[\rho^2\frac{d^2v}{d\rho^2} + 2\rho\frac{dv}{d\rho}\right] + \frac{1}{u}\left[\sec^2\theta\frac{\partial^2 u}{\partial\psi^2} + \frac{\partial^2 u}{\partial\theta^2} - \tan\theta\frac{\partial u}{\partial\theta}\right].$$

Since the first member on the right of this formula is constant as to (θ, ψ) whereas the second is constant as to ρ, they must both be absolute constants, if the product vu is to satisfy the differential equation of (9). The functions v and u must, therefore, be respectively solutions of the equations

$$\rho^2\frac{d^2v}{d\rho^2} + 2\rho\frac{dv}{d\rho} - \lambda u = 0, \qquad (10)$$

and

$$\sec^2\theta\frac{\partial^2 u}{\partial\psi^2} + \frac{\partial^2 u}{\partial\theta^2} - \tan\theta\frac{\partial u}{\partial\theta} + \lambda u = 0. \qquad (11)$$

Our immediate concern is with the differential equation (11). It has singular points where $\theta = \pm\pi/2$, and these in general call forth corresponding singularities in the solutions. To be physically useful, however, the solutions $u(\theta, \psi)$ must be

free from singularities. This requisite and the fact can be shown to be compatible if and only if λ has a value from a certain peculiar set. These values, it need hardly be said, are the characteristic values. The determination of ϕ with the use of them, is again a problem in the related expansion theory.

5. Eigen-werte in quantum mechanics. Characteristic values are to be met with in modern physical theory no less than in those more classical. In the theory of quantum mechanics, for instance, they appear in connection with the wave equation. To illustrate this it suffices to regard the mechanical system which consists of only a single particle of mass m, when the potential energy U is a function of the space coördinates alone. The probability that at any specified time this particle be located within a given portion of space, is taken to be computable from a distribution function ϕ, which is postulated to be a solution of the wave equation

$$\frac{h^2}{8\pi^2 m}\nabla^2\phi - U\phi = \frac{h}{2\pi i}\frac{\partial\phi}{\partial t}. \tag{12}$$

If ϕ is assumed to be the product of two factors v and u, which are respectively free from the space and time variables, the equation (12) constrains these functions to be solutions of the equations

$$\frac{dv}{dt} + \frac{2\pi i W}{h} v = 0,$$

$$\nabla^2 u + \frac{8\pi^2 m}{h^2}\{W - U\}u = 0,$$

in which W is a parameter, such as has heretofore been designated by λ. In the instance of many coefficient functions U the requisite, which is physically inspired, that the solution u be bounded throughout the space, constrains the parameter W to take one of a set of *Eigen-werte*. These are then interpreted as the characteristic total energy values of the system.

6. The boundary problem in the theory of ordinary differential equations. Many physical phenomena (the examples considered can suggest their diversity only very inadequately) thus yield to mathematical formulation as boundary problems, in which a linear differential equation involving a free parameter appears in conjunction with a complement of auxiliary conditions. As might well be expected such problems have been the object of much study, some of this motivated by the applicability of the results, but much of it, on the other hand, inspired by the purely mathematical interest which inheres in them. Insofar as ordinary differential equations are concerned, the type of such problems may be taken as of the form

$$\begin{aligned} &L(u) - \lambda u = 0, \\ &W_k(u) = 0, \qquad\qquad k = 1, 2, \cdots, n, \end{aligned} \tag{13}$$

in which

$$L(u) \equiv u^{[n]} + p_1 u^{[n-1]} + \cdots + p_n u,$$

and

$$W_k(u) = \sum_{j=0}^{n-1} \left\{ \alpha_{jk} u^{[j]}(a) + \beta_{jk} u^{[j]}(b) \right\},$$

the symbol $u^{[j]}$ designating the jth derivative of u. In the commonest cases the coefficients α_{jk} and β_{jk} are constants, and the p_j are suitably differentiable functions of x.

The differential equation of the system (13) considered by itself, admits of n linearly independent solutions $u_1(x, \lambda), \cdots, u_n(x, \lambda)$. In terms of these, every solution is expressible in the manner.

$$u(x, \lambda) = \sum_{r=1}^{n} c_r u_r(x, \lambda), \tag{14}$$

with constant coefficients c_r. The substitution of this expression into the boundary conditions yields the set of relations

$$\sum_{r=1}^{n} c_r W_k(u_r) = 0, \qquad k = 1, 2, \cdots, n,$$

which constitutes a linear algebraic system that must be satisfied by the coefficients c_r if the function (14) is to solve the boundary problem. Since these coefficients may not all be zero, inasmuch as the solution is to be non-trivial, it is evidently requisite that

$$\left| W_i(u_j) \right| = 0, \tag{15}$$

the symbol on the left designating the determinant of which the general element is shown. The equation (15) is known as the "characteristic equation." Its variable is the parameter λ, which is involved through the functions $u_r(x, \lambda)$, and its roots are the characteristic values.

It must suffice here to observe that the mathematical literature includes discussions of many different generalizations of the problem (13) as described. The characteristic values in themselves, inasmuch as they are explicitly determinable only in the very simplest cases, have called forth many deductions relative to their approximate or asymptotic representation. In these studies the notation of vectors and matrices has often been found advantageous. Thus the boundary problem

$$u' = \left\{ \rho P(x) + Q(x) \right\} u,$$
$$W_1 u(a) + W_2 u(b) = 0,$$

in which the capitals stand for square matrices, in which $\rho^n = \lambda$, and in which u

is a vector, is in scalar terms more explicitly

$$u_i' = \sum_{r=1}^{n} \{\rho p_{ir}(x) + q_{ir}(x)\} u_r,$$

$$\sum_{s=1}^{n} \{w_{1,i,s}u_s(a) + w_{2,i,s}u_s(b)\} = 0, \qquad i = 1, 2, \cdots, n$$

and to this the problem (13) is transformable.

7. Eigen-werte in the theory of integral equations. If the differential system

$$\begin{aligned} L(u) &= f(x), \\ W_k(u) &= 0, \qquad k = 1, 2, \cdots, n, \end{aligned} \tag{16}$$

admits of only the trivial solution in the instance that the function $f(x)$ is identically zero, it may be solved in the instance of all other suitably integrable functions by a formula

$$u(x) = \int_a^b K(x, s) f(s) ds,$$

in which $K(x, s)$ is theoretically constructible, and is known as the Green's function. With the formal replacement of $f(x)$ by λu ,the system (16) reverts to the boundary problem (13), and this latter thus appears, insofar as its solution is concerned, as coextensive with the "integral equation"

$$u(x) - \lambda \int_a^b K(x, s) u(s) ds = 0. \tag{17}$$

Many physical problems thus clearly admit of indirect formulation in the manner (17). The indirect route is, however, by no means the only one. Thus, in the case of the vibrating string, a straightforward application of mechanical principles may be made to lead directly to the equation (17), the specific "kernel" in that case being

$$K(x, s) = \begin{cases} \dfrac{s(l - x)}{lT}, & \text{when} \quad 0 \leqq s \leqq x, \\[2ex] \dfrac{x(l - s)}{lT}, & \text{when} \quad x \leqq s \leqq l, \end{cases}$$

in which T denotes the tension. The most direct and elementary means will show in this case that the equation (17) admits of a non-trivial solution only if λ is a characteristic value. This suggests, and it is a fact, that the appearance of characteristic values is an important phenomenon in the theory of integral equations. Their rôle in this theory is completely analogous to that which they fill in the theory of boundary problems in differential form.

8. Eigen-werte in Hilbert spaces. The integral equation serves conveniently here as a point of departure toward a domain of greater abstractness. If the kernel $K(x, s)$ is envisaged as any specific function that is of integrable square the relation

$$\Psi(x) = \int_a^b K(x, s)\psi(s)ds, \tag{18}$$

may be looked upon as associating with any function $\psi(x)$ of the space L_2 (*i.e.* of the aggregate of functions whose squares are integrable) another such function $\Psi(x)$. The relation, therefore, effectively defines a transformation from ψ to Ψ, and this may be indicated concisely by the symbolism

$$T\psi = \Psi. \tag{19}$$

Now with appropriate adjustments the space L_2 is includable in a class of spaces known as "Hilbert spaces," and the relation (19) is then significant of a particular linear transformation in such a space. By abandoning the particularization of the space, and the restriction whereunder the relation (19) is a mere restatement of (18), the symbol T is emancipated, and may be taken to stand for the general linear transformation in abstract Hilbert space. In the theory of such transformations the analogue of the equation (17) is the relation

$$I\psi - \lambda T\psi = 0, \tag{20}$$

in which I stands for the identical transformation under which $I\psi = \psi$, for all elements ψ. If λ is a constant for which the transformation $I - \lambda T$ possesses an inverse, only the trivial solution is admitted by the relation (20). The demand for a non-trivial solution can be met, therefore, only if the transformation in question has no inverse. The values of λ for which that is so are the characteristic values.

The simplest of the Hilbert spaces is that in which the elements are the number sequences $\psi \equiv (x_1, x_2, x_3, \cdots)$, for which the series $\sum_{k=1}^{\infty} |x_k|^2$ are convergent. In this space a transformation T is defined by a system of equations

$$y_r = \sum_{k=1}^{\infty} a_{r,k}x_k, \qquad r = 1, 2, 3, \cdots,$$

with constant coefficients a_{ij}. With this interpretation the relation (20) takes on the form

$$x_r - \lambda \sum_{k=1}^{\infty} a_{r,k}x_k = 0, \qquad r = 1, 2, 3, \cdots, \tag{21}$$

and such a system of equations accordingly has a theory of characteristic values associated with it.

9. The reduction of a quadratic form. To bring this discussion to a close, let the system of algebraic equations

$$x_r - \lambda \sum_{k=1}^{m} a_{r,k} x_k = 0, \qquad r = 1, 2, \cdots, m, \tag{22}$$

which is a finite analogue of the infinite system (21), be briefly considered. These equations present themselves in particular in the determination of relative extrema of the quadratic form

$$\sum_{k=1}^{m} x_k^2 \tag{23}$$

in the face of the auxiliary condition

$$\sum_{i,j=1}^{m} a_{i,j} x_i x_j = 1, \tag{24}$$

the matrix $(a_{i,j})$ being symmetric. The parameter λ appears in this instance as a Lagrange multiplier. Since the form (23) is geometrically interpretable as the square of the distance from the center of the quadric surface (24), the problem is clearly that of locating the end points of the principal axes of that surface. If λ_i is any characteristic value of the system (22), and $x_r = x_r^i$ is the corresponding characteristic solution, the multiplication of the equations by the respective x_r and their summation relative to r, yields by virtue of the condition (24) the evaluation

$$\sum_{r=1}^{m} x_r^{(i)^2} = \lambda_i.$$

This shows the characteristic values to be the squares of the semi-axes. Under an orthogonal transformation which lays the coördinates along these axes, the equation (24) would accordingly be replaced by

$$\sum_{k=1}^{m} \frac{y_k^2}{\lambda_k} = 1.$$

From this it will be evident that the characteristic values are significant in the theory of the normalization of quadratic forms.

The writer will plead limitations of space and the purpose of this note as excuses for the many omissions, which will be all too obvious, and for the cursoriness with which all matters have been touched upon. The subject is a vast one. Nor has its closing chapter yet been written. Its significance will maintain far into the future.

A MAGIC CUBE WITH $6n^3$ CELLS

R. V. HEATH, New York Stock Exchange

An *ordinary magic cube of the nth order* is here defined as an arrangement of n^3 different positive integers at the vertices of an $n \times n \times n$ (three dimensional) lattice work in such a manner that the sum of the numbers in every row, every column, every file, and in each of the four main diagonals* shall be the same number S. When n is greater than three, it is also required that the sum along the diagonals of each horizontal layer shall have the value S. The number S will be called the *characteristic* of the magic cube.

It may be noted in passing that earlier writers on this topic have been content with a definition of a magic cube which imposed no restriction on the sums along the diagonals of the horizontal layers. It seems to the author that the definition above is more appropriate and more interesting. As an indication of the possibilities of further development along the lines which will be suggested by this paper, it may be said that the writer has recently constructed ordinary magic cubes of orders six and ten which conform to the definition in the first paragraph.

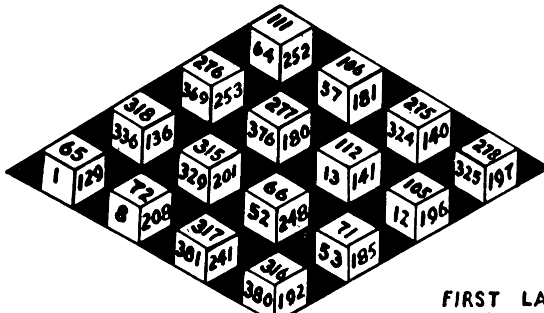
It is the purpose of this paper to explain the construction of a "generalized magic cube" which will now be defined. Consider a cube as made up of n^3 "cubelets" (of equal dimensions) arranged in layers of n^2 cubelets each. On each face of every one of these cublets let there be written one of the integers from 1 to $6n^3$ inclusive, without repetition. Let it be required that the numbers written on the top faces of the cubelets form an ordinary magic cube. Likewise those on the bottom faces, those on the north faces, those on the south faces, those on the east faces, and those on the west faces. (In the four cases last mentioned, it is understood that the $n \times n$ arrays whose diagonals are required to have the sum S lie in vertical planes, instead of the horizontal planes as specified in the first paragraph.) It is also required that all of these ordinary magic cubes have the same characteristic. Such an arrangement of the first $6n^3$ positive integers will be called a generalized magic cube.

A generalized magic cube can be obtained from an ordinary magic cube by multiplying all of the numbers by 6, and then judiciously adding to or subtracting from each of them a suitably chosen number from the set 0, 1, 2, 3, 4, 5, so as to obtain the proper combination of six magic cubes with no duplication of numbers.

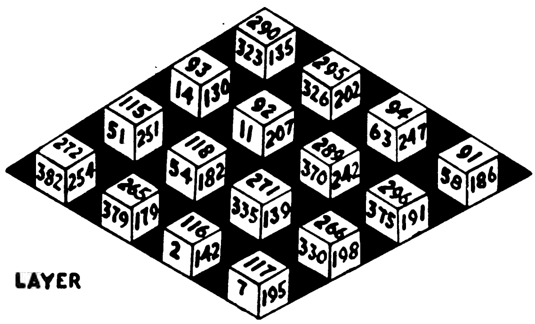
Such a generalized magic cube with $n=4$ was constructed by the author in 1931. In the present paper there is presented a generalized magic cube with $n=4$ which has the additional property that the sum of the integers on each of the cubelets is equal to 1155. The manner in which this result was obtained will now be explained.

First, a magic square was constructed by use of the first sixteen positive

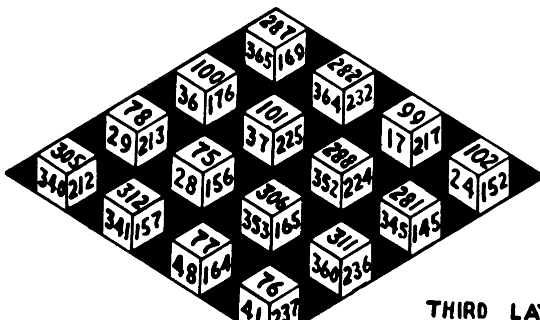
* A main diagonal of the cube is understood to be one of the four lines joining two vertices which do not lie in the same face of the cube.



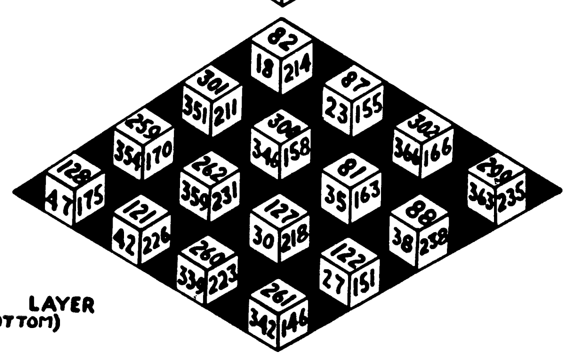
FIRST LAYER
(TOP)



SECOND LAYER



THIRD LAYER



FOURTH LAYER
(BOTTOM)

integers. Then, by a natural extension of the methods which have been developed in connection with magic squares, a magic cube was formed by use of the first sixty-four positive integers. Care was exercised, however, to form the cube in such a way that in each horizontal layer every integer in the first two rows was *complementary* to the corresponding integer in the last two rows. (Two integers are said to be complementary if their sum is equal to the sum of the least and the greatest of the integers employed. In this case, two integers whose sum is 65 are complementary.)

Then, using the first 384 positive integers, six magic cubes were formed from the original one by the method explained in the fourth paragraph of this paper. Each of these cubes was so constructed that the elements were complementary in pairs as explained in the preceding paragraph. (In this connection, two integers whose sum is 385 are complementary.) The next step was to effect a rearrangement so as to obtain two ordinary magic cubes such that each integer in one of them would be the complement of the corresponding integer in the other. This purpose was accomplished as follows. Two of the six cubes were selected, say cubes A and B. Cube A, keeping the first two rows in each of its four layers, had the numbers in the third and fourth rows of each layer replaced by the integers occupying corresponding positions in cube B. Let the new cube so obtained be denoted by C. Then a cube D was formed whose first and second rows in each layer were the third and fourth rows of the corresponding layers of A; and whose third and fourth rows in each layer were the first and second rows of the layers of B. Thus were obtained two magic cubes C and D such that each term in one of them was complementary to the corresponding number in the other.

The same procedure was applied to another pair of the six cubes, and finally to the remaining pair. Thus were obtained three pairs of magic cubes, each cube being complementary, term by term, to the one with which it was paired. It was found that this could be done in such a way that each of the six cubes had the same characteristic.

Finally, the terms of each of the six ordinary magic cubes were inscribed on the faces of the cubelets in such a manner that the integers on opposite sides of the cubelets were in all instances complementary. The figure shows only three faces of each cubelet. But the integers not shown can be obtained by subtracting the visible numbers from 385.

In the generalized magic cube shown in the figure, the sum 770 will be obtained by adding the four integers in any row, column, file, main diagonal, or any one of the previously specified diagonals of squares. The sum of the six numbers on any one of the cubelets is 1155, as can easily be proved by means of the fact that the numbers on opposite faces are complementary. The sum 770 will also be obtained by adding the four numbers visible on any one of the cubelets when any two opposite faces are covered. This fact is obvious since the sum will be $1155-385$. It is easily verified that the sum 770 can be obtained in 552 ways from the figure.

The figure exhibits other uniformities not pointed out here. The reader may enjoy discovering them for himself.

The results obtained in this paper suggest the problem of constructing a cube similar to the one in the figure, but with the property that each of the twenty-four layers is a pan-diagonal magic square; or with the property that each of the six ordinary magic cubes is a pan-diagonal magic cube. In the latter connection see W. W. R. Ball, *Mathematical Recreations and Essays* (1939), page 221.

LEGENDRE FUNCTIONS OF THE SECOND KIND AND RELATED FUNCTIONS*

DUNHAM JACKSON, University of Minnesota

1. Introduction. A reviewer of my recent Carus Monograph† has noted as an omission the absence of any reference to Legendre functions of the second kind.‡ The purpose of the present paper is to remedy the omission in some degree by an introductory treatment of these functions which is comparable as to its elementary character with the treatment of the Legendre polynomials in the Monograph, and which in particular makes no use of the theory of functions of a complex variable. It is further pointed out that the same method is applicable to a discussion of second solutions of the differential equations of Hermite, Laguerre, Jacobi, and Bessel.

The Legendre functions of the second kind are met with in the problem of solving completely the Legendre differential equation (formula (10) below), which is of importance in mathematical physics and elsewhere. For each non-negative integral value of n the equation has as one solution the Legendre polynomial of corresponding degree. A second solution is readily found for any particular value of n by a well known standard procedure. From the point of view of a comprehensive theory the properties of the second solutions are in many respects materially less simple than those of the polynomials. There is however one striking resemblance: the familiar and fundamentally important relation of recurrence (formula (1)) connecting any three successive Legendre polynomials is satisfied also by the corresponding second solutions, when the arbitrary constants are suitably chosen. This property appears further deserving of particular emphasis in an introductory account because it makes possible the rapid tabulation of complete solutions of the differential equation for successive values of n in explicit form.

It is natural to inquire how directly it is possible to trace a formal connection between the differential equation itself and the recurrence formula. In

* Presented to the Minnesota Section of the Association at Northfield, Minn. May 9, 1942.

† Fourier Series and Orthogonal Polynomials, Carus Mathematical Monograph No. 6, 1941.

‡ See Review of Scientific Instruments, vol. 13, 1942, pp. 180–181.

particular, if $y_0, y_1, y_2, \cdots$ form a sequence of functions connected by the recurrence relation, and if the initial functions y_0, y_1, which suffice to determine the rest of the sequence, are solutions of the differential equation for the values of n equal to their respective subscripts, can it be inferred from these facts alone that the later functions necessarily satisfy the differential equation for the corresponding values of the parameter? The answer to this question is in the negative; if y_{n-1}, y_n, y_{n+1} are connected by the recurrence formula, and if y_{n-1} and y_n satisfy the differential equations appropriate to their subscripts, the expression

$$(1 - x^2)y''_{n+1} - 2xy'_{n+1} + (n + 1)(n + 2)y_{n+1}$$

reduces, not to zero, but to

$$\frac{2(2n + 1)}{n + 1}[(1 - x^2)y'_n + nxy_n - ny_{n-1}].$$

The quantity in brackets vanishes identically when the y's are the Legendre polynomials of the degrees indicated by the subscripts, or second solutions with the constants suitably determined. Furthermore it can be shown that if this identity holds for $n = 1$ and for $n = 2$ it must continue to hold by virtue of the recurrence relation. Therefore a process of mathematical induction can be set up for the differential equation in conjunction with this additional identity.

It may be pointed out also that if the first *four* functions y_0, y_1, y_2, y_3 are solutions of the respective differential equations, the recurrence relation holding throughout, then the supplementary identity is necessarily fulfilled for $n = 1$ and for $n = 2$, and the whole induction goes through accordingly.

It is not the purpose of this paper to ask the reader to follow through the particular process of calculation that has been outlined. When the possibility of the reasoning by mathematical induction has been recognized it becomes apparent that the manipulation of the formulas admits almost endless variation, and there are alternative arrangements of the work which simplify the details appreciably. On comparison of a number of different procedures, one growing naturally out of another, it seems most advantageous to make use of expressions relating y'_n and y_n respectively to the pair of functions y'_{n-1} and y_{n-1} (formulas (2), (3), (8), (9)), and this arrangement is followed below, without further protracted account of the successive stages of experiment leading to the form which is ultimately adopted.

It is perhaps not surprising that the method of treatment which is effective for the Legendre differential equation can be generalized to the equation satisfied by arbitrary Jacobi polynomials, among which those of Legendre are included as a special case, and can be applied also to the differential equations for the closely related orthogonal polynomials of Hermite and Laguerre. The place of these various systems of orthogonal polynomials in a single comprehensive theory is set forth in detail in the Monograph referred to at the beginning of the paper. Especially noteworthy is the fact that the type of treatment under pres-

ent consideration applies likewise to Bessel's equation, which is associated with that of Legendre through its significance in mathematical physics, but is otherwise less similar in its formal theory. Still other applications are omitted from discussion here. The ultimate scope of the method is not superficially apparent.

2. Recurrence formulas and the Legendre differential equation. Let $y_0(x)$, $y_1(x)$, $y_2(x)$, $\cdots$, be a sequence of functions such that

$$(1)\qquad (n+1)y_{n+1}(x) - (2n+1)xy_n(x) + ny_{n-1}(x) = 0, \qquad n = 1, 2, \cdots.$$

Let it be supposed that each of these functions has first and second derivatives over the range of values of x to be considered. Let

$$(2)\qquad u_n(x) = y_n'(x) - ny_{n-1}(x) - xy_{n-1}'(x), \qquad n = 1, 2, \cdots,$$

$$(3)\qquad v_n(x) = ny_n(x) - nxy_{n-1}(x) + (1-x^2)y_{n-1}'(x), \qquad n = 1, 2, \cdots.$$

From (1),

$$(4)\qquad (n+1)y_{n+1} = (2n+1)xy_n - ny_{n-1},$$

and consequently

$$(5)\qquad (n+1)y_{n+1}' = (2n+1)xy_n' + (2n+1)y_n - ny_{n-1}'.$$

Substitution of (5) in the relation

$$(n+1)u_{n+1} = (n+1)y_{n+1}' - (n+1)^2y_n - (n+1)xy_n',$$

obtained from (2) on replacement of n by $n+1$ and multiplication by $n+1$, gives

$$(n+1)u_{n+1} = nxy_n' - n^2y_n - ny_{n-1}'.$$

By combination of (2) and (3), on the other hand,

$$xu_n - v_n = xy_n' - ny_n - y_{n-1}'.$$

Consequently

$$(6)\qquad (n+1)u_{n+1} = n(xu_n - v_n).$$

Similarly, from (3) and then (4),

$$v_{n+1} = nxy_n - ny_{n-1} + (1-x^2)y_n',$$

while the combination $(1-x^2)u_n + xv_n$ is found from (2) and (3) to reduce to the same expression, so that

$$(7)\qquad v_{n+1} = (1-x^2)u_n + xv_n.$$

The relations (6), (7), holding for $n \geqq 1$, show that if $u_1 \equiv v_1 \equiv 0$ then $u_2 \equiv v_2 \equiv 0$, and by induction $u_n \equiv v_n \equiv 0$ for all n. That is to say, *if the y's, connected by* (1), *are such that the identities*

$$y_n' - ny_{n-1} - xy_{n-1}' = 0, \tag{8}$$

$$ny_n - nxy_{n-1} + (1 - x^2)y_{n-1}' = 0 \tag{9}$$

hold for $n=1$; they hold for all positive integral n.

By differentiation of (3) and combination with (2),

$$v_n' - nu_n = (1 - x^2)y_{n-1}'' - 2xy_{n-1}' + (n - 1)ny_{n-1}.$$

The identities (8) and (9), stating that $u_n \equiv v_n \equiv 0$, imply that the right-hand member vanishes for $n \geqq 1$, which is equivalent to saying that

$$(1 - x^2)y_n'' - 2xy_n' + n(n + 1)y_n = 0 \tag{10}$$

for $n \geqq 0$.

The conclusions of the last two paragraphs can be combined to yield the following:

THEOREM I. *If $y_0(x)$, $y_1(x)$ are any two functions (possessing first and second derivatives) such that*

$$y_1' - y_0 - xy_0' = 0, \tag{11}$$

$$y_1 - xy_0 + (1 - x^2)y_0' = 0, \tag{12}$$

and if other functions $y_2(x)$, $y_3(x)$, $\cdots$ are calculated successively by (4), *the general function $y_n(x)$ of the sequence thus defined satisfies the Legendre differential equation* (10).

It is to be repeated also that under the conditions stated (8) and (9) hold separately, and furthermore, naturally, any other identities that can be deduced from these. For example, since

$$u_{n+1} = y_{n+1}' - (n + 1)y_n - xy_n',$$

$u_{n+1}+xu_n-v_n$ reduces to $y_{n+1}'-(2n+1)y_n-y_{n-1}'$, and as $u_{n+1}\equiv u_n\equiv v_n\equiv 0$,

$$(2n + 1)y_n = y_{n+1}' - y_{n-1}'. \tag{13}$$

3. Legendre polynomials and Legendre functions of the second kind. If the functions $y_0(x)=1$, $y_1(x)=x$ are substituted in (11) and (12) it is immediately apparent that both identities are satisfied. Calculation of y_2, y_3, $\cdots$, from (4) with these values of y_0 and y_1 for a beginning gives for y_n a polynomial of the nth degree. By Theorem I this polynomial, conventionally denoted by $P_n(x)$, is a solution of (10). The polynomials P_0, P_1, P_2, $\cdots$, are the *Legendre polynomials.** A substantial introduction to their theory is contained in the recurrence formula (1) or (4), the differential equation, and the identities (8), (9), and (13), from which further properties can readily be deduced.

For another sequence of y's let

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* See for example the Monograph already cited, Chapter II.

$$y_0 = \tfrac{1}{2}\log\frac{1+x}{1-x},$$

$$y_1 = -1 + \tfrac{1}{2}x\log\frac{1+x}{1-x} = -1 + xy_0.$$

These also are found to satisfy (11) and (12). The variable x is to be restricted now to the interval $-1<x<1$, so that the logarithm shall be real. As indication of the source of these functions, let it be noted that the differential equation

$$(1-x^2)y'' - 2xy' = 0,$$

to which (10) reduces for $n=0$, can be solved by the most elementary means, being an equation of the first order for y':

$$\frac{y''}{y'} = \frac{2x}{1-x^2}, \qquad \log y' = c - \log(1-x^2), \qquad y' = \frac{k_1}{1-x^2},$$

$$y = k_1\int\frac{dx}{1-x^2} = k_1\cdot\tfrac{1}{2}\log\frac{1+x}{1-x} + k_2,$$

so that the general solution is a linear combination of the particular solution 1 used in the preceding paragraph and the present y_0. If then y_1 is defined in terms of this y_0 by means of (12), the form obtained for y_1 is that given above; and (11) is found to be satisfied also.

By a different arrangement of the work the whole sequence of y's can be built up from the single function y_0 by successive applications of (9), and the accompanying theory developed accordingly, without separate special consideration of y_1. For any sequence of functions $y_0, y_1, y_2, \cdots$, possessing first and second derivatives, but not assumed at the outset to satisfy (1), let v_n and u_n have the meanings assigned to them by (3) and (2), and let w_n denote the left member of (10). Then

$$u_n = (v_n' - w_{n-1})/n,$$
$$w_n = (1-x^2)u_n' - 2xu_n + (n+1)v_n + xw_{n-1},$$

as may be verified by evaluation of the right-hand members. From these identities it follows that if y_0 is a function for which $w_0\equiv 0$, and if $y_1, y_2, \cdots$ are defined successively by setting $v_n\equiv 0$ for $n=1, 2, \cdots$, then $u_1\equiv 0$, $w_1\equiv 0$, and by further induction $u_n\equiv 0$ and then $w_n\equiv 0$, that is, (10) is satisfied, for all positive integral n. Also

$$(n+1)y_{n+1} - (2n+1)xy_n + ny_{n-1} = v_{n+1} - xv_n - (1-x^2)u_n,$$

so that the identical vanishing of all the u's and v's implies the relation (1). This scheme of operations is perhaps simpler in plan than that followed in the earlier paragraphs, but the actual verification in detail appears to be more laborious.

Let $Q_n(x)$ denote the general function y_n of the sequence defined by (4) with

the logarithmic y_0 and y_1 of the second paragraph preceding. It has the form $\phi_n(x)y_0+\psi_n(x)$, in which $\phi_n(x)$ and $\psi_n(x)$ separately satisfy the recurrence relation

$$(n+1)\phi_{n+1} = (2n+1)x\phi_n - n\phi_{n-1},$$
$$(n+1)\psi_{n+1} = (2n+1)x\psi_n - n\psi_{n-1}.$$

For recognition of the last fact it suffices to substitute $\phi_n y_0+\psi_n$ and $\phi_{n-1}y_0+\psi_{n-1}$ in (4) for y_n and y_{n-1}. Since $\phi_0=1=P_0(x)$ and $\phi_1=x=P_1(x)$, the ϕ's are identical with the Legendre polynomials, while the initial determinations $\psi_0=0$, $\psi_1=-1$ make ψ_n a polynomial of degree $n-1$, which for the sake of putting its degree explicitly in evidence may be denoted alternatively by $R_{n-1}(x)$. Then*

$$Q_n(x) = \tfrac{1}{2}P_n(x)\log\frac{1+x}{1-x} + R_{n-1}(x).$$

By Theorem I, the function $y_n=Q_n(x)$ satisfies (10). As it is clearly not a constant multiple of $P_n(x)$, it is a second solution independent of $P_n(x)$, and the general solution of (10) is of the form

$$AP_n(x) + BQ_n(x).$$

The Q's are *Legendre functions of the second kind.*

Any particular Q_n can also be calculated without the necessity of building up from Q_0 and Q_1 through the recurrence formula if the corresponding P_n is known, by assuming a solution of (10) in the form

$$y_n = \tfrac{1}{2}P_n(x)\log\frac{1+x}{1-x} + z_n,$$

with z_n as a new unknown function. Then

$$y_n' = \tfrac{1}{2}P_n'(x)\log\frac{1+x}{1-x} + \frac{P_n(x)}{1-x^2} + z_n',$$
$$y_n'' = \tfrac{1}{2}P_n''(x)\log\frac{1+x}{1-x} + \frac{2P_n'(x)}{1-x^2} + \frac{2xP_n(x)}{(1-x^2)^2} + z_n'',$$

and by virtue of the fact that $P_n(x)$ satisfies (10)

$$(1-x^2)y_n'' - 2xy_n' + n(n+1)y_n = (1-x^2)z_n'' - 2xz_n' + n(n+1)z_n + 2P_n'(x).$$

If a polynomial of degree $n-1$ is substituted for z_n in the right-hand member, its coefficients can be uniquely determined in descending order so that the expression shall reduce to zero.

* For another method of deriving this form of solution, and for further particulars concerning the polynomials $R_{n-1}(x)$, see E. W. Hobson, The Theory of Spherical and Ellipsoidal Harmonics, Cambridge, 1931, pp. 49–51, 53–55.

The work of the preceding section shows further that the Q's satisfy the relations (8), (9) and (13).*

4. Associated functions. The differential equation

$$\frac{d^2T}{d\theta^2} + \cot\theta\,\frac{dT}{d\theta} + \left[n(n+1) - \frac{k^2}{\sin^2\theta}\right]T = 0 \tag{14}$$

is met with in connection with the solution of Laplace's equation in spherical coordinates.† By the substitution $x = \cos\theta$ this becomes

$$(1 - x^2)\frac{d^2T}{dx^2} - 2x\frac{dT}{dx} + \left[n(n+1) - \frac{k^2}{1 - x^2}\right]T = 0. \tag{15}$$

A further change of dependent variable, defined by the relation $T = (1 - x^2)^{k/2}z$, gives for z the differential equation

$$(1 - x^2)\frac{d^2z}{dx^2} - 2(k+1)x\frac{dz}{dx} + [n(n+1) - k(k+1)]z = 0. \tag{16}$$

On the other hand, the result of differentiating (10) k times in succession is

$$(1 - x^2)y^{(k+2)} - 2(k+1)xy^{(k+1)} + [n(n+1) - k(k+1)]y^{(k)} = 0,$$

which is the same as (16) if $y^{(k)}$ is identified with z. That is to say, if y is any solution of (10), $z = y^{(k)}$ satisfies (16). In particular, $P_n^{(k)}(x)$ and $Q_n^{(k)}(x)$ are solutions of (16).

Consequently (15) has for solutions the *associated Legendre functions*

$$(1 - x^2)^{k/2}P_n^{(k)}(x), \qquad (1 - x^2)^{k/2}Q_n^{(k)}(x),$$

and corresponding solutions of (14) are

$$\sin^k\theta P_n^{(k)}(\cos\theta), \qquad \sin^k\theta Q_n^{(k)}(\cos\theta).$$

5. The Hermite differential equation. In modification of the initial hypotheses of §2 let $y_0, y_1, y_2, \cdots$ now be a sequence of functions satisfying the recurrence relation

$$y_{n+1} - xy_n + ny_{n-1} = 0, \qquad n \geqq 1. \tag{17}$$

In place of (2) and (3) let

$$u_n = y_n' - ny_{n-1}, \qquad n \geqq 1,$$
$$v_n = y_n - xy_{n-1} + y_{n-1}', \qquad n \geqq 1.$$

* For additional information about the functions $Q_n(x)$ reference may be made for example to Hobson, *op. cit.*, especially pp. 49–72, and Whittaker and Watson, A Course of Modern Analysis, Cambridge, 1920, and other editions, Chapter XV.

† See Monograph, pp. 118–120.

Formulas (4) and (5) are replaced by

$$
\begin{aligned}
y_{n+1} &= xy_n - ny_{n-1}, \\
y'_{n+1} &= xy'_n + y_n - ny'_{n-1}.
\end{aligned} \tag{18}
$$

Further calculation corresponding to that of §2 gives

$$u_{n+1} = y'_{n+1} - (n+1)y_n = xy'_n - ny_n - ny'_{n-1} = xu_n - nv_n, \tag{19}$$

$$v_{n+1} = y_{n+1} - xy_n + y'_n = y'_n - ny_{n-1} = u_n, \tag{20}$$

$$v'_n - u_n = y''_{n-1} - xy'_{n-1} + (n-1)y_{n-1}. \tag{21}$$

The recurrence relation expressed by (17) or (18) being assumed to hold throughout, (19) and (20) show that the identical vanishing of u_1 and v_1 implies that of u_n and v_n for all positive integral n, and (21), holding for $n \geqq 1$, means then that

$$y''_n - xy'_n + ny_n = 0 \tag{22}$$

for $n \geqq 0$. Thus the following analogue of Theorem I is obtained:

THEOREM II. *If $y_0(x)$, $y_1(x)$ are any two (twice differentiable) functions such that*

$$y'_1 = y_0, \tag{23}$$

$$y_1 = xy_0 - y'_0, \tag{24}$$

and if other functions $y_2(x)$, $y_3(x)$, $\cdots$ are calculated successively by (18), *the general function $y_n(x)$ of the sequence thus defined satisfies the Hermite differential equation* (22).

Under the same hypotheses, *the y's satisfy also the relations*

$$y'_n = ny_{n-1}, \qquad y_n = xy_{n-1} - y'_{n-1}, \tag{25}$$

which express the vanishing of u_n and v_n.

If $y_0 \equiv 1$, $y_1 \equiv x$, satisfying (23) and (24), the function $y_n(x)$, which is seen from (18) to be a polynomial of the nth degree, is the *Hermite polynomial** $H_n(x)$. It is a solution of (22), by Theorem II; and successive H's satisfy the relations (25).

A second particular solution of (22) for $n=0$ is obtained by writing:

$$y''_0/y'_0 = x, \qquad \log y'_0 = x^2/2, \qquad y'_0 = e^{x^2/2},$$

$$y_0 = \int_0^x e^{x^2/2}dx.$$

With this y_0 let y_1 be defined by means of (24) as

$$y_1 = x\int_0^x e^{x^2/2}dx - e^{x^2/2}.$$

* See Monograph, Chapter IX

It is apparent that (23) is satisfied also. Further definition of $y_2, y_3, \cdots$ by means of (18) gives for y_n an expression of the form

$$y_n = H_n(x)\int_0^x e^{x^2/2}dx + r_{n-1}(x)e^{x^2/2},$$

where $H_n(x)$ is the Hermite polynomial, and $r_{n-1}(x)$ is a polynomial of degree $n-1$. Let this y_n be denoted by $G_n(x)$. It is obviously not a constant multiple of $H_n(x)$, if the fact that $\int e^{x^2/2}dx$ is not a function of "elementary" form is assumed as known; in any case the conclusion is assured by the observation that the G's satisfy the relations (25), from the first of which it follows by repeated application that if G_n were a polynomial of the nth degree, G_{n-1} would have to be a polynomial of degree $n-1$, and ultimately G_0 would have to be a constant, which it certainly is not. The general solution of (22) is of the form

$$AH_n(x) + BG_n(x).$$

6. The Laguerre differential equation. For the relations (4), (2), (3) let the following now be substituted:

$$\begin{aligned} y_{n+1} &= (x - \alpha - 2n - 1)y_n - n(\alpha + n)y_{n-1}, \\ u_n &= y_n' - ny_{n-1} + ny_{n-1}', \\ v_n &= y_n - (x - \alpha - n)y_{n-1} + xy_{n-1}'. \end{aligned} \tag{26}$$

Straightforward carrying out of the indicated operations verifies the identities

$$\begin{aligned} u_{n+1} &= (x - \alpha - n)u_n - nv_n, \\ v_{n+1} &= xu_n - nv_n, \\ v_n' - u_n &= xy_{n-1}'' + (\alpha + 1 - x)y_{n-1}' + (n - 1)y_{n-1}. \end{aligned}$$

From these may be read off, in analogy with Theorems I and II:

THEOREM III. *If $y_0(x)$, $y_1(x)$ are any two (twice differentiable) functions such that $u_1 \equiv v_1 \equiv 0$, that is to say, such that*

$$y_1' = y_0 - y_0', \tag{27}$$

$$y_1 = (x - \alpha - 1)y_0 - xy_0', \tag{28}$$

and if other functions $y_2(x)$, $y_3(x)$, $\cdots$ are calculated successively by (26), the general function $y_n(x)$ of the sequence thus defined satisfies the Laguerre differential equation

$$xy_n'' + (\alpha + 1 - x)y_n' + ny_n = 0. \tag{29}$$

Successive y's are connected by the relations

$$y_n' = ny_{n-1} - ny_{n-1}', \tag{30}$$

$$y_n = (x - \alpha - n)y_{n-1} - xy_{n-1}'. \tag{31}$$

If $y_0=1$, $y_1=x-\alpha-1$ (the form of y_1 being prescribed by (28) when y_0 is given), both (27) and (28) are satisfied. The corresponding y_n is the *Laguerre polynomial** $L_n(x)$ or $L_n^{(\alpha)}(x)$ of the nth degree, for the value of α which appears in (29). The polynomials of the sequence thus obtained satisfy (29), (30), and (31).

For $n=0$, a non-constant solution of (29) is given by the formulas

$$\frac{y_0''}{y_0'} = 1 - \frac{\alpha+1}{x}, \qquad \log y_0' = x - (\alpha+1)\log x,$$
$$y_0 = \int_1^x \frac{e^x dx}{x^{\alpha+1}};$$

any other positive number would serve equally well instead of 1 as lower limit of integration. The corresponding y_1 from (28) is

$$y_1 = (x-\alpha-1)\int_1^x \frac{e^x dx}{x^{\alpha+1}} - \frac{e^x}{x^\alpha},$$

and this function together with y_0 satisfies (27). The general function y_n of the sequence now defined by (26) is of the form

$$y_n = L_n(x)\int_1^x \frac{e^x dx}{x^{\alpha+1}} + \rho_{n-1}(x)\frac{e^x}{x^\alpha},$$

in which $L_n(x)$ and $\rho_{n-1}(x)$ are the Laguerre polynomial and a polynomial of degree $n-1$ respectively. If this $y_n=M_n(x)$ were a polynomial it would follow from the identity

$$xy_n' - ny_n = n(\alpha+n)y_{n-1},$$

which is obtained from (30) and (31) by elimination of y_{n-1}', that $y_{n-1}, \cdots, y_0$ must be polynomials, in contradiction with the definition of y_0. Consequently $M_n(x)$ as a solution of (29) is independent of $L_n(x)$, and the general solution is of the form

$$AL_n(x) + BM_n(x).$$

7. The Jacobi differential equation. For a corresponding treatment of the general Jacobi polynomials and the accompanying second solutions, the key formulas are the differential equation and the recurrence formula on pages 174 and 173 of the Monograph, and the combinations

$$\begin{aligned} u_n &= 2(\alpha+\beta+n)y_n' - (\alpha+\beta+n)(\alpha+\beta+2n)y_{n-1} \\ &\quad - [(\alpha+\beta+2n)x + (\beta-\alpha)]y_{n-1}', \\ v_n &= 2n(\alpha+\beta+n)y_n - (\alpha+\beta+n)[(\alpha+\beta+2n)x + (\alpha-\beta)]y_{n-1} \\ &\quad + (\alpha+\beta+2n)(1-x^2)y_{n-1}'. \end{aligned}$$

* See Monograph, Chapter X.

The calculations are naturally somewhat laborious, and will not be given here at length. With the understanding that $\alpha>-1$, $\beta>-1$, the reader who has the patience to verify the identities

$$\begin{aligned}
&2(n+1)(\alpha+\beta+2n)u_{n+1}\\
&\qquad=(\alpha+\beta+2n+2)\{(\alpha+\beta+2n)(xu_n-v_n)+(\beta-\alpha)u_n\},\\
&2(\alpha+\beta+n)(\alpha+\beta+2n)v_{n+1}\\
&\qquad=(\alpha+\beta+2n+2)\{(\alpha+\beta+2n)[(1-x^2)u_n+xv_n]+(\beta-\alpha)v_n\}
\end{aligned}$$

will have little difficulty with the rest. In making comparisons with §2 it is to be noted that if the expressions defined by (2) and (3) are denoted now by s_n and t_n, the present u_n and v_n reduce for $\alpha=\beta=0$ not to s_n and t_n, but to $2ns_n$ and $2nt_n$ respectively, while u_{n+1} and v_{n+1} become $2(n+1)s_{n+1}$ and $2(n+1)t_{n+1}$.

Exceptional circumstances arise if $\alpha+\beta+1=0$, so that the factor $\alpha+\beta+n$ vanishes for $n=1$. In this case the differential equation has for $n=1$ the non-polynomial solution $y_1=(1-x)^{-\alpha}(1+x)^{-\beta}$. A corresponding solution for $n=2$ is obtained from this y_1 by setting $v_2=0$, and solutions $y_3, y_4, \cdots$ for higher values of n are then given by the recurrence formula.

8. The Bessel differential equation. For Bessel functions the formulas are comparatively simple once more, except for the less elementary definition of the initial functions in the sequences. The relations to be considered are those satisfied by the functions $J_n(x)$ for non-negative integral n and the functions of the second kind associated with them.

Let $y_0, y_1, y_2, \cdots$ be a sequence of functions connected by the recurrence formula

$$y_{n+1}=(2n/x)y_n-y_{n-1} \tag{32}$$

for $n\geqq 1$. Let

$$u_n=y_n'-\left(1-\frac{n(n-1)}{x^2}\right)y_{n-1}-\frac{n}{x}y_{n-1}', \qquad n\geqq 1,$$

$$v_n=y_n-\frac{n-1}{x}y_{n-1}+y_{n-1}', \qquad n\geqq 1.$$

Then

$$u_{n+1}=\frac{n-1}{x}u_n-\left(1-\frac{n(n-1)}{x^2}\right)v_n, \tag{33}$$

$$v_{n+1}=u_n+\frac{n}{x}v_n, \tag{34}$$

$$v_n'-u_n=y_{n-1}''+\frac{1}{x}y_{n-1}'+\left(1-\frac{(n-1)^2}{x^2}\right)y_{n-1}, \tag{35}$$

for the same values of n.

Let y_0 be any function satisfying the differential equation

$$y_0'' + (1/x)y_0' + y_0 = 0, \tag{36}$$

which is obtained by setting the right-hand member of (35) equal to zero with $n=1$. Let y_1 be defined by making $v_1=0$, with this choice of y_0. The whole sequence of y's is then determined. It follows from (35) that $u_1=0$, and then by an induction based on (33) and (34) that $u_n=v_n=0$ for all $n\geqq 1$. Consequently the right-hand member of (35) vanishes for each value of n, which means, on replacement of $n-1$ by n, that y_n is a solution of the Bessel differential equation

$$y_n'' + \frac{1}{x}y_n' + \left(1 - \frac{n^2}{x^2}\right)y_n = 0. \tag{37}$$

If $y_0=J_0(x)$, setting $v_1=0$ makes $y_1=-J_0'(x)=J_1(x)$, and it is seen that $y_n=J_n(x)$ for each n, since this is true for $n=0$, 1, and the y's are connected by the recurrence relation which is known to be satisfied by the functions $J_n(x)$ as otherwise defined.*

Now let y_0 be instead the function

$$y_0 = J_0(x)\log x + \frac{x^2}{2^2} - (1+\tfrac{1}{2})\frac{x^4}{2^2 4^2} + \cdots$$

which is a second solution† of (36). The relation $v_1=y_1+y_0'=0$ gives

$$y_1 = -J_0'(x)\log x - (1/x)J_0(x) - \tfrac{1}{2}x + \cdots,$$

or, since $-J_0'(x)=J_1(x)$, while $J_0(x)$ has the form $1-x^2/2^2+\cdots$,

$$y_1 = J_1(x)\log x - (1/x)\pi_1(x),$$

in which $\pi_1(x)$ is a power series in non-negative powers of x with constant term unity. It is found by application of (32) for the evaluation of $y_2, y_3, \cdots$ that

$$y_n = J_n(x)\log x - 2^{n-1}(n-1)!x^{-n}\pi_n(x),$$

where $\pi_n(x)$ is in each case a power series in non-negative powers of x with 1 as constant term. This y_n is a second solution of (37).

It will be observed that the results which have been presented in this and the preceding sections do not by any means exhaust the generality of the method employed. For example, the sequence of indices 0, 1, 2, $\cdots$ may with appropriate modifications of statement be replaced by a non-integral sequence proceeding at unit intervals; and in §§6 and 7 the parameters α and β need not be restricted, as in the ordinary theory of orthogonal polynomials, to values greater than -1, though the admission of values outside this range in §7 requires further attention to exceptional cases.

* See Monograph, p. 86.

† See Monograph, p. 219.

MATHEMATICS INSTRUCTION IN ENGINEERING

J. W. CELL, North Carolina State College

1. Introduction. As chairman of three different committees of the Mathematics Division of the Society for the Promotion of Engineering Education, the writer of this article has been engaged for the past four years in collecting engineering problems which illustrate mathematics and which are suitable for use in teaching freshman and sophomore courses in engineering mathematics. The Mathematics Division has entered into a non-royalty agreement with the McGraw-Hill Book Company for the publication of the resulting collection of 510 problems which will be available in book form this spring.

The actual work of collecting these problems and a survey of the final collection have suggested many ideas about mathematics instruction. It is the purpose of these notes to point out some of them. The writer has recently engaged in teaching mathematics for advanced radio training, and that experience also may color the following remarks. But in order to avoid any possible misunderstanding, the writer does not intend his remarks to suggest or to imply that the teaching of engineering mathematics should be solely utilitarian, or that we should teach the engineering student only such mathematics as he will need in his junior and senior engineering courses. It does stand to reason, however, that if certain required instruction in engineering mathematics would bring out the same general mathematical abilities as does some of the traditional mathematical subject matter, then it would be well to consider teaching the former in place of the traditional material. On the other hand, if a change in teaching methods should be suggested by engineering requirements, and if this change could be shown to indicate improvement in the mathematics instruction itself, then the teacher of mathematics would profit by such suggestions. It is in the light of these two possibilities that the following remarks are to be read. It is certain, however, that some mathematical topics are necessary for the proper understanding of mathematics itself, irrespective of possible later applications.

2. College algebra. The topic of completing the square in college algebra is an example of a mathematical procedure which has little direct engineering application. We teach it for its technique value and as the means of proof of the quadratic formula. Incidentally, we should correlate it with later mathematical needs as exemplified by the two following problems:

1. Complete the square under the radical $\sqrt{3-4x-2x^2}$ and give your result as the square root of a sum or difference of two quantities.

2. Complete the square on both x and y in the equation

$$2x^2 + 4y^2 - 4x - 12y = 7.$$

It is almost a truism that if a quadratic equation occurring in an engineering problem is solvable by the method of factoring, then the student has made an error.

The topic of radical equations likewise has certain manipulative values but

few actual direct applications in engineering. Further, it presents the first opportunity to discuss multiple-valued functions and the choice of a principal value. Here, too, we should introduce subsequent needs in analytic geometry in the form of problems, such as the elimination of the radicals from the equation

$$\sqrt{(x-4)^2+y^2}+\sqrt{(x+4)^2+y^2}=10.$$

A third topic the value of which is chiefly in the development of algebraic skill is the topic of equations in quadratic form. While I was gathering the problems for this collection I observed many applications to engineering of the general process of introducing a letter for a combination of letters or for a function of a given variable.

The investigation of engineering texts convinced me that most of the simultaneous linear equations found in engineering should be solved by methods of addition and subtraction or of substitution and not by the use of determinants. Where the coefficients are decimal numbers, some check method should be used in the process of addition and subtraction, such as the check on the algebraic sum of all coefficients and the constant term. Applications for determinants were found in proving theorems, in generating equations whose solutions would ensure non-trivial solutions of a set of linear homogeneous equations, in solving simultaneous linear equations with literal coefficients, and in testing the character of the roots of a given rational integral equation in one variable. The applications involved the theory of determinants rather than the use of determinants for solving simultaneous equations with numerical coefficients.

Most problems in engineering courses are word problems. According to engineering educators, one of the fundamental abilities to be achieved by the time of graduation from college is a thorough understanding of the engineering method and elementary competence in its use. The engineering or scientific method for the solution of problems comprises the four steps:

1. Sketching figures, such as free-body diagrams; labeling all relevant points, lines, etc.;
2. Determining what fundamental principles apply (Newton's laws of motion, Ohm's law, basic theorems from plane geometry, etc.); applying the principles and obtaining a mathematical problem;
3. Solving the mathematical problem;
4. Discussing the engineering implications of the mathematical results, the limitations that were originally imposed, and their consequences.

In the first quarter of the freshman year we teachers of engineering mathematics should initiate the development of this ability by our teaching of word problems.

The collection contains few higher-degree equations requiring solution by Horner's method, for in engineering problems the data seldom warrant accuracy to more than three significant figures, and so it is customary to solve higher-degree equations by cut-and-try methods and by graph. The approximate solution by slide rule of a cubic equation in one variable is easy if the second-degree

term is first eliminated by a process taught in the chapter on the theory of equations. In some cases where complex roots are required, as in electrical and in aeronautical engineering, Graeffe's method or some other approximate method is used. It seems clear that the reasons for teaching this chapter on the theory of equations, as well as the chapter on determinants, lie in the illumination of general theory, possible transformations, and manipulative technique.

3. Trigonometry. Word problems in trigonometry require adequate figures and serve as a second opportunity to teach the translation of them from English to mathematics and the general use of the engineering method. Incidentally, the mathematics teacher should distinguish carefully between word problems which need only common sense as the guide to sketching the required figures, and the other word problems which entail later scientific concepts as aids in figure sketching.

The accuracy of the given data in the majority of engineering problems is such that a solution by the slide rule is sufficient. It would seem appropriate, therefore, to introduce instruction in the slide rule early in the freshman year and to require the slide-rule solution of many problems in trigonometry. At North Carolina State College, at the specific request of the engineering college, we have introduced instruction in the slide rule into the course in trigonometry. We give rules for accuracy simultaneously and expect the student to choose between the slide rule and tables. We have found that the use of the slide rule increases the effectiveness of subsequent teaching, creates active interest, and does not necessitate the sacrifice of other topics in the course. One great advantage of the use of the slide rule is that we can drop the artificial numbers and special angles so common in analytic geometry and calculus problems and dispel the illusion that any involved result is automatically incorrect.

The uses of complex numbers in electrical engineering, especially in polar and exponential form, are many, as an inspection of any text on alternating currents will show. The collection of problems contains many of these applications.

The chapter on multiple angles has many applications, both direct and indirect. It was an amazing experience to me to learn that the really important multiple-angle formulas for the electrical engineer are those which transform a product to a sum or difference.

4. Analytic geometry. Locus derivation problems in analytic geometry serve as the third opportunity to teach the engineering method. Indeed, the steps required in a locus derivation are basically those of this method.

An examination of the problems on conics will show that most of the applications concern either sketching or finding the equation from simple data, and that there are few problems that require information about the foci or eccentricity. An equation from an engineering problem that requires the sketch of some one of the conics often has decimal coefficients, so that the use of the latus rectum is laborious and inferior to the method of finding the coordinates of a few well-chosen points.

The increasing use of transcendental, parametric, and polar coordinate graphs in engineering suggests that we should devote more time to their study. For example, the 2.3 relation between logarithms with the base e and logarithms with the base 10 is easily remembered after it has been visualized on a graph. Further, modern radio texts make use of a graphical solution for one parametric graph $y=f(t)$ from the graphs of $g(x, y)=0$ and $x=h(t)$. The solution consists in considering all three graphs as three views of a space curve and in using engineering drawing methods to complete the solution. I have learned that a somewhat similar approach makes easy the graphical construction of $g(x, y)=0$ (to use the preceding notation). Rapid sketching methods may be used to sketch the two parametric graphs, and then the required graph is easily completed.

Engineering teachers would be eternally grateful if instructors in mathematics taught the students a little about log log and semilog graph paper, since much of the modern engineering data are plotted directly on one of these two types of paper. (Note the relation to the question of decimal versus significant-figure accuracy.) One place to introduce this topic is in empirical equations in analytic geometry.

5. Calculus. One of the most recurrent suggestions in the correspondence about the collection of problems was that we devote more time to graphical and numerical differentiation and integration. Many engineering problems require the derivative or integral curve when the original function is given as a graph. I am convinced, especially by my experiences in this advanced radio training, that the teaching of mathematics would be much more effective if we placed greater emphasis on graphical instruction. Sophomore calculus students should be required to construct accurate, undistorted graphs for several standard functions such as $y=\sin x$ and $y=e^x$, and to determine both the derivative and integral curves by graphical and numerical methods.

Maximum-minimum and related-rate problems in the differential calculus serve as a fourth opportunity to teach the engineering method as well as mathematical methods. Some engineering applications require the largest or smallest value of a function in a given interval, which implies that the student must determine the end values as well as the intermediate extremum values.

The applications of integral calculus will show that there are few uses of volumes, lengths of arc, etc. In junior and senior engineering texts many definite integrals are obtained by the same general methods that we use to obtain the definite integrals in integral calculus. The chief application of these topics, then, consists in the art of setting up definite integrals and not in the use of the formulas in heavy print to be found in most calculus texts. (Sometimes I wish these formulas in heavy print were to be found only in the footnotes.) Although the solution of the usual calculus problems would be longer if the students were required to sketch and label the figure and typical element, to set up the differential statement for that typical element, to sum for all of the elements to obtain a definite integral, and finally to perform the integration; nevertheless this sequence of steps is indicated by engineering needs.

Whether we like it or not we teachers of mathematics must admit that most junior and senior engineering students are going to use integral tables for all but the simplest integrals. It would seem proper, therefore, for us to devote at least one assignment in integral calculus to the proper use of these tables.

6. **Multiple-step problems.** One ability that is especially difficult for the student to acquire is the identification of the successive mathematical steps required in a multiple-step problem. If the instructor has told him to solve a quadratic equation by the quadratic formula, the procedure is obvious. But if the engineering problem is to determine the positions of zero horizontal stress in a built-in horizontal beam, a subsequently derived quadratic equation may be quite mysterious. Again, the calculus problem of finding the area under the curve $y = \sin x$ from $x = 1$ to $x = 3$ would cause difficulty because the student would probably not remember that the angles are in radian measure. While single-step problems are necessary in each assignment, we should give at least one problem in each assignment that requires the use of some step learned in a preceding course, the use of which is not indicated in the problem. Mathematics texts should have many such problems, not only for their value as review problems but also for the development of ability in identification.

7. **Conclusion.** The difficult elements in the engineering method are the translation from engineering to mathematics and the determination of the required mathematical steps. Starting with word problems in college algebra and continuing through setting up definite integrals in calculus, we in mathematics have the opportunity to inaugurate the development of the engineering method. Most of the suggestions in this paper relative to methods of instruction are substantiated by problems in this S.P.E.E. collection. A few comments result from fruitless search for illustrative problems, whether for inclusion in this collection or for use in my own teaching.

APPLIED MATHEMATICS*

H. B. PHILLIPS, Massachusetts Institute of Technology

This paper is a brief report concerning the development of work in applied mathematics at the Massachusetts Institute of Technology. Applications of mathematics had always been emphasized at M.I.T. but plans to cover the various applied fields systematically were first seriously considered in 1934. At that time it was decided to have in the mathematics department at least one specialist in each important applied field, these specialists to be responsible for the program of education and research in their various fields. In particular it seemed desirable to include industrial statistics, analytical statistics, mechani-

* Presented at the twenty-fifth summer meeting of the Mathematical Association of America, at Poughkeepsie, New York, September 7, 1942.

cal applications, electrical applications, and fluid dynamics. An effort was accordingly made to find specialists in these fields.

Unfortunately this was during the depression and so far as we were concerned conditions were becoming steadily worse. Our only opportunity of obtaining new men was therefore through the choice of instructors to replace professors reaching retirement age. In some cases these instructors had little previous training in the fields they were expected to represent but seemed to have the right qualities and were willing to try.

The problem of industrial statistics is principally one of inspection and quality control in mass production industries. It soon became evident that this is as much a problem of economics as of mathematics. Arrangements were therefore made to have the work handled in cooperation with the economics department. Courses, seminars, and industrial surveys were conducted jointly by members of the two departments. After a few years problems in industry were investigated, first as a part of the educational program and then on a contract basis for pay to the Institute. At the outbreak of the war these contracts showed prospect of supporting our entire program in applied mathematics and at the same time supplying problem work in an important field.

Work in analytical statistics was much more easily provided since this had long been one of Professor Wiener's principal fields. The most interesting problem is that of extrapolation in time series. When relations exist it is a question of carrying these forward. The methods first used for doing this were too complicated but these methods have recently been simplified and are now becoming part of the general statistical technique.

In the electrical field it was recognized that mathematicians work at a particular disadvantage due to electrical engineers being mathematically the best trained of all the engineers. For success in competition with them it is necessary to use mathematical methods which they have not sufficiently exploited. Of such methods integral equations seemed to have best prospect of success. The principal difficulty of using integral equations is in reducing results to numerical form. As his first task our representative in this field, Professor Crout, therefore developed methods for obtaining numerical solutions of integral equations. This was accomplished by use of polynomial approximations somewhat analogous to Simpson's Rule. These methods were then used on problems first for practice and then on work handled under contract for particular firms.

Mathematical applications to mechanics consist mainly in the determination of the strength and vibration characteristics of suggested structures. As a basis for this, graduate courses in mechanics and theory of elasticity were given in the department of mathematics. The principal applications made so far have been in the field of aeronautics.

In the meantime work of a very mathematical character was also being done in the science and engineering departments. In 1941 arrangements were, therefore, made to consolidate the mathematical resources of all departments in a research center designed to handle graduate work and research in applied mathe-

matics. This center is under the charge of an interdepartmental committee consisting of representatives of the mathematics department and of all departments in which important applications occur. Graduate students in applied mathematics work directly under the committee, the only limitation being that their study programs include the essentials of one or more fields of application and the mathematics needed to analyze those fields. This type of interdepartmental cooperation has been found particularly satisfactory. Members of each department, being busy with their own affairs, give attention only to matters in which they are particularly interested with the result that each detail is handled by those most competent to handle it.

Funds have been provided for fellowships in applied mathematics but the demand for this type of training is so great that men holding these fellowships have usually been employed in research at the end of one term. The further development of this work is thus even in peace time limited almost entirely by the supply of properly qualified graduate students.

SOME HISTORICAL NOTES ON THE CYCLOID*

E. A. WHITMAN, Carnegie Institute of Technology

1. Introduction. In this paper our interest is not in a renowned mathematician, a celebrated school, or a famous problem, but in a curve, the cycloid. More particularly, our interest is to center around its relation to the mathematics of the seventeenth century, one of the great centuries in the history of the subject. This curve had the good fortune to appear at a time when mathematics was being developed very rapidly and perhaps mathematicians were fortunate that so useful a curve appeared at this time. A new and powerful tool for the study of curves was furnished by the analytic geometry, whose year of birth is commonly given as 1637. New methods for finding tangents to curves, the areas under curves, and the volumes of solids bounded by curved surfaces were being discovered at a rapid pace, and a new subject, the calculus, was in the making. In these developments the cycloid was the one curve used preeminently and nearly every mathematician of the time used it in a trial of some of his new theory, even to the extent that much of the early histories of analytic geometry, calculus, and the cyloid are closely interwoven.

In the history that follows we shall not be concerned with historical minutiae, but only with the broad outlines of the story of this curve.

2. Early history of the curve. The original discoverer of the cycloid appears to be unknown. Paul Tannery has discussed a passage by Iamblichus referring to double movement and has remarked that it is difficult to see how the cycloid

* Presented to the Allegheny Section of the Mathematical Association of America, October 25, 1941.

could have escaped the notice of the ancients.* John Wallis in a letter of 1679, ascribed the discovery to Nicolas Cusanus in 1450 and also mentioned Bouelles as one who in 1500 advanced the study of this curve. In the case of Cusanus, however, historians are agreed that Wallis was mistaken unless, says Cantor, he had access to some manuscript now lost. Now Bouelles mentions that he had observed a rolling wheel yet he seems to have considered the generated arch as a part of a circle whose radius was five-fourths that of the generating circle. The history of the cycloid becomes more definite when we come to Galileo. This scientist and teacher, famed for his telescope and microscope and as the discoverer of the isochronism of the vibrations of a pendulum, this Galileo attempted the quadrature of a cycloidal arch in 1599, at least so writes his pupil Torricelli in a publication of 1644. We here learn that Galileo had sought to measure its area and for this purpose used a balance upon which he placed a material cycloidal arch and a generating circle of like material. Always the arch was about three times as heavy as the circle, wherefore Galileo had given up his experiment since he believed that an incommensurable ratio was in question. Cantor writes of Galileo that he was the first to make this curve well known and that it was he who gave it its name. The curve was also known as a roulette and as a trochoid.

3. The work of Roberval. The scene now shifts to France, to the activities of Gilles Persone de Roberval, and to the problem of the quadrature of the cycloid. Going up to Paris in 1628, Roberval soon became a member of that small group of scientists and mathematicians who were wont to gather twice a week, generally at the home of Père Marin Mersenne, to discuss matters of common interest. Now Mersenne had brought the cycloid to the attention of French mathematicians at various times and Roberval soon learned of this curve but could not immediately effect the quadrature. However, a new method of finding the areas under curves was made known in 1629 when Cavalieri submitted his notes on the theory of *indivisibles* to show his fitness for the chair of mathematics at the University of Bologna, where he was a candidate. This new theory, and its extensions later, exerted an enormous influence upon the subject of finding the areas under curves, hence on the development of the calculus. In this paper we are concerned only with one part of this theory which is known as Cavalieri's Theorem† and which says that if two areas are everywhere of the same width one to the other, then the areas are equal.

About 1634 Roberval effected the quadrature of the cycloid, or trochoid as he called this curve. The first publication of his proof seems to have been in 1693 when his *Traité des Indivisibles*‡ appeared. To explain the long delay in

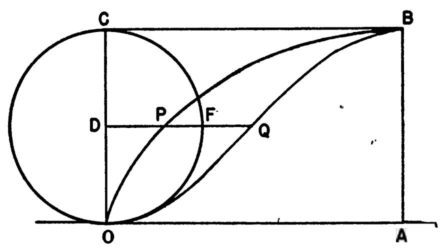
* Pour l'histoire des lignes et surfaces courbes dans l'antiquité; Bulletin des sciences mathématique, Paris, 1883, p. 284.

† A translation of this theorem and its proof is given in Smith's Source Book in Mathematics, also by Professor G. W. Evans in the American Mathematical Monthly for December, 1917.

‡ A Study of Roberval's Traité des Indivisibles, Columbia University, 1932, by Professor Evelyn Walker, gives an extended account of Roberval's works and discusses at some length the subject of indivisibles.

publication of this important discovery, it may be noted that the Chair of Ramus at the Collège Royale which Roberval had won in 1634, automatically became vacant every three years, to be filled again by open competition. As the incumbent set the questions it seems plausible that Roberval should conceal his methods. In this way he would have a set of questions whereby he should win the coming contests. Professor Walker states that the accident of occupying this chair caused Roberval to lose credit for many of his discoveries.

Roberval's quadrature depends upon a so-called cycloid companion curve and an application of Cavalieri's Theorem. Professor Walker gives a translation of this quadrature, but we shall describe it only in a general way. This is among the very earliest of the quadratures.



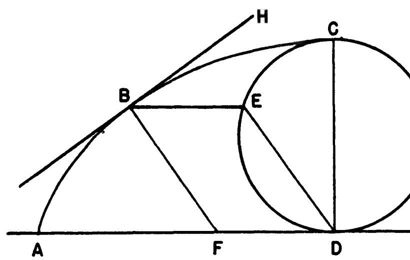
Let $OABP$ be the area under the half arch of the cycloid whose generating circle has the diameter OC. Take P any point on the cycloid and take PQ equal to DF. The locus of Q will be the companion curve to the cycloid. This curve OQB is the sine curve $y=a \sin x/a$ where a is the radius of the generating circle, if we take the origin at the midpoint of the arc OQB, and the x-axis parallel to OA. Now by Cavalieri's Theorem, the curve OQB divides the rectangle $OABC$ into two equal parts since to each line as DQ in $OQBC$, there corresponds an equal line in $OABQ$. The rectangle $OABC$ has its base and altitude equal respectively to the semicircumference and diameter of the generating circle, hence its area is twice that of the circle. Thus $OABQ$ has the same area as the generating circle. Also the area between the cycloid OPB and the curve OQB is equal to the area of the semicircle OFC since these two areas are everywhere of the same width one to the other. Hence the area under the half arch is one and one-half times the area of the generating circle, and the area under the arch is three times that of the generating circle.

4. Construction of the tangent. Early in 1638, Mersenne wrote to Fermat and Descartes presenting for their consideration the problem of the quadrature of the cycloid and the construction of a tangent to the curve. For a year or more previously Roberval and Fermat had been in correspondence, with Senator Carcavy as intermediary. The subjects discussed included tangents, cubatures, and centers of gravity. Mersenne's letters, however, brought to a focus the question of tangents for in August of this year Roberval, Fermat, and Descartes each gave Mersenne a method of drawing a tangent and each had a different method. In the ensuing dispute between Fermat and Descartes over the relative merits of their constructions, Roberval sided with Fermat. In turn Descartes wrote

several letters to Mersenne bitterly ridiculing some of Roberval's tangent constructions which Mersenne had transmitted to him.

The question of priority in the matter of tangents we leave as unsettled and also unimportant, since each could not have borrowed from the others, so different were the methods. Part of the dispute over the relative merits of the constructions arose from different ideas as to the meaning of tangents to curves other than circles. The definition of a tangent as the limiting position of a secant had not yet been generally accepted.

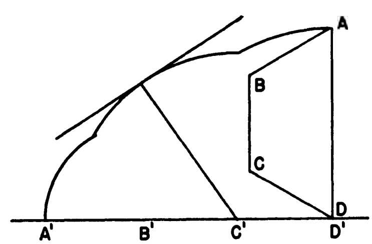
We proceed to describe each of the three tangent constructions. Descartes' method is that which we now call instantaneous centers of curvature.



Let B be any point on the half arch of the cycloid ABC and let it be required to draw a tangent to the cycloid at B.

Draw BE parallel to the base AD cutting the circle at E. Draw BF parallel to ED and BH perpendicular to BF. BH is the required tangent.

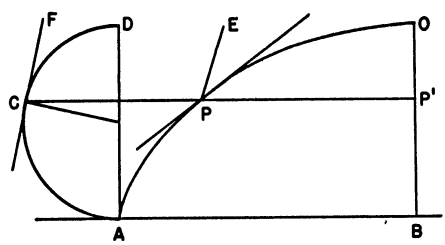
The proof is based on the following considerations:



If a polygon $ABCD$ rolls on a straight line $A'D'$, any point A will describe a number of segments of circles whose centers will be at B', C', D', etc. The tangents to these segments will always be perpendicular to the line joining the point of tangency to the center of the circle. Consequently if the generating circle is considered as a polygon which has an infinite number of sides, the tangent at a given point will be the line perpendicular to the line joining this point to the point where the generating circle touches the base at the same instant it passes through the point.

Roberval's tangent construction makes use of the composition of forces and is easily understood in connection with his particular way of stating the definition of the cycloid.

Let the diameter AD of the circle move always parallel to its original position with A on the line AB until it takes the position BO with AB equal to a semicircumference. At the same time let the point A move on the semicircle ACD in such a way that the speed of AD along AB may be equal to the speed of A along the semicircle, thus allowing A to reach the point D at the same time AD reaches BO. The point A is carried along by two motions, its own on the semicircle and that of the diameter AD. The path of A due to these two motions is the half cycloid APO.



To construct a tangent at any point P on the cycloid, draw PP' parallel to AB cutting the semicircle at C. Then draw CF tangent to the semicircle and draw PE parallel to CF. The bisector of the angle EPP' is the required tangent since it is the resultant of two equal motions.

While finding the two components may be difficult for many curves, yet the cycloid is said to be the eleventh curve for which Roberval thus found tangents.

Fermat's construction is not unlike that of Descartes, but the proof appears to the casual reader to be quite as complicated as that of Descartes is simple. In the course of the proof a straight line is replaced by the arc of a circle. This is equivalent to assuming that an arc of a circle approaches coincidence with a certain straight line, making the method essentially one of limits. To one interested in the early approaches to the calculus, Fermat's method will be more interesting than that of Roberval or Descartes. The methods of the latter show what can be done in special cases and without the calculus. As Fermat's proof is quite long and is readily available elsewhere,* it will not be shown here.

With the area under the cycloidal arch and the tangent construction well mastered by his fellow Frenchmen, Mersenne announced these results to Galileo in 1638. Galileo, now old and blind, passed them on to his pupils Torricelli and Viviana, adding the suggestion that this curve would give a graceful form for the arch of the bridge that was projected for the nearby Arno River at Pisa. These pupils responded with a quadrature and a tangent. The interest thus kindled led Torricelli to a considerable study of the curve. In 1644 he made public his quadrature and a method of drawing a tangent. This was the earliest printed article on the cycloid.

* See Cantor's Vorlesungen über Geschichte der Mathematik, Volume II, pages 861–863, or Walker's Traité des Indivisibles, pages 132–134.

Roberval was angered at seeing another print proofs that he considered his own discoveries. He wrote a letter to Torricelli charging plagiarism. More specifically, Roberval charged that a certain Frenchman had written out Fermat's method of maxima and minima and Roberval's propositions on the cycloid, that these papers had come into Torricelli's hands after the death of Galileo, and that Torricelli had published them as his own. This dispute was cut short by Torricelli's early death in 1647, a death caused, according to Cajori, by this charge of plagiarism.

5. Pascal's mathematical contest. Our next episode in this history centers around Blaise Pascal, known for his *Pensées* and his *Lettres Provinciales* as well as for his mathematical works. After a brilliant early career in mathematics he had turned to theology. But suddenly the old mathematical propensity reasserted itself. Ball writes that Pascal was suffering from sleeplessness and toothache when the idea of an essay on the cycloid occurred to him. To his surprise the tooth ceased to ache. Regarding this as a divine intimation to proceed with the problem, he worked incessantly at it for eight days and completed a tolerably full account of the geometry of the cycloid. As certain questions about this curve had never been publicly answered, a prize was now offered by Pascal under the nom de plume of Amos Dettonville.

The year was 1658 when Newton was sixteen years old. The prizes were two in number, forty and twenty Spanish doubloons. The time allotted was June first to October first. Senator Carcavy was made recipient of the solutions offered and he, Pascal, and Roberval were the judges. The problems were as follows:

(1) The area and the center of gravity of that part of a cycloidal arch above a line parallel to the base.

(2) The volume and center of gravity of the volume generated when the above area is revolved about its base and also about its axis of symmetry.

(3) The center of gravity of the solids formed when each body is cut by a plane parallel to its axis of revolution.

Only two contestants, Wallis and Lalouvère, had submitted offerings when time was called. Ball says that Wallis did not submit solutions for the centers of gravity, and Cajori says that Wallis made many mistakes. Both historians agree that Lalouvère was quite unequal to the task. The judges declared that neither contestant was entitled to a prize.

At the time of this contest, Sir Christopher Wren sent to Pascal his proposition on the rectification of the cycloid, not, however, including any proof. When Pascal showed this to Roberval the latter is said to have proved the proposition immediately, claiming to have known it for many years. To Wren goes the credit for the first publication and its proof,* when Wallis published it as Wren's a year later in his *Tractatus duo.*

While the contest was on, Pascal published his *L'Histoire de la Roulette* and after the decision of the judges, his solution of the problems. With Pascal's and

* The general character of the proof is given in Cantor, Volume II, page 904.

Wallis's publications at this time, the problems of quadrature, tangents, rectification, cubature, and centers of gravity are substantially completed in so far as the cycloid, or roulette as it was better known to the Frenchmen, was concerned. All this was accomplished in a period of about twenty-five years and before Newton's work in the calculus. The principle of *indivisibles*, or *infinites*, or whatever they had used, had in the hands of Roberval, Fermat, Torricelli, Wren, and Wallis led to important results. The cycloid curve was always being used; it was the pre-eminent curve, and its importance was to be seen later.

6. The brachistochrone problem. In another fifteen years, Huygens was using the cyloidal pendulum in an attempt to get a better chronometer and made use of the property that the evolute of the cycloid is another equal cycloid. This same Huygens discovered that a heavy particle reached the bottom of an inverted cycloidal arch in the same length of time no matter from what point on the arch it began its descent. In 1686, Leibniz wrote the equation for the curve, thus showing the rapid progress that was being made in analytic geometry. This equation is given here as Leibniz wrote it since his form shows interesting variations from those employed at present:

$$y = \sqrt{2x - xx} + \int dx/\sqrt{2x - xx}.$$

In the decade following the publication of this equation, the Bernoulli brothers, Jacques and Jean, published several articles on the cycloid. But we shall hurry on to one final episode in the history of the curve.

In June, 1696, Jean Bernoulli proposed a new problem which mathematicians were invited to solve: If two points A and B are given in a vertical plane, to assign to a mobile particle M the path AMB along which, descending under its own weight, it passes from the point A to the point B in the briefest time. In later amplifying the problem Bernoulli says to choose such a curve that if the curve is replaced by a thin curve or groove and a small sphere placed in it and released, then this sphere will pass from one point to the other in the shortest possible time. Thus the famous brachistochrone problem appeared on the scene. The solution is the inverted cycloidal arch. An elaborate model of the brachistochrone formed a considerable part of the mathematics exhibit at the Golden Gate International Exposition in 1940, from which we may conclude that there is still considerable interest in the problem.

In giving out his solution,* Jean Bernoulli wrote that a new kind of maxima and minima is required. In this solution we see that mathematics had advanced at this time as far as the calculus of variations. In a few more years there began to appear articles on general methods for determining the nature of curves formed by other rolling circles and on curves of descent under activating forces other than gravity. As the cycloid thus loses its pre-eminence, this seems a proper place to close this recital of its history.

* See Smith's Source Book in Mathematics, pages 644–655.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Allegheny College, Meadville, Pa.

ANNUAL REPORTS

Some annual club reports for the year 1942–43 have already been received. The editor will be glad to receive the others as soon as possible, so that plans can be made for the ensuing year. Reports should consist of a single paragraph giving a summary of the activities of the club and ending with the names of the officers. A good length is about one double spaced typewritten page. Printed programs and other material of interest, such as bibliographies for club talks, should be included with the club report whenever possible. The best student speaker for the year might be encouraged to write up his talk and submit it to the editor for possible publication.

CLUB REPORTS 1941–42

Mathematics Club, University of Dayton

Semi-monthly meetings were held at which members of the club presented papers: *The extraction of roots of numbers* by Robert Stacy, *Impossibilities and unsolved problems* by Joseph Overwein, *Non-Euclidean geometries* by William Fitzgibbon, *The Hatchet planimeter* by Harold Raybould, *Probabilities* by Robert Mantey, *Maps and charts* by Paul Herking. Three members of the faculty also presented papers: *Nomographs* by Mr. C. G. Peckham, *Operations with a slide rule* by Professor A. R. Weber, *Life and work of Niels Henrik Abel* by Professor K. C. Schraut. In December the Club sponsored a *Symposium on Applied Mathematics* at which Mr. W. E. Restemeyer of the University of Cincinnati spoke on *The Laplace transformation*, Mr. C. A. Ludeke of the University of Cincinnati spoke on *The mathematical and physical conditions causing resonance*, and Professor K. C. Schraut of the University of Dayton spoke on *A method of approximating the sum of a convergent series.* After the Symposium a testimonial dinner was held at the Dayton Engineers' Club in honor of the speakers. Toward the close of the year a steak fry was held at Hills and Dales Park. At the final meeting of the year the guest speaker was Professor C. N. Moore, Director of Graduate Studies at the University of Cincinnati, who delivered a very interesting lecture on *The relationship between pure and applied mathematics.* At this meeting the Annual Dean of Science Book Award was conferred upon William Fitzgibbon for having delivered before the Club the most interesting paper of the year. The book chosen for this year was *Men of Mathematics* by E. T. Bell. At the close of the year the Alumni of the Club established the *Mathematics Club Alumni Award for Excellence in Advanced Mathematics* which is to be conferred each year at commencement upon that senior who, in the opinion of the Faculty of the Department of Mathematics, has distinguished himself most in courses beyond the calculus. The award was conferred upon Robert Stacy. The officers of the Club were: President, Robert Stacy; Vice-President, Joseph Overwein; Secretary, William Fitzgibbon. Professor K. C. Schraut served as Faculty Adviser.

Pi Mu Epsilon, Oklahoma A. and M. College

Twenty new members were initiated at a banquet held April 9, 1942, at which time two prizes were awarded, one to the outstanding freshman in mathematics, and one to the best initiate, both selected by competitive examination. The printed menu for the banquet announced a variety of mathematical solids including *ellipsoids of carbohydrates, latticed solids*, and a volume bounded by surfaces whose equations suggested a piece of pie; followed by an after dinner program of *Harmonics*

by Julia Herman, *Differentiation* by Ramez Saab, *Integration* by George Goddard, and *Logic in non-mathematical fields* by Edward Robinson. Other papers presented at club meetings during the year were: *Ciphers and cryptograms* by Peter Messenger, *Mathematics in war* by Lt. Col. Yost of the U. S. Army, *Finite differences* by Dr. Diamond, and *Astronomical distances* by Dr. Mindenhall. The officers for 1942–43 are: Director, Patrick Butler; Vice-Director, William Schierman; Secretary-Treasurer, Lawrence Hanna; Corresponding Secretary, E. F. Allen.

Delta Rho, Southern Illinois Normal University

Interesting lectures were presented at every meeting of the club, mostly by members of the club. Among these were *Books of mathematics* by Vernon Snead, *Adding and subtracting logarithms* by Bill Fisher, and *History of geometry* by Betty Johnson. In February every member of the club cooperated in scoring the tests given for the Mathematics Field Day by the Southern Illinois Council of Mathematics Teachers. A Founder's Day banquet was held in March at which Claude Pyle, an outstanding senior chosen by the department, spoke on *Vector analysis.* A six page *Mathematics News* sheet, printed in March, included many items about the *Delta Rho* honorary fraternity, whose student membership is limited to mathematics majors with averages of at least B in mathematics and to sophomores enrolled for their third term of calculus with at last one A in the calculus. This publication also included news and pictures of mathematics classes in several nearby high schools, and articles written by students about the importance of mathematics in the war effort. The officers of *Delta Rho* for the year were: President, Robert Clendenin; Vice-President, Claude Pyle; Secretary, Mary Downen; Treasurer, William Hentze; Program Chairman, Eugene Ulrich.

Kappa Mu Epsilon, Illinois State Normal University

The activities of the chapter were recorded in a mimeographed booklet dedicated to the alumni of *Kappa Mu Epsilon* who are in the armed forces of the United States. At a homecoming breakfast thirty-two members heard a talk on *The bright and dark side of teaching mathematics* by an alumna, Mildred Schulze. Book reviews by Nancy Hightower and Harold Gambrel were given at the next meeting, and in December the members presented a play *The evolution of numbers,* written by the late Professor H. E. Slaught of the University of Chicago. In January a panel discussion was held on the subject *Student teachers in mathematics,* and in March a radio program entitled *Mathematics for victory* was broadcast over *WJBC* by the following cast of members: Gene Weed, Mildred Bauer, Warren Buck, Don Reeves, Melvin Meisinger. At the last regular meeting in April, Dr. C. N. Mills spoke on *The future in mathematics,* discussing in detail methods of discovering prime numbers. Commanding Officers were: Sponsor, Edith I. Atkin; President Gauss, Shirley Isaacson; Vice-President Pascal, Nancy Hightower; Recording Secretary Ahmes, Geneva Meers; Treasurer Napier, Leo Montgomery; Historian Cajori, Warren Buck; Social Chairman Lilivati, Dorothy Johnson; Corresponding Secretary Descartes, C. N. Mills.

Pi Mu Epsilon, University of Wisconsin

A talk on the subject of *Gyroscopes* was given at the opening meeting by Professor Rudolph Langer. The later meetings included talks on *Methods of apportionment in Congress* by Churchill Eisenhart, *Vector diagrams in air navigation* by Mr. Hillis, *Planimeters* by Walter Sivley, *The use of small scale research models in engineering* by Robert McBurney, *The Gamma function* by Professor R. D. James, and *The arithmetic of infinite numbers* by Mr. Tompkins. A three hour examination consisting of thirteen mathematical problems designed to test the intelligence and resourcefulness of the undergraduate student was sponsored by the chapter, with a first prize of fifteen dollars and a second prize of five dollars. Henry Rogers, son of a professor of French at the University, was winner of the first prize. The officers elected for 1942–43 are as follows: President, Arne Larson; Vice-President, Anne Braun; Secretary, Betty Lohr; Treasurer, Henry Rogers. The corresponding secretary for the past year was Keith LeRoy Clark.

Mathematics Club, University of Kansas

Meetings were held on the first and third Thursdays of each month at 4:45 P.M. following a half-hour period of refreshments. The following topics were announced in advance on a printed program: *Mathematics in Coast Artillery* by Lieutenant Baker, C.A.C.; *Three famous problems of ancient mathematicians* by Jean Bartz, *Some problems in synthetic geometry* by Clark Moots, *A history of the development of algebraic symbols* by Harwood Kolsky, *Gambling* by Howard Barnett, *Computing machines* by John Yarnell, *Summation of some interesting numerical sines* by William Luby, University of Kansas City, *The mechanics of rocket flight* by Howard Gadberry, *The special theory of relativity* by John Ise, Jr., *Mathematics in Economics* by Louise Polson. Bibliographies for some of these topics were prepared and have appeared in the September 1942 issue of this MONTHLY. Officers of the club were: President, Merle DeMoss; Vice-President, Harwood Kolsky; Secretary-Treasurer, Jean Bartz; Social Chairman, Irene McClune; Faculty Adviser, Dean Gilbert Ulmer.

The Square Circle Woman's College, University of North Carolina

A lesson in the mathematics used in the army, taught by Lieutenant E. S. Brown, proved to be the highlight of our programs this year. Another interesting program was the demonstration of the slide rule, varieties of graph paper, an instrument for drawing the ellipse, and a model to represent the fourth dimension. The instrument and model were made by students of the Charlotte, N. C., high school. The traditional ceremony was used for the initiation program. Mathematical games and puzzles, including mathematical bingo, were played at the Christmas party. The last meeting was a picnic; new officers were elected. Officers for 1941–42 were: President, Mary Lou Mackie; First Vice-President (Program Chairman), Janice Pickard; Second Vice-President (Assistant Program Chairman), Zabelle Corwin; Secretary-Treasurer, Shirley Elliot.

The Archimedeans, Winthrop College

Three talks on the lives and works of great mathematicians were presented at the October meeting: *Thales* by Sarah Parks; *Ptolemy* by Lillie Belle Evans; *Desargues* by Jessie Cockfield. Other program topics were *Mathematical logic* by Callie Hartley, *Mathematical recreations* by Frances Pinckney and *Paper folding* by Margaret Rickman. Officers for 1942–43 were chosen as follows: President, Lillie Belle Evans; Vice-President, Kathlyn Bomar; Secretary, Jessie Cockfield; Treasurer, Nellie McGill; Social Chairman, Frankie Cole.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A NOTE ON THE DIVERGENCE OF A SERIES

RICHARD BELLMAN, University of Wisconsin

The purpose of this note is to add another elementary proof of the well-known theorem that the series whose elements are the reciprocals of the primes diverges. This proof is very close to the proof of Hermite, elegant and elementary, that the number of primes is infinite.

Assume $\sum_p 1/p < \infty$. Then there would exist a P such that

$$\sum_{p \geqq P} \frac{1}{p} = \alpha < 1.$$

Hence

$$\left(\sum_{p \geqq P} \frac{1}{p}\right)^2 = \sum_{p,q \geqq P} \frac{1}{pq} = \alpha^2,$$

where p, q, range over all primes $\geqq P$, and similarly

$$\left(\sum_{p \geqq P} \frac{1}{p}\right)^n = \sum_{p,q,\cdots \geqq P} \frac{1}{p\,q\cdots} = \alpha^n.$$

Hence, since a double series of positive terms can be summed in any manner, $\sum 1/N$, where N ranges over all numbers representable as a product of powers of primes greater than or equal to P, is convergent by comparison with $\sum_1^\infty \alpha^n$.

But

$$\left(\sum \frac{1}{N}\right) \frac{1}{\left(1 - \frac{1}{2}\right) \cdots \left(1 - \frac{1}{p}\right)},$$

where p is the greatest prime less than P, is equal to

$$\left(\sum \frac{1}{N}\right)\left(\sum \frac{1}{M}\right),$$

where N has the same connotation as before, and M ranges over all numbers representable as a product of powers of primes less than P. From the unique factorization theorem

$$\left(1 + \sum \frac{1}{N}\right)\left(\sum \frac{1}{M}\right) = \sum_1^\infty \frac{1}{n}$$

as the product of two convergent series of positive terms and with permissible rearrangements. But $\sum_1^\infty 1/n$ diverges. Hence there is a contradiction, and $\sum_p 1/p$ diverges.

AN ANALOG OF PASCAL'S ARITHMETICAL TRIANGLE

S. F. Bibb, Illinois Institute of Technology

The purpose of this note is to express the coefficients of

$$(1) \qquad y_n = \sum_{s=0}^{r} (-1)^s {}_nD_s x^{n-2s},*$$

* For another form of the coefficients of this equation see Elementary Theory of Equations, first edition, by L. E. Dickson, p. 83.

($r=n/2$ if n is even and $(n-1)/2$ if n is odd), obtained by eliminating the parameter t from

$$x = t + t^{-1}, \qquad y_n = t^n + t^{-n}, \qquad (t \neq 0,\ n = 1, 2, 3, \cdots), \tag{2}$$

as determinants, and to show how they may be arranged as a triangle analogous to that of Pascal, thus:

n	Absolute value of coefficients of x				
1	1				
2	1	2			
3	1	3			
4	1	4	2		
5	1	5	5		
6	1	6	9	2	
7	1	7	14	7	
8	1	8	20	16	2
9	1	9	27	30	9
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$n-2$	${}_{n-2}D_0$	${}_{n-2}D_1$	${}_{n-2}D_2$	${}_{n-2}D_3$	${}_{n-2}D_4 \cdots {}_{n-2}D_r$
$n-1$	${}_{n-1}D_0$	${}_{n-1}D_1$	${}_{n-1}D_2$	${}_{n-1}D_3$	${}_{n-1}D_4 \cdots {}_{n-1}D_r$
n	${}_nD_0$	${}_nD_1$	${}_nD_2$	${}_nD_3$	${}_nD_4 \cdots {}_nD_r$

Except for the first element in each row and the last element when n is even, the law of formation of this triangle is given by the formula,

$${}_nD_s = {}_{n-1}D_s + {}_{n-2}D_{s-1}.$$

The D-symbols are defined in the following way:

$${}_nD_0 = 1, \qquad {}_nD_1 = {}_nC_1,$$

$${}_nD_2 = \begin{vmatrix} {}_nC_1 & {}_nC_2 \\ 1 & {}_{n-2}C_1 \end{vmatrix}, \qquad {}_nD_3 = \begin{vmatrix} {}_nC_1 & {}_nC_2 & {}_nC_3 \\ 1 & {}_{n-2}C_1 & {}_{n-2}C_2 \\ 0 & 1 & {}_{n-4}C_1 \end{vmatrix}, \cdots,$$

$${}_nD_r = \begin{vmatrix} {}_nC_1 & {}_nC_2 & {}_nC_3 & {}_nC_4 \cdots & & {}_nC_r \\ 1 & {}_{n-2}C_1 & {}_{n-2}C_2 & {}_{n-2}C_3 \cdots & & {}_{n-2}C_{r-1} \\ 0 & 1 & {}_{n-4}C_1 & {}_{n-4}C_2 \cdots & & {}_{n-4}C_{r-2} \\ 0 & 0 & 1 & {}_{n-6}C_1 \cdots & & {}_{n-6}C_{r-3} \\ \cdots & \cdots & \cdots & \cdots \quad \cdots & & \cdots \\ 0 & 0 & \cdots & 1 & {}_{n-2r+4}C_1 & {}_{n-2r+4}C_2 \\ 0 & 0 & \cdots & & 1 & {}_{n-2r+2}C_1 \end{vmatrix}. \tag{3}$$

The C-symbols are binomial coefficients.

The binomial theorem may be used to obtain powers of x in terms of y_j, $j=1, 2, \cdots, n$, thus:

(4) $x^n = (t + t^{-1})^n = y_n + {}_nC_1 y_{n-2} + {}_nC_2 y_{n-4} + \cdots + {}_nC_{n/2},$ if n is even,

(5) $x^n = (t + t^{-1})^n = y_n + {}_nC_1 y_{n-2} + {}_nC_2 y_{n-4} + \cdots + {}_nC_{(n-1)/2} y_1,$ if n is odd.

If n is even the $n/2$ equations (4), $n=2, 4, \cdots, n$, and if n is odd, the $(n+1)/2$ equations (5), $n=1, 3, \cdots, n$, may be solved for $y_n = f(x)$. For example,

$$y_6 = x^6 - {}_6C_1 x^4 + \begin{vmatrix} {}_6C_1 & {}_6C_2 \\ 1 & {}_4C_1 \end{vmatrix} x^2 - \begin{vmatrix} {}_6C_1 & {}_6C_2 & {}_6C_3 \\ 1 & {}_4C_1 & {}_4C_2 \\ 0 & 1 & {}_2C_1 \end{vmatrix},$$

$$y_7 = x^7 - {}_7C_1 x^5 + \begin{vmatrix} {}_7C_1 & {}_7C_2 \\ 1 & {}_5C_1 \end{vmatrix} x^3 - \begin{vmatrix} {}_7C_1 & {}_7C_2 & {}_7C_3 \\ 1 & {}_5C_1 & {}_5C_2 \\ 0 & 1 & {}_3C_1 \end{vmatrix} x.$$

THEOREM 1. *The general form for y_n is given in equation* (1), *where the D-coefficients are defined in* (3).

A proof of this theorem by induction will be given. Assume that equation (1) holds for $n=k-2$ and $k-1$. To show that the law is valid for $n=k$, the left member of (1), obtained by setting $n=k-1$, may be multiplied by $t+t^{-1}$ and the right member by x. By transposing and simplifying this resulting equation the proof is completed by demonstrating that the D-coefficients must be connected by precisely the same formula as that which was observed to be the probable law of formation of the above triangle. Namely,

$$(6) \qquad {}_{k-2}D_{s-1} + {}_{k-1}D_s = {}_kD_s.$$

To derive formula (6), the law for constructing the triangle of binomial coefficients,

$$(7) \qquad {}_nC_r + {}_nC_{r+1} = {}_{n+1}C_{r+1},$$

points the way. Hence, in ${}_{k-1}D_s$ add each adjacent column from right to left and apply (7). To this equivalent determinant add ${}_{k-2}D_{s-1}$ written as a determinant of order s, and the result is ${}_kD_s$.

THEOREM 2. *In absolute value* ${}_nD_{n/2}=2$ and ${}_nD_{(n-1)/2}=n$.

The proof of this theorem follows at once from equation (1) by setting $t=\sqrt{-1}=i$; for then $x=t+t^{-1}=0$ and $|y_n| = |t^n+t^{-n}| = 2$, if n is even. However, if n is odd, both $t+t^{-1}$ and $t^n+t^{-n}=0$. In this case, the right member of (1) may be divided by x and the left member by $t+t^{-1}$ and then note that $|\lim_{t\to i}(t^n+t^{-n})/(t+t^{-1})| = n$.

The following four facts are of some interest. If the columns of elements in the above triangle are numbered $0, 1, 2, \cdots, k$, then beginning with $k=1$ it is seen that (a) the first term in any column is ${}_{2k}D_k$, (b) the number of terms in any column is $n-(2k-1)$. Furthermore, since the first terms of the successive orders of differences of the terms in any column k are the 2nd term in column $k-1$, the

3rd term in column $k-2$, the 4th term in column $k-3$, $\cdots$ (this follows easily from (6)), then, (c) the jth term, ${}_{2k-1+j}D_k$, in any column may be written as

$$ {}_{2k}D_k + \sum_{u=1}^{k} {}_{j-1}C_u \, {}_{2k-u}D_{k-u}, $$

(d) the sum of j terms may be written as

$$ j \, {}_{2k}D_k + \sum_{u=1}^{k} {}_jC_{u+1} \, {}_{2k-u}D_{k-u}. $$

Note: If the parameter t is eliminated from equations (2) by differentiation, a result will be

$$ (4 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + n^2 y = 0. $$

A solution of this equation in series gives at once the coefficients of equation (1) in the form as shown by Dickson. See the preceding reference.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

Logarithms, Trigonometry, Statistics. By H. R. Cooley, P. H. Graham, F. W. John, and A. Tilley. New York, McGraw-Hill Book Company, Inc., 1942. 12+280 pages. $2.00.

As the title of this book indicates, it is a compilation of material for a course in first year college mathematics. However, it contains far more material than can be covered in the usual one semester course for college freshmen, and could be easily used for a two-semester course. Roughly half deals rather briefly with logarithms, including the use of the four principal scales of a simple slide rule, logarithmic and exponential functions, polar coordinates, empirical functions, and statistics.

It is a very well written text book, with clear concise discussions and explanations. It presupposes very little preliminary knowledge of mathematics on the part of the student. On the other hand it is not too elementary for any college class. The reviewer thinks, however, that except for the chapters on trigonometry, which are very complete, it might be improved for classroom use by supplementary exercises.

The chapter on statistics, which covers approximately thirty pages, contains an excellent discussion of classification of data and of central tendencies and dis-

persion. There is a very brief discussion of the normal curve and its uses in statistics, but there is no mention of sampling or of correlation.

BEATRICE L. HAGEN

Differential Equations. By R. P. Agnew. New York, McGraw-Hill Book Company, 1942. 7+341 pages. $3.00.

This text is intended for a "first course" in ordinary differential equations with physical applications. It places unusual emphasis on careful analytical argument and on incentives for the student to reason. Anyone who teaches in this field should find much of interest in the original comments and viewpoints presented frequently with humor, in the book. A poll of a half dozen qualified examiners revealed the unanimous opinion that it would be very desirable to have students who *could* absorb this type of treatment in the usual brief course in elementary differential equations, but that the rather symbolic and critical viewpoint employed would demand a considerable background. We can at least hope that the standards of elementary mathematical education will improve.

As to the content, an "epsilon-delta treatment" of continuity, differentiation, and Riemann and Cauchy integration is presented after a preliminary discussion of some basic definitions. Then first order equations and some physical applications are considered in terms of standard "tricks." (The unifying principle of exact differentials is developed much later in the text.) Division by variables is avoided as a trap for the unwary. The physical basis for the radioactive disintegration equation and the mathematical basis for n-parameter families of curves are considered in an almost annihilatingly critical way which is interesting to those who are aware of silver linings but which is probably shattering to the uninitiated. A factored-operator treatment is given for linear equations of the second and higher orders, supplemented by a chapter on power series in relation to the hypergeometric, the Bessel and the Legendre differential equations. Complex exponentials and impedances and reciprocal operators are used to discuss n-th order ordinary differential equations with constant coefficients, Heaviside operational methods receiving a fairly extensive treatment in terms of general symbols. Application is made to mechanical and electrical problems with special reference to resonance. Unlike most texts in the field, this one does not neglect the important topic of eigenvalues. Some of the more important relevant theorems are stated. There is a chapter on special methods and singular solutions for some first order equations, two chapters on Picard's methods and existence theorems, and finally an appendix on the simple differential equations for slightly bent beams.

The book seems to be remarkably free from misprints, though the following errors may be noted: on page 81 c should be $\sqrt{c}$ in the double inequality; on page 186 (at the bottom) "all $f(x)$" is too inclusive; on page 195, lines 9 and 10 in 9.01, at least one "in" should be dropped. The decimal notation for paragraphs and equations is used and is convenient except when a defenceless little equation is loaded down with a label such as 12.5831 in small type. Numerous informal

problems are provided, the usual "drill" being contrary to the author's tenets.

Frequent discouraging reference to possible misrepresentation of physical phenomena due to the introduction of approximations, seems to the reviewer to be misleading, since nonsingular variation of coefficients can be shown to produce nonsingular variations in solutions. Furthermore, idealizations (approximations guided by experimental knowledge) are the rule rather than the exception in science. However, we can all sympathize with the author's reaction against the usual uncritical attitude of most elementary texts.

J. K. L. MacDonald

Differential Equations. By Max Morris and O. E. Brown. Revised edition. New York, Prentice-Hall, 1942. 12+355 pages. $3.00.

The first edition of this book was reviewed in this Monthly, vol. 41 (1934), pp. 183–4. After five printings the authors and publishers have now issued a revised edition. By comparing it with the first edition, it may be seen that the book is more changed in appearance than in content. A differently-colored cover, a new style of type, and a larger page are examples of alterations in its physical appearance. Changes in the text itself are few. The most extensive are in the chapter on Linear Differential Equations, which has been entirely rewritten in the interests of clarity. New material includes Milne's method of numerical integration and Laplace's equation. A table of integrals and a table of natural logarithms add to the usefulness of the book. Many new problems have been included.

The reviewer noted few misprints, the most serious being the omission of equation (15) from the Existence Theorem on page 8. In equation (19) on page 87, students may be confused by the use of k in two different senses.

The revised edition will appeal, as did the first edition, to those teachers who desire a text in differential equations which emphasizes physical applications and at the same time gives the student an introduction to some advanced topics in algebra and analysis.

H. M. Gehman

Spherical Trigonometry. By R. W. Brink. New York, D. Appleton-Century Company, 1942. 8+62 pages. $0.75.

This little book begins with a review of the geometry of the sphere, featuring spherical distances, spherical angles and triangles, and polar triangles. Then comes the trigonometry of a right spherical triangle, including Napier's Rules and the discussion of species. This is followed by a similar discussion of oblique triangles. The law of sines, law of cosines, both for sides and for angles, and the half-angle and half side formulas are fully treated. The discussion of Napier's analogies is followed by a rather full treatment of the ambiguous cases. A short final chapter is devoted to the celestial sphere. Appendices explain the mil, and the haversine. The latitude and longitude of a number of places is included, but the typography is rather crowded, and not clear. Otherwise the type and the

figures are good. The rather brief treatment is not essentially different from that in other texts, hardly justifying the claims of originality in the preface.

VIRGIL SNYDER

A Mathematics Refresher. By A. Hooper. New York, Henry Holt and Co., 1942. 10+342 pages. $1.90.

This is not only an excellent book for candidates for the air forces who have inadequate mathematical background, but it is also valuable for any teacher of mathematics. Mr. Hooper's long and successful experience in teaching air-crew candidates in the R.A.F. and R.C.A.F. has eminently qualified him to write a preparatory text for those who have forgotten their school mathematics, and who must, in the shortest possible time, become completely familiar with the mathematics essential to the aviator. Mr. Hooper knows what to omit and what to emphasize—that is, he understands the difficulties of young people who do not "take to" mathematics—and the result is a minimum, yet thorough foundation, forcefully and enthusiastically presented.

As to the use of *A Mathematics Refresher* in "civilian" classrooms, how many of us have frequently sighed for such a volume! This is the book that we have looked for to knit together arithmetic, algebra, geometry, trigonometry, and the beginnings of the calculus. While we shall regret that it is too brief to use as a class text, we can discover in it innumerable suggestions for our teaching and questions (which, with their answers, have all been tested and approved in aviation schools) for review. And this is the book to hand to John and Mary as a "reference" when they tell us that they have forgotten how to "do" per cent, or that they never did understand what "angle" really means or why anyone ever bothers with logarithms or that mysterious dy/dx!

Mr. Hooper has indeed written "mathematics without tears," but in so doing he has not sacrificed substance. His arrangement is logical, well balanced, and sufficiently detailed for his purpose, and the easy charm of his style adds to the value of the book for all who use it.

MURIEL BOWDEN

NEW BOOKS RECEIVED

A Start in Meteorology. By A. N. Spitz. New York, Norman W. Henley Publishing Co., 1942. 95 pages. $1.50.

Air Navigation for Beginners. By S. G. Lamb. New York, Norman W. Henley Publishing Co., 1942. 103 pages. $1.50.

Learning to Navigate. By P. V. H. Weems and W. C. Eberlee. Second edition. New York, Pitman Publishing Corp., 1943. 8+135 pages. $2.00.

A Handbook of Perspective Drawing. By J. C. Morehead and J. C. Morehead, Jr. Pittsburgh, James C. Morehead, 1941. 6+166 pages. $4.50.

Empirical Equations and Nomography. By D. S. Gale. New York and London, McGraw-Hill Book Co., Inc., 1943. 9+200 pages. $2.50.

Differential Equations. By H. W. Reddick. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1943. 9+245 pages. $2.50.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 571. *Proposed by Sol Mitchell, University of Toronto*

Let O be the midpoint of a chord AB of a circle, and let CD, EF be any two other chords through O. Prove synthetically that CE and DF meet AB in points equidistant from O.

E 572. *Proposed by V. Thébault, San Sebastián, Spain*

What scale of notation admits perfect squares ab, bb, and cd, where a, b, c, d are consecutive integers?

E 573. *Proposed by N. A. Court, University of Oklahoma*

Given two (three) vertices of a triangle (tetrahedron), determine the remaining vertex so that a given point and a given line (plane) shall be harmonic for the triangle (tetrahedron).

E 574. *Proposed by W. E. Buker, Pittsburgh Public Schools*

If a quadrilateral with sides a, b, c, x is inscribed in a semicircle of diameter x, show that

$$x^3 - (a^2 + b^2 + c^2)x - 2abc = 0.$$

E 575. *Proposed by P. S. Donchian, Hartford, Connecticut*

Let (n^r) denote the standard residue of n^r modulo p (an odd prime), so that $(n^r) \equiv n^r \pmod{p}$ and $0 \leq (n^r) < p$. Prove that the $p-2$ numbers

$$\sum_{n=1}^{p-1} (n^r) \qquad (r = 1, \cdots, p-2)$$

are a permutation of the $p-2$ numbers

$$\sum_{r=0}^{p-2} (n^r) \qquad (n = 2, \cdots, p-1),$$

and that all these numbers are divisible by p.

SOLUTIONS

Stirling's Numbers of the First Kind

E 520 [1942, 257]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Given

$$u_n = n!\sum_{1}^{n} \frac{1}{r}, \quad \text{prove that} \quad \lim_{r\to\infty} \frac{u_{r+1}}{\Delta^r u_1} = e.$$

Solution by H. W. Becker, Mare Island Submarine Base. In the notation of Jordan (*Tohoku Mathematical Journal*, vol. 37, 1933, pp. 255–283), Stirling's numbers of the first kind are given by

$$S_{n+1}^{m} = S_{n}^{m-1} - nS_{n}^{m}, \qquad S_{n}^{0} = 0 \ (n > 0), \qquad S_{n}^{n} = 1.$$

On comparing this recurrence formula with the proposer's

$$u_n = (n-1)! + nu_{n-1}, \qquad u_1 = 1,$$

we see that $u_n = |S_{n+1}^2|$. Thus the desired result can be obtained by setting $m=2$ and $n=1$ in the more general relation

$$\lim_{r\to\infty} \frac{\Delta^r |S_{n+1}^m|}{|S_{r+n+1}^m|} = \frac{1}{e},$$

which we proceed to prove.

From Jordan's asymptotic expression

$$(1) \qquad |S_{n+1}^m/S_n^m| \sim n$$

(*loc. cit.*, p. 261), we deduce

$$|S_{n+1}^m/S_{n+1-j}^m| \sim (n)_j = n(n-1)\cdots(n+1-i),$$

whence (for large r)

$$\frac{\Delta^r|S_{n+1}^m|}{|S_{r+n+1}^m|} = \frac{\sum_{j=0}^{r}(-1)^j\binom{r}{j}|S_{r+n+1-j}^m|}{|S_{r+n+1}^m|} \sim \sum_{j=0}^{r}\frac{(-1)^j\binom{r}{j}}{(r+n)_j}$$

$$(2) \qquad = \sum_{j=0}^{r}\frac{(-1)^j}{j!}\frac{(r)_j}{(r+n)_j} \sim \sum_{j=0}^{r}\frac{(-1)^j}{j!}\left(\frac{r}{r+n}\right)^j$$

$$\sim e^{-r/(r+n)}.$$

Thus the limit, as r tends to ∞, is e^{-1}, as required.

Since $|S_{n+1}^1| = n!$, another special case of this asymptotic formula is

$$\frac{\Delta^r n!}{(r+n)!} \sim e^{-r/(r+n)} \qquad (r \to \infty).$$

Instead of keeping n fixed, we may let it be a constant multiple of r; *e.g.*

$$(3) \qquad \lim_{r\to\infty} \frac{\Delta^r r!}{(2r)!} = e^{-1/2} = .6065\ldots.$$

The convergence is quite rapid; when $r=3$ we already have

$$\frac{\Delta^3 3!}{6!} = \frac{426}{720} = .591\dot{6}.$$

Editorial Note. The approximation (2) can be justified as follows. Stirling's formula $n! \sim (2\pi)^{1/2} n^{n+1/2} e^{-n}$ gives

$$\begin{aligned}
\frac{(r)_j}{(r+n)_j} &= \frac{r!(r+n-j)!}{(r+n)!(r-j)!} \\
&\sim \left(\frac{r}{r+n}\right)^j \left(1-\frac{j}{r}\right)^{-r+j-1/2} \left(1-\frac{j}{r+n}\right)^{r+n-j+1/2} \\
&\sim \left(\frac{r}{r+n}\right)^j \left(\frac{1-j/r}{1-j/(r+n)}\right)^{j-1/2} \sim \left(\frac{r}{r+n}\right)^j \left(1-\frac{nj(j-\frac{1}{2})}{r(r+n)}\right).
\end{aligned}$$

Hence the error in (2) is approximately

$$\begin{aligned}
&-\sum_{j=0}^{\infty} \frac{1}{j!}\left(-\frac{r}{r+n}\right)^j \frac{nj(j-1+\frac{1}{2})}{r(r+n)} \\
&= -\sum_{j=2}^{\infty} \frac{1}{(j-2)!}\left(-\frac{r}{r+n}\right)^{j-2} \frac{rn}{(r+n)^3} \\
&\quad + \sum_{j=1}^{\infty} \frac{1}{(j-1)!}\left(-\frac{r}{r+n}\right)^{j-1} \frac{n}{2(r+n)^2} \\
&= -e^{-r/(r+n)}\left(\frac{rn}{(r+n)^3} - \frac{n}{2(r+n)^2}\right) = -e^{-r/(r+n)} \frac{n(r-n)}{2(r+n)^3}.
\end{aligned}$$

In the special case when $m=1$, (1) is exact, so we can say confidently that, for large r,

$$\frac{\Delta^r n!}{(r+n)!} \sim e^{-r/(r+n)}\left(1-\frac{n(r-n)}{2(r+n)^3}\right).$$

The presence of the factor $r-n$ explains the rapid convergence of (3).

Eight Consecutive Digits forming a Square

E 538 [1942, 546]. *Proposed by R. V. Heath, Wall St., New York City*

Find a perfect square whose digits form one of the permutations of eight consecutive digits. (Cf. E 532.)

Solution by E. P. Starke, Rutgers University. We consider three possible cases:

I. Let the digits of N^2 be 0, 1, 2, 3, 4, 5, 6, 7, in some order. Then

(1) $$N^2 \equiv 1, \qquad N \equiv \pm 1 \pmod 9).$$

Also, from consideration of the final two digits of N^2, we have

(2) $$\pm N \equiv 1, 2, 4, 5, 6, 8, 11, 15, 16, 18, 19, 21, 24, \text{ or } 25 \pmod{50}.$$

Furthermore, if we do not permit an initial zero, we have

(3) $$3199 < N < 8749.$$

Now put $N = 100a + b$, $0 < b < 100$. To each value of b which is satisfactory according to (2), there correspond a number of values of a which satisfy (1) and (3). The possibilities for N are thus restricted to something over six hundred. Testing these (or referring to a table of squares, *e.g.* Barlow's) discloses six squares of the proposed form:

$$3698^2 = 13675204, \qquad 4175^2 = 17430625, \qquad 4616^2 = 21307456,$$
$$5968^2 = 35617024, \qquad 6596^2 = 43507216, \qquad 7532^2 = 56731024.$$

II. Let the digits of N^2 be 1, 2, 3, 4, 5, 6, 7, 8. Then

$$N \equiv 0 \pmod 3), \qquad 3513 < N < 9363,$$
$$\pm N \equiv 4, 5, 6, 8, 9, 11, 15, 16, 18, 19, 21, 22, 24, 25 \pmod{50}.$$

Proceeding as in Case I, we find five squares:

$$3678^2 = 13527684, \qquad 5904^2 = 34857216, \qquad 8082^2 = 65318724,$$
$$8559^2 = 73256481, \qquad 9024^2 = 81432576.$$

III. Let the digits of N^2 be 2, 3, 4, 5, 6, 7, 8, 9. Then

$$N^2 \equiv 8 \pmod 9).$$

Since 8 is not a quadratic residue modulo 9, this last case produces no results.

Also solved by W. E. Buker.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4082. *Proposed by H. S. M. Coxeter, University of Toronto*

In hyperbolic geometry, obtain the condition

$$\cos \tfrac{1}{2}A < \sin \tfrac{1}{2}B + \sin \tfrac{1}{2}C$$

for the triangle ABC to have a proper excircle beyond BC.

4083. *Proposed by P. Erdös, Princeton, N. J.*

Let $a_1 < a_2 < \cdots < a_x \leqq n$ be an arbitrary sequence of positive integers such that no a_i divides the product of the others, then $x \leqq \pi(n)$, where $\pi(n)$ denotes the number of primes not exceeding n.

4084. *Proposed by Otto Dunkel, Washington University*

On the sides A_jA_k of a given triangle $A = A_1A_2A_3$ as bases, directly similar triangles $B_iA_jA_k$ are constructed interiorly giving the triangle $B = B_1B_2B_3$. Show that, if B has the maximum area when the sense of rotation of its vertices is opposite to that for A, the triangles $B_iA_jA_k$ must be isosceles with $\cot \alpha \cot V = 3$, where α is the base angle and V is the Brocard angle for A. Determine the form of the triangle B giving the maximum.

4085. *Proposed by V. Thébault, San Sebastián, Spain*

Given an equilateral hyperbola (H) and a circle (O) passing through the center ω of (H), show that the necessary and sufficient condition for the existence of an infinite number of triangles inscribed in (H) and circumscribing the circle is that the center O of the circle lies on (H). Consider the envelope of the sides of these triangles.

SOLUTIONS

Tetrahedron, Hyperbolic Set of Lines

3986 [1941, 152]. *Correction. Proposed by V. Thébault, San Sebastián, Spain*

The six points $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are taken respectively on the edges BC, CA, AB, DA, DB, DC of the tetrahedron $ABCD$, and the radical planes of the circumsphere of $ABCD$ with the spheres $(A\beta\gamma\alpha')$, $(B\gamma\alpha\beta')$, $(C\alpha\beta\gamma')$, $(D\alpha'\beta'\gamma')$ meet the planes of the faces BCD, CDA, DAB, ABC in the straight lines $\Delta_1, \Delta_2, \Delta_3, \Delta_4$. Show that these four straight lines are generators of the same system of a quadric surface if the straight lines $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ are concurrent, and conversely.

Generalization by R. Bouvaist, Nantes, France

THEOREM I. *The circles (ω_1), (ω_2), (ω_3) pass respectively through the vertices A, B, C of the triangle ABC whose circumcircle is (O). The radical axis of (ω_2) and (ω_3) meets BC in the point α; the radical axis of (ω_3) and (ω_1) meets CA in β; the radical axis of (ω_1) and (ω_2) meets AB in γ. The radical axis of (O) and (ω_1) meets BC in α'; the radical axis of (O) and (ω_2) meets CA in β'; the radical axis of (O) and (ω_3) meets AB in γ'. If the straight lines $A\alpha$, $B\beta$, $C\gamma$ are concurrent, the points α', β', γ' are collinear, and conversely.*

If $S=0$ is the equation of (O) and $\Delta=0$ that of the line at infinity, we have the following equations

$$(1)\qquad (\omega_1):\ S+\Delta(m_1y+n_1z)=0;\qquad (\omega_2):\ S+\Delta(l_2x+n_2z)=0;\qquad (\omega_3):\ S+\Delta(l_3x+m_3y)=0.$$

The radical axes of (ω_2) and (ω_3), of (ω_3) and (ω_1), of (ω_1) and (ω_2) have the equations

$$(2)\qquad (l_2-l_3)x-m_3y+n_2z=0;\qquad l_3x+(m_3-m_1)y+n_1z=0;\qquad -l_2x+m_1y+(n_1-n_2)z=0.$$

If $A\alpha$, $B\beta$, $C\gamma$ are concurrent, we find that $m_1n_2l_3=n_1l_2m_3$. If the straight lines whose equations are

$$(3)\qquad m_1y+n_1z=0;\qquad l_2x+n_2z=0;\qquad l_3x+m_3y=0$$

meet BC, CA, AB in the collinear points α', β', γ', we have the same relation $m_1n_2l_3=n_1l_2m_3$.

THEOREM II. *The four spheres (ω_1), (ω_2), (ω_3), (ω_4) pass respectively through the vertices A, B, C, D of the tetrahedron $ABCD$ whose circumsphere is (O). The radical planes of (ω_1) and (ω_2), of (ω_3) and (ω_4), of (ω_1) and (ω_3), of (ω_2) and (ω_4), of (ω_1) and (ω_4), of (ω_2) and (ω_3) meet respectively AB, CD, AC, DB, DA, BC in the points γ, γ', β, β', α', α. The radical planes of sphere (O) with each of the spheres (ω_1), (ω_2), (ω_3), (ω_4) meet respectively the planes of the faces BCD, CDA, DAB, ABC in the straight lines Δ_1, Δ_2, Δ_3, Δ_4. If the straight lines $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ are concurrent, then Δ_1, Δ_2, Δ_3, Δ_4 are on the same hyperboloid, and conversely.*

The radical plane of (ω_1) and (ω_2) has the equation

$$(4)\qquad -l_2x+m_1y+(n_1-n_2)z+(p_1-p_2)t=0$$

and the equations of the five other radical planes are written in a similar manner. If the point $P(x_0, y_0, z_0, t_0)$ is the point of concurrency of $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$, we find from the equations (4) that $l_2x_0=m_1y_0$, and in the same way five other equations. We write these in the form

$$(5)\qquad l_2=m_1y_0/x_0;\qquad l_3=n_1z_0/x_0;\qquad l_4=p_1t_0/x_0;\qquad n_2=m_3y_0/z_0;\qquad p_2=m_4y_0/t_0;\qquad p_3=n_4z_0/t_0.$$

The equations of the radical planes of (O) and (ω_i) may then be written

$$(6)\qquad m_1y+n_1z+p_1t=0;\qquad m_1x/x_0+n_2z/y_0+p_2t/y_0=0;\qquad n_1x/x_0+n_2y/y_0+p_3t/z_0=0;\qquad p_1x/x_0+p_2y/y_0+p_3z/z_0=0;$$

and we see that they are polar planes of A, B, C, D with respect to the quadric surface

$$(7)\qquad m_1xy/x_0+n_1xz/x_0+p_1xt/x_0+n_2yz/y_0+p_2yt/y_0+p_3zt/z_0=0.$$

The straight lines Δ_i are therefore on the same hyperboloid. Conversely, if they are on the same hyperboloid, we have the relations

$$(8)\qquad \begin{gathered} n_2p_3m_4 = p_2m_3n_4; \qquad n_1p_3l_4 = p_1l_3n_4; \qquad m_1p_2l_4 = p_1l_2m_4; \\ m_3l_2n_1 = m_1l_3n_2. \end{gathered}$$

The first, for example, means that Δ_2, Δ_3, Δ_4 meet CD, DB, BC in three points which are collinear. The relations above are seen to be satisfied by means of the equations (5); and this proves the proposition.

Note by the Proposer. The second theorem results immediately from the first by observing that the planes of the faces cut circles on the spheres (ω_i), and that the straight lines Δ_i meet each of four straight lines $d_i \equiv (\alpha_i', \beta_i', \gamma_i')$ which correspond to them in the faces of the tetrahedron.

Editorial Note. The proposer stated in regard to the original problem that its theorem is easily deduced from a theorem by Roy in the *Nouvelles Annales de Mathématiques*, 1926, p. 277, which is the special case of Theorem I where the circles (ω_i) intersect in pairs on the sides of ABC. For Theorem II there are a number of special cases which we shall separate by assuming that no sphere (ω_i) passes through two vertices of $ABCD$; then the point $P(x_0, y_0, z_0, t_0)$ in the above proof does not lie in a face and no line Δ_i passes through a vertex. As the proposer's note to the above solution states, it is easy to show synthetically from Theorem I that, if $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ meet in a point P, then there exist four straight lines d_i each lying in a corresponding face which meet each of the four lines Δ_i. The synthetic proof of the converse is long and complicated with exceptional cases, and even when we make the above assumption it is not simple. In the case now considered one method is to show that the straight lines AP_1, BP_2, CP_3, DP_4, defined below, meet in a point P and then that $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ are concurrent in this point. It is simpler to prove together the two theorems analytically as follows. If in the two sets of equations for the radical axes we set $t=0$, we find from (4) and similar equations of the solution that the equations of the traces $A\alpha$, $B\beta$, $C\gamma$ in the plane of ABC of the planes $A\alpha'\alpha$, $B\beta'\beta$, $C\gamma'\gamma$ meeting in the point P are

$$(1)\qquad m_3y - n_2z = 0; \qquad l_3x - n_1z = 0; \qquad l_2x - m_1y = 0.$$

The necessary and sufficient condition that they meet in the point P_4 which is the trace of DP in the face ABC is $l_2m_3n_1 - l_3m_1n_2 = 0$. The traces of Δ_1 on BC, of Δ_2 on CA, of Δ_3 on AB are the points

$$(2)\qquad \alpha_1:\ 0, n_1, -m_1, 0; \qquad \beta_2:\ n_2, 0, -l_2, 0; \qquad \gamma_3:\ m_3, -l_3, 0, 0.$$

The necessary and sufficient condition that α_1, β_2, γ_3 lie on a straight line d_4 is the same as above. By cyclic permutations of the letters and subscripts we then find for the existence of d_1, d_2, d_3 the necessary and sufficient conditions in the first three equations of (8) in the above solution, the last being for d_4. These four equations are not independent. Conversely, if there exist four such lines d_i, then we have shown that for d_4 the equations (1) are consistent; and by cyclic permu-

tation we have those for d_1, d_2, d_3 which are also consistent. We then have twelve equations with duplication of each; the six distinct equations are in pairs the equations of $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ and as pointed out they are consistent by reason of the equations (8). Hence these six equations have a unique solution (x_0, y_0, z_0, t_0). and we have now proved that $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ meet in a point P with these coordinates. Thus the necessary and sufficient conditions that $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ meet in a point are the same as those for the existence of the straight lines d_i. The four lines d_i are distinct; for, if two coincide they coincide with an edge of $ABCD$ and then two of the lines Δ_i pass respectively through the vertices on that edge, and this is at present excluded.

But in order for the four lines Δ_i to form a hyperbolic set we must impose the condition that no two lie in a plane; and we have then to consider the two special cases where they do not form a hyperbolic set. First there is the case where only two determine the same plane, say Δ_1 and Δ_2. It will then be seen synthetically that d_1, d_2, Δ_1, Δ_2 are concurrent in γ_{34}' on CD, the planes (Δ_1, Δ_2), (d_1, d_2) meet face ABC in d_4, Δ_4 respectively. It then follows that d_3, d_4, Δ_3, Δ_4 are concurrent in γ_{12} on AB, the point γ_{34}' lies in the plane (Δ_3, Δ_4) and γ_{12} in the plane (Δ_1, Δ_2), that is the planes (Δ_1, Δ_2) and (Δ_3, Δ_4) intersect in the straight line through γ_{12} and γ_{34}'. In this case there are an infinite number of straight lines in the plane (Δ_1, Δ_2) that meet Δ_3 and Δ_4, in γ_{12}, and similarly for plane (Δ_3, Δ_4). We now have the case where, for example, Δ_1 and Δ_2 lie in a plane and also Δ_1 and Δ_3 lie in a plane. It then follows from the above that $d_1 = \Delta_1$, $d_3 = \Delta_3$, $d_2 = \Delta_2$, and finally $d_4 = \Delta_4$. Thus the four lines Δ_i lie in the same plane, and we now have the condition lacking in the original statement of the problem. The remaining special cases relate to the number of distinct lines d_i. In the case where there are precisely three distinct lines d_i and no two of the lines Δ_i lie in a plane, the four lines Δ_i form a hyperbolic set. The discussion of the remaining cases is tedious, and it is simpler to assume that the theorem excludes them.

Properties of a Special Triangle

4026 [1942, 128]. *Proposed by V. Thébault, San Sebastián, Spain*

(1) Construct a triangle ABC knowing a, A and given that the median and symmedian from A are perpendicular and parallel to two given directions. (2) Indicate the properties of this special triangle. (3) Let B' and C' be the projections of B and C on a variable straight line AP which cuts BC in P. The locus of the harmonic conjugate of P with respect to B' and C' is a right strophoid having the vertex A for a double point and tangent to the bisectors of angle A.

Editorial Note. The proposer stated: (1) The given elements $a = BC$ and the angle A suffice to determine the circumradius R of ABC. Let A_m be the midpoint of BC; let M be the point of intersection of BC with the symmedian from A whose direction is known; and let F be the other point of intersection of AM and the circumcircle (O). The bisectors of angle A and A_mAM intersect (O) again at the midpoints of the arcs BC, and the median AA_m meets (O) again at the point E diametrically opposite to F, since A_mAM is a right angle. The points E, F be-

ing known as also the directions of AA_m, AF, the vertex A is determined from the points E, F.

(2) This special triangle is rather curious. The circle with A_mM as diameter is orthogonal to the Apollonian circle for BC. We have the relations

$$a^2 = -2bc\sqrt[3]{\cos A}\,(\sqrt[3]{\cos A} + 1), \qquad A > 90°,$$
$$(b^2 + c^2)/2bc = -\sqrt[3]{\cos A} = m_a/s_a,$$

where m_a and s_a denote the lengths of AA_m and AM.

(3) Consider the parabola tangent to BC and the interior bisector AI, and having AA_m for directrix. The locus of the harmonic conjugate of P with respect to B', C' is the pedal of the parabola with respect to the point A, and it is therefore a right strophoid. The double point is A, and the bisectors of angle A are the tangents at that point. The curve passes through the isogonal centers of ABC.

Boolean Algebras

4027 [1942, 202]. *Proposed by Morgan Ward, California Institute of Technology*

In the projective plane P, a triangle with sides A, B, C and vertices D, E, F is the only linear configuration forming a Boolean algebra of order eight with respect to the operations of union and cross-cut. Here the union of A and B is P, the cross-cut of E and D is the null space Z, and so on. In a three dimensional space P, two types of Boolean algebra of order eight are possible; (i) the configuration of three planes A, B, C through a point Z meeting in three lines D, E, F; and (ii) the configuration of two planes A and B through a line D and a line C skew to D meeting A and B in points E and F. Here Z again is the null space.

Show that in a projective space of n dimensions, the total number of distinct types of linear configuration forming a Boolean algebra of order eight is asymptotically equal to $n^3/3!3!$. Also show that the corresponding number for a Boolean algebra of order 2^r is asymptotically equal to $n^r/r!r!$.

Solution by the Proposer. The actual number $u(n)$ of types of linear configurations in a projective space of $n-1$ dimensions forming a Boolean algebra of order eight is given by the formula

$$u(n) = \left[\frac{2n^3 + 3n^2 - 6n}{72}\right] \qquad \text{if} \quad n \equiv 0, 2, 4, 5 \pmod 6$$
$$= \left[\frac{2n^3 + 3n^2 - 6n}{72}\right] + 1 \quad \text{if} \quad n \equiv 1, 3 \pmod 6.$$

For small n this gives the table

Dimensions:	1	2	3	4	5
Number:	0	1	2	4	7

which it is easy to verify directly.

The asymptotic formula follows immediately but one can establish it for a Boolean algebra of order 2^r by a simple argument which I sketch here. One simply observes that the Boolean algebra as a configuration is completely determined by the dimensions $\pi_1, \cdots, \pi_r$ of the linear manifolds $P_1, \cdots, P_r$ which constitute the "points" of the lattice algebra. If the space is of n dimensions, then

$$\pi_1 + \pi_2 + \cdots + \pi_r = n + \epsilon \tag{1}$$

where

$$0 \leqq \epsilon \leqq n - r; \qquad \pi_i \geqq \epsilon + 1, \tag{2}$$

and ϵ is the dimension of the null element of the Boolean algebra. (For simplicity of notation I count a point as of dimension one, a line of dimension two, and so on, so that the "dimension" here is one more than the ordinary dimension I used in stating the problem.)

Since we are interested in distinct types, we add the restriction

$$\pi_1 \geqq \pi_2 \geqq \cdots \geqq \pi_r. \tag{3}$$

Let $u(n)$ be total number of solutions of the diophantine system (1), (2), (3). Then

$$u(n) - u(n-1) = \phi(n)$$

where $\phi(n)$ is the total number of solutions of

$$\pi_1 + \pi_2 + \cdots + \pi_r = n \tag{4}$$

$$\pi_1 \geqq \pi_2 \geqq \cdots \geqq \pi_r \geqq 1. \tag{5}$$

Now the total number of solutions of (4) with (5) replaced by

$$\pi_i \geqq 1 \tag{6}$$

is evidently

$$(n-1)(n-2) \cdots (n-r+1)/(r-1)!.$$

But if $n > 1$, solutions of (4) and (6) with all π unequal outnumber all others. Hence we infer $\phi(n)$ is asymptotic to

$$\frac{(n-1)(n-2)\cdots(n-r+1)}{r!(r-1)!}.$$

Hence $u(n)$ is asymptotic to

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!r!} \quad \text{or} \quad u(n) \sim \frac{n^r}{r!r!}.$$

The details of a rigorous proof are easily provided since we are dealing with *polynomials* in n essentially.

Geodesics

4033 [1942, 263]. *Proposed by P. D. Thomas, Norman, Oklahoma*

Points on a surface with the linear element $ds^2 = [(u+a)^2+(v+b)^2](du^2+dv^2)$ correspond to points in the xy-plane by $x=u$, $y=v$. Show that geodesics on the surface correspond to equilateral hyperbolas in the xy-plane.

Solution by Thomas Bauserman, Camp Davis, N. C. By making the transformation $(u+a)=\rho\cos\theta$, $(v+b)=\rho\sin\theta$, the linear element becomes

$$ds^2 = \rho^2(d\rho^2 + \rho^2 d\theta^2).$$

Then by the transformation, $\rho\, d\rho = dr$, and $2\theta = \phi$

$$ds^2 = dr^2 + r^2 d\phi^2.$$

Since this is the polar form of the linear element of a straight line, the equation of the geodesics may be written

$$2Ar\cos\phi + 2Br\sin\phi = C,$$

which becomes

$$A\rho^2\cos 2\theta + B\rho^2\sin 2\theta = C,$$

or

$$\rho^2\cos 2(\theta - \alpha) = K,$$

and finally

$$\rho^2\cos^2(\theta-\alpha) - \rho^2\sin^2(\theta-\alpha) = K.$$

In the xy-plane this is an equilateral hyperbola where the origin has been translated to the point $(-a, -b)$ and the axes rotated through an angle α.

A set of geodesics is also given by ϕ = constant which are represented in the xy plane by straight lines through $(-a, -b)$.

Solved also by the proposer using the differential parameter $\Delta_1 f$ (Eisenhart's *Differential Geometry*, 1909, p. 217).

FIFTH ANNUAL MEETING OF THE NORTHERN CALIFORNIA SECTION

The fifth annual meeting of the Northern California Section of the Mathematical Association was held at the University of San Francisco on Saturday, January 30, 1943. Professor E. B. Roessler, vice-chairman of the Section, presided at both morning and afternoon sessions in the absence of Dean Fredrick Wood, the chairman, who was unable to be present. During the noon recess, members and visitors were the guests of the University of San Francisco at a luncheon served in "The Lounge" of the University.

The attendance at the two sessions was seventy-five, including the following twenty-four members of the Association: H. M. Bacon, G. A. Baker, G. C.

Evans, E. L. Fitzgerald, S. A. Francis, Emma V. Hesse, D. H. Lehmer, Sophia H. Levy, A. L. McCarty, E. D. Miller, F. R. Morris, E. J. Moulton, E. J. Phillips, George Polya, Edris P. Rahn, E. B. Roessler, Ethel Spearman, Gabor Szegö, Ruth G. Sumner, K. J. Waider, R. K. Wakerling, L. A. Walker, Harriet A. Welch, A. R. Williams.

The following officers were elected for the coming year: Chairman, E. B. Roessler, University of California at Davis; Vice-Chairman, Gabor Szegö, Stanford University; Secretary-Treasurer, H. M. Bacon, Stanford University. Mrs. Ruth G. Sumner, Oakland High School, was re-elected to represent the Section as associate editor of the *California Journal of Secondary Education*. A resolution of thanks to the University of San Francisco for its generous hospitality was unanimously adopted for transmission to the Very Reverend William J. Dunne, President of the University. The report of Professor Levy regarding the activities of the Committee on Mathematical Education of the Northern and Southern California Sections was followed by the adoption of a resolution endorsing the following statement recently adopted by the Subcommittee on Mathematics of the California Committee for the Study of Education:

It is the opinion of the Subcommittee on Mathematics that our California schools and their teachers can make an essential and invaluable contribution to the war effort by continuing to teach all of the mathematics which has formed and now forms a most important part of their work. Students are leaving school at every grade and age to enter either industry or the armed forces in capacities where they find mathematics a prime requirement.

A fourth-year high school student without skill in arithmetic should, at the time, be given arithmetic intensively, followed by as much algebra as can be included. A third-year high school student without suitable skill in arithmetic should be started with intensive training in this subject. For him there will be time, also, for algebra through quadratics and some geometry. It will be noted that these students are more mature than the usual beginner and also know their immediate need for mathematical training when they start these basic subjects. Hence, they can cover the materials at a relatively faster pace.

For students in the earlier grades, there will be time for the usual sequence of algebra, geometry, and trigonometry. For these students the question of time is not pressing, and the mathematics courses already being given in many schools are suitable.

This Subcommittee agrees with the opinions of many representatives of the Armed Services and industry that the mathematics courses in schools must continue to place emphasis on basic and fundamental parts of the subject treated. Such applications as are given, and there should be as many as possible, should be primarily for the purpose of illustrating and giving training in the basic mathematics. The specific uses of mathematics in the various branches of the Service and industry are best given by appropriate instructors after entrance into service. These are technical subjects and are therefore adequately taught only by the technicians themselves.

When courses are given in basic principles of mathematics, it is clear that such courses fit the needs of both boys and girls, whether for service in the Armed Forces or in industry.

There is general agreement on the point that few students can use arithmetic satisfactorily. It is therefore urged that all teachers beginning with those in our primary schools contribute their utmost towards the training of pupils in the fundamentals of arithmetic. Skills attained at any level should be maintained by cumulative review and use at later levels. Teachers of algebra and geometry should utilize the many opportunities which exist in their courses for continued use of arithmetic. Also, the teachers of geometry should further develop algebraic skill through having their students use it in geometry.

It is the unanimous opinion of this Subcommittee on Mathematics that our schools will not be

rendering their greatest possible service to the Army, Navy, and war industries, unless they conscientiously urge their students to take basic mathematics, including arithmetic, algebra at least through quadratics, concepts and constructions of plane and solid geometry, and trigonometry.

Mrs. Sumner reported that the *California Journal of Secondary Education* would publish a symposium of articles on mathematics in the near future, probably in the May issue.

By invitation of the Section, Professor J. H. McDonald of the University of California gave an hour's address during the morning session.

The following papers were read:

1. "Analogy, a course of discovery" by Professor George Polya, Stanford University.

2. "The mathematical requirements of a course in college physics" by Professor K. J. Waider, University of San Francisco.

3. "Conformal mapping" by Professor J. H. McDonald, University of California, introduced by Professor Levy.

4. "Mathematics in our schools and its contribution to war" (second paper) by Professor Sophia H. Levy, University of California.

5. "The difference-equation method in heat conduction problems" by Professor W. P. Berggren, University of California at Davis, introduced by Professor Roessler.

6. "A solution of $x^2+y^2+z^2=w^2$" by A. L. McCarty, San Francisco Junior College.

7. "Elementary derivation of series for sin x and cos x" by Professor J. V. Uspensky, Stanford University, introduced by the Secretary.

8. "The new military training programs in colleges and universities" by Professor F. R. Morris, Fresno State College.

Abstracts of these papers follow:

1. Professor Polya discussed a well-known but not too commonplace question of elementary solid geometry in order to illustrate the typical way of using analogy in the solution of problems. If the proposed problem does not seem to be easily accessible, we may look out for some more accessible analogous problem. If we succeed in finding and solving such a simpler analogous problem, we may use the *method* of its solution, or we may use its *result* for our original problem, or we may use *both the method and the result* as it happens in the example discussed. The aim of the speaker was to show that methods of discovery are not at all above the level of the classroom, provided the teacher has some real interest and knowledge, and does not spend all his time on technicalities. The subject was taken from a chapter of the book on the solution of problems which the speaker is preparing.

2. Professor Waider stated that of the two lower division courses in general physics commonly offered, the one taken by premedical and letters students requires nothing beyond high school mathematics; while the other, intended for science and engineering majors, involves the use of some calculus. A thorough grounding in the fundamentals of algebra, geometry, and trigonometry con-

stitutes the basic preparation for successful work in a college physics course. Factors of special importance because of their continued employment by the student are: firstly, facility in carrying out numerical computations with accuracy and dispatch, a feat which can scarcely be accomplished without the aid of a slide rule; and secondly, familiarity with graphical representation, for the purpose of depicting the effect of physical laws.

3. Professor McDonald gave in outline three methods of determining the map function for simply connected surfaces. Two of these do not lead to an effective construction of that function, and the third, while suggesting a convergent sequence, proves to be without utility for arithmetic calculation. A fourth method was proposed and illustrated in a manner which shows it to have considerable availability in that regard.

4. Professor Levy stated that during the year many schools have increased the amount of work given in mathematics. The armed services and industry have asked for the teaching of fundamentals, and statements are heard on all sides that fundamentals are now being taught. Inquiries to two-hundred fifty schools have confirmed the belief that the meaning of *fundamentals* has undergone drastic change. Apparently what is fundamental for a navigator is different from what is fundamental for a bombardier! Trigonometry on land and sea is different from trigonometry in the air . . . everything must be "preflight" to be important! It is one thing to do patchwork for the senior in high school who does not know his mathematics, for he needs it quickly. It is another thing to do patchwork for the student who has time in which he could be given real fundamentals. The first duty of our schools is to undertake thorough programs. There is great danger that the impression of being extremely patriotic and extremely busy on patchwork may lead people to ignore the essential job, and on this essential job there is no waste. When the war ends, the fundamental training in mathematics which the boys and girls will have had will be of far more benefit than anything else upon which their time and effort could have been spent.

5. Professor Berggren described a method of solving transient problems to which the Poisson-Fourier (parabolic) partial differential equation is applicable. The procedure, conveniently carried out graphically, is based on repetitive solution of difference equations. The most useful feature of the method is its great versatility with regard to the application of boundary conditions.

6. Mr. McCarty showed that $x^2+y^2+z^2=w^2$ may be written in the form $(w+x)(w-x)=(y+iz)(y-iz)$ or

$$\frac{w+x}{y-iz}=\frac{v+iz}{w-x}=u+iv.$$

This latter form leads to four equations which yield

$$x = ku^2 + kv^2 - k, \qquad y = 2ku, \qquad z = 2kv,$$

and

$$w = ku^2 + kv^2 + k.$$

7. Professor Uspensky gave an elementary derivation of inequalities equivalent to sine and cosine series which does not require even the notion of the limit.

8. Professor Morris stated that in December of 1942 the Army, Navy, and War Manpower Commission issued a joint statement giving preliminary plans for placing the enlisted reserves on active duty and utilizing colleges and universities for giving these men further training of a collegiate grade. The two branches of this plan are known as the Army Specialized Training Program and the Navy Collegiate Training Program. Details of these programs had not been completed, but the membership of advisory committees and executive officers indicate that college faculties and laboratories will be effectively used in the war effort.

H. M. BACON, *Secretary-Treasurer*

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

The Society for the Promotion of Engineering Education will hold its Fiftieth Anniversary meeting at the Drake Hotel in Chicago on Friday, Saturday and Sunday, June 18–20, 1943. The following men will speak on the program of the Mathematics Division: Professors W. L. Ayres, G. A. Bliss, J. M. Dobbie, L. R. Ford, A. N. Milgram, M. A. Sadowsky, Evan Thomas, J. H. Zant. Professor L. R. Ford, Illinois Institute of Technology, is Chairman and Professor J. W. Cell, North Carolina State College, is Secretary of the Mathematics Division.

Professor J. M. Howie of Nebraska Wesleyan University, Lincoln, Nebraska, wishes to dispose of a complete set (49 volumes) of the AMERICAN MATHEMATICAL MONTHLY, bound and in good condition.

Dr. P. H. Anderson of John Carroll University is also serving as a statistician with the regional office of the War Production Board.

Dr. Henry Barone of The Citadel has been appointed assistant professor at Pennsylvania State College.

Associate Professor F. C. Gentry, on leave from Louisiana Polytechnic Institute, is at present a lecturer in the pre-meteorological program of the University of New Mexico.

Dr. J. F. Heyda of Michigan State College has been appointed assistant professor at Denison University.

Assistant Professor Tryphena Howard of Western Kentucky State Teachers College has resigned. She was married in December to Mr. A. C. Scibiorski.

Professor Mabel I. Nowlan, on leave from Bethel Woman's College of Kentucky, is assisting with the work of the War Department.

Associate Professor C. A. Rupp of Pennsylvania State College is now a captain in the Signal Corps in Washington.

The following appointment to an instructorship is announced:
Denison University (Pre-Meteorological Program): Paul Cramer

Professor Emeritus Lilian Hackney of Marshall College died on February 4, 1943.

The death is announced of Professor David Hilbert of the University of Göttingen. He was eighty-one years old.

Professor H. F. Minssen, head of the mathematics department and vice-president of San Jose State College, died on February 7, 1943.

Professor H. H. Mitchell of the University of Pennsylvania died on March 13, 1943.

Professor M. H. Tyler of Rhode Island State College died on December 16, 1942.

Professor J. J. Westemeier of Dowling College (Des Moines, Iowa) died on February 1, 1943.

SUMMER COURSES

The following institutions announce courses in mathematics for the summer of 1943:

Brown University. Semester I of the three semester academic year for regular students will begin about July 1 and extend until about October 23. Essentially no formal courses in pure mathematics will be offered. However, the Program of Advanced Instruction and Research in Mechanics enters on its third year on June 14 and will extend to September 4. Courses in the following subjects are planned: partial differential equations, advanced theoretical mechanics, theory of structures, elasticity, fluid dynamics, differential and integral equations of physics, practical analysis, mathematical optics. The faculty includes Professors Stefan Bergman, L. N. Brillouin, Willy Feller, Witold Hurewicz, P.W. Ketchum, Willy Prager, J. D. Tamarkin and S. P. Timoshenko.

The University of California. From *July 5 to October 23* the following graduate courses will be offered. By Professor McDonald: theory of functions of a complex variable. By Professor Buck: differential equations. By Professor Sperry: differential geometry. By Professor Foster: algebra. By Professor Neyman: probability. By Mr. Eudey: statistical problems in experimentation. By Professors Bernstein, Foster and Tarski: seminar on topics in higher algebra. By Professor McDonald: seminar on function geometry and complex variable. By Professor Evans: seminar on elasticity, wave theory, shock waves, characteristics. By Professor Lehmer: seminar on advanced computation for pure and applied

mathematics. By Professor Neyman: seminar on probability and statistics. From *June 28 to August 7* the following courses will be offered. By Professor Tarski: famous problems of elementary algebra. By Dr. Wakerling: survey of mathematics.

The Catholic University of America. In addition to the elementary courses the following will be offered: *First term, June 28 to August 7.* By Professor Rice: analytic geometry of three dimensions. By Professor Finan: modern algebraic theories. By Professor Ramler: college geometry, differential equations, analytic projective geometry. *Second term, August 9 to September 18.* By Professor Rice: numerical mathematical analysis. By Professor Finan: abstract algebra. By Professor Ramler: advanced analytic geometry.

The University of Chicago. In addition to courses in trigonometry, college algebra, analytical geometry and calculus, theory of equations and differential equations, the following will be offered beginning June 22: By Professor Albert: higher algebra. By Professor Logsdon: analytic projective geometry. By Professor Graves: functions of a complex variable. By Professor Schilling: applications of differential equations to physics.

Columbia University. The following graduate courses will be given from July 7 to August 13: By Professor Kasner: general introduction to modern mathematics, transformations and curves. By Professor Koopman: probability. By Professor Murray: differential equations. By Professor Ritt: functions of a real variable.

The University of Illinois. From June 14 to August 7 graduate courses will be offered in the following subjects: theory of statistics, differential equations of mathematical physics, algebra, analysis, geometry. From June 14 to October 2 the following intermediate courses will be offered: differential equations, advanced calculus, statistics, fundamental concepts, introduction to higher algebra, introduction to higher analysis.

The State University of Iowa. In addition to the undergraduate courses the following will be offered over the period from June 5 to July 30: By Dr. Price: studies in high school mathematics. By Dr. Berg: differential equations. By Professor Knowler: elements of statistics. By Professor Chittenden: advanced calculus (definite integrals), seminar in analysis. By Professor Wylie: applied astronomy. By Professor Woods: solid analytic geometry, constructive geometry. By Professor Conkwright: group theory, navigation and maps. From August 2 to August 20 there is an independent study unit for graduate students.

The University of Michigan. In addition to elementary courses and courses in differential equations, theory of equations and determinants, advanced calculus, the following courses will be offered in the period from June 28 to August 20: By Professor Anning: modern geometry. By Professor Carver: introduction to air navigation. By Professor Copeland: statistics. By Professor Craig: theory of statistics I and II. By Professor Dwyer: computational methods. By Professor Hildebrandt: real variables, algebraic theory. By Professor Karpinski: teaching of geometry, history of arithmetic and algebra. By Professor Nesbitt: theory

of probability. By Professor Rainich: topics in higher geometry. By Professor Steenrod: introduction to the theory of sets. By Professor Wilder: introduction to the foundations of mathematics. In addition there will be a seminar in pure mathematics and one in statistics conducted by Professor Craig.

The University of North Carolina. In addition to the elementary courses, the following will be offered in the period from June 10 to August 27: By Professor Browne: theory of equations. By Professor Garner: the history and teaching of mathematics. By Professor Henderson: differential equations. By Professor Hobbs: partial differential equations. By Professor Trimble: mechanical drawing. By Professor Winsor: college geometry. By Mr. E. T. Hodges, Lieut., U. S. Coast Guard (Ret.): marine navigation. By Dr. Seebeck: aerial navigation.

The Ohio State University. In addition to the elementary courses the following will be offered in the period from June 21 to September 3: By Professor Bamforth: partial differential equations. By Professor Wylie: advanced mathematics for engineering students, fundamental ideas in geometry. By Dr. Albert: advanced calculus. By Dr. Mickle: theory of equations. By Dr. Helsel: differential equations.

The University of Texas. The following advanced courses will be offered. *First term, June 1 to July 22.* By Professor R. L. Moore: introduction to the foundations of geometry. By Professor H. J. Ettlinger: differential equations, selected topics in mathematical physics. By Professor E. F. Beckenbach: analytic functions. By Professor P. M. Batchelder: advanced calculus. By Professor H. V. Craig: vector and tensor analysis. By Professor E. W. Titt: advanced applied mathematics. By Professor R. N. Haskell: dynamics. By Dr. N. Coburn: meteorology. *Second term, July 23 to September 11.* By Professor H. J. Ettlinger: advanced calculus, descriptive geometry. By Professor R. G. Lubben: non-Euclidean geometry. By Professor A. E. Cooper: theory of functions of a complex variable. By Professor E. W. Titt: dynamics. By Dr. N. Coburn: meteorology.

The University of Wisconsin. From June 18 to September 18 the following advanced courses will be offered: By Professor Cohen: higher mathematics for engineers. By Professor Ulam: differential equations. During the six-weeks session the following courses will be given: By Professor Kenney: theory of equations, mathematics of educational statistics. By Mrs. Sokolnikoff: advanced calculus, mathematical applications. By Professor Trump: college geometry.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

THE MINNESOTA PROGRAM IN PRE-FLIGHT MATHEMATICS

In September, 1942, a committee was organized in Minneapolis, Minnesota, to foster free instruction in mathematics for men about to become aviation cadets in the Army or the Navy. Courses were outlined, and the plan was pushed vigorously all over Minnesota by the Committee on Education of the Minnesota Civilian Defense Council.

To inaugurate the program, the names of boys who were eligible for the training were obtained from the Aviation Cadet Examining Board. A card and a letter were then sent to each boy; the card giving pertinent information upon the boy's training and upon his interest in the proposed course in mathematics was to be returned. Shortly thereafter, a letter was sent to all the superintendents of schools throughout the state, and to local educational representatives on the Civilian Defense Council. The letter contained a list of the boys in the community who were interested in the course in mathematics, and asked for assistance in finding volunteers who would serve as instructors. Almost without exception, favorable replies were received, and outlines of the courses were then sent to the persons volunteering for the teaching. When the project was well under way, ninety-five towns in the state were giving instruction without charge to the men enrolled.

The courses in mathematics were designed to provide as much review in the fundamentals of arithmetic, algebra, and plane geometry as each student needs, and then to teach as much new material, including trigonometry, as is necessary to solve standard problems in aviation. Outlines of Courses A and B used in the mathematics instruction may be obtained by addressing The Office of Civilian Defense, State Capitol Building, St. Paul, Minnesota.

Minneapolis started its courses October 1, 1942, and thus became the first community in the state to have the project in operation. The instruction was done entirely by volunteers; in fact, the staff at the start included three retired teachers, three housewives, three members of the faculty of the University of Minnesota, one industrial chemist, one public school teacher, two secretaries and one life underwriter. Classes met six hours each week. Enrollment was limited to men who had enlisted in the Air Corps of either the Army or the Navy. Students were assigned to classes on the basis of their previous training in mathematics, and within a month after the start of the program, nine classes were in operation with a total enrollment of 334. The project originally was intended to extend over a period of six weeks, and to be repeated every six weeks as new enlistments occurred. It was found, however, that a more fluid organiza-

tion was desirable. New enlistees, therefore, were admitted as soon as they applied, without requiring them to wait for the opening of a new term. At the end of a period of six weeks, a reassignment of students was made within the school. Under this plan, the students progressed from class to class, and continued in classes until they were called into service, thus acquiring the maximum amount of preparation for their future needs at the aviation training centers.

The foregoing project was an outgrowth of the experience of other agencies with pre-flight courses. As early as July, 1941, the Minneapolis Public Schools conducted review courses for boys who expected to take the qualifying examinations for the Army Air Corps. These courses were discontinued in January, 1942, when the qualifying examinations in specific subjects were replaced by a screening test. Then during the summer months of 1942, the Extension Service of the University of Minnesota gave courses in mathematics to nearly five hundred boys who were awaiting assignment to the schools of the Air Corps. Professor W. L. Hart became very much interested in this training, and advocated the idea that boys should not be required to pay for this service. Accordingly, he and Dr. Clifford Archer of the College of Education of the University of Minnesota formulated the preliminary proposals for the present program.

A BIBLIOGRAPHY OF CRYPTOGRAPHY

The following bibliography on cryptography was compiled from several sources; in particular, thanks are due Lieutenant Commander H. T. Engstrom of the United States Navy for his assistance. Titles in the list which are preceded by asterisks are known to be available at the present time.

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The American Cryptogram Association publishes a bi-monthly journal, *The Cryptogram*. For information upon the association and its publications, communicate with Mr. George C. Lamb, Burton, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-sixth Summer Meeting, New Brunswick, N. J., September 11–13, 1943.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, Jan. 29, 1944
OHIO
OKLAHOMA
PHILADELPHIA, Philadelphia, Nov. 27, 1943
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN

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THREE HUNDRED YEARS OF MATHEMATICS AT HARVARD

J. L. COOLIDGE, Harvard University

1. The first century. The beginning of mathematical study at Harvard was contemporaneous with the beginning of the college, and both beginnings were indeed small. It was on October 28, 1636 that the Great and General Court of Massachusetts "agreed to give 400 £ towards a schoale or colledge." The first classes seem to have been held in July, 1638, but the real founding was in 1640 when an unusual man, Henry Dunster, assumed the duties of the presidency. The grant of 400 £ was generous, and so was the bequest of John Harvard who, at his death in 1638, left one half of his estate, £779 17s 2d, and all of his library to the infant institution. However, the need for additional funds was immediately evident. The most likely source for such aid was that body of people in Old England who were deeply interested in the creation of a Commonwealth in New England, which should conform to the principles of the Puritan creed. Accordingly in 1643 an appeal was addressed to them entitled "*New England's First Fruits*." The object of this was ostensibly to set forth what had been accomplished in Massachusetts Bay. The tone is pious and edifying; not so the comment in Morison,* " '*New England's First Fruits*' was a promotion pamphlet; one half expects to find in it a return post card on receipt of which our representative will call." But the document pays a good deal of attention to the college, and we read that the faculty, that is to say President Dunster, at 10:00 A.M. on Mondays and Tuesdays read arithmetic and geometry to the students of the third (and last) year during the first three quarters, and astronomy in the last quarter. And so mathematics began in our oldest college.

It is probable that there was little change in the next eighty years. By 1653 mathematical theses appeared on the Commencement programme. Dunster taught geometry in English, though Latin was the language of the college, basing his teaching on Euclid and Ramus. In astronomy, surveying and navigation, progress was more marked as we see in Morison.†

With the beginning of the eighteenth century some progress had been made. In a thesis of 1711 there is mention of the conic sections, in 1719 there is a reference to fluxions, and in 1721 we find algebra. These theses were apparently statements which the candidate was prepared to defend. "A fluxion is the velocity of an increasing or decreasing flowing quantity." "Algebra is the art of reasoning with unknown quantities in order to define their relation to known quantities." There must have been enough substance to all this to stimulate that unusual student to whom I now turn.

2. Isaac Greenwood. The greatest step forward came in 1726 with the appointment of the first mathematical professor, Isaac Greenwood. There lived in

* Samuel Eliot Morison, The Founding of Harvard College, Cambridge, 1935, p. 304.

† Harvard in the Seventeenth Century, Chapter X.

London at that time a man named Thomas Hollis who was deeply interested in Harvard for reasons that are not now evident. In 1721 he established a professorship of Divinity which the college proceeded to accept, and then to violate the spirit of the gift.* He planned to establish a professorship of mathematics and natural philosophy by his will, but meeting Greenwood in London changed his mind and gave the sum, £1200, outright, suggesting that Greenwood be appointed.

Isaac Greenwood graduated from Harvard in 1721, and received an A.M. degree three years later. He then went to London, studied divinity and preached, but I judge did some mathematics also. At any rate he somehow acquired more knowledge of that subject than was easily attainable in Cambridge, Massachusetts. For a time he fascinated the good Hollis, but doubts crept in for we find Hollis writing to the Harvard Corporation in July 1726,† "I advise you to make due trial of him for your own satisfaction. He has not pleased me of late. Only you may know that if you recommend I accept, not else." Some little time later he writes in the same vein, "Mr. Greenwood has left us on a sudden without paying his debts, or taking leave of his landlord, his tutor or me;" and again that the money Greenwood had spent in a ramble of a few weeks and his debts to his tradesmen and others amounted to three hundred pounds sterling. Among other instances of extravagance were three pairs of pearl colored stockings.

One would naturally expect that the Harvard authorities would hesitate to appoint a professor with such a record. Perhaps they did hesitate, but they surely appointed him "under the strong apprehension, probably entertained by both boards, that if they hesitated a Baptist might be forced on them."‡

Hollis's fears proved well grounded. Quincy tells us "The proceedings in respect to this unhappy individual were marked with consideration and firmness. On the 26th of April, 1737 Mr. Greenwood was called before the Overseers charged with intemperance which he confessed, and casting himself on the lenity of the Board professed his resolution to reform. Several lapses and warnings followed. On November 25 he was required to exhibit a humble confession and received a public warning. He lapsed twice between then and December 7 when a formal vote of expulsion was passed, but six months of further repentance and lapses followed till he finally left on July 13, 1738.§

So much for Greenwood the citizen. As a mathematician he was a decided success. He was interested and wide awake, publishing in 1726 a course of philosophical lectures explaining Newton's theories and in the same year a "*Course of mechanical philosophy.*" In 1729 appeared his "*Arithmetic vulgar and decimal,*" the earliest work on that subject by an English speaking American. He also offered to give private instruction in the novel subject of flux-

* Quincy, History of Harvard University, Cambridge, 1840, Vol. I, pp. 238 ff.

† Quincy, Vol. II, pp. 11 ff.

‡ Quincy, Vol. II, p. 21.

§ Quincy, Vol. I, pp. 11 ff.

ions. We get a close view of Greenwood's influence on his pupils from the manuscript notes of two of them which have been recently published in Simons.* These are the notebooks of Samuel Langdon, subsequently President of the university, and John Diman. The subject is algebra. They cover all of the usual present school topics, with quite a little attention to cubic equations, and a short account of approximate determination of the roots of equations of any order. There are also some geometrical problems solved by algebraic manipulation. There are references to the works of Raphson, Oughtred and Halley, probably accessible to Greenwood, but not to his pupils.

The College might well have been embarrassed to find an adequate successor to Greenwood had not Fate kindly provided a thoroughly capable one in the person of John Winthrop, who served as Hollis's professor for forty-one years. He has been called the father of American astronomy, and did notable work in that science, being a Fellow of the Royal Society, and contributing to their Transactions. He had a wide interest in various branches of science, among them physics, astronomy and seismology. I can not find that his interest in pure mathematics was outstanding, but between 1735 and 1750 certain higher plane curves appear in the Harvard mathematical theses, and after 1751 fluxions occupied a leading place.†

3. The first decline. The decline in the eminence of the holder of the Hollis Professorship, which was fortunately averted by the appointment of Winthrop, finally set in at his departure. Samuel Williams next took the chair. I can not find out much about him, and little more concerning his successor Samuel Webber. It is true that the latter published two or three volumes of mathematical textbooks "taken from the best authors" and managed to be elected president of the College. Cajori‡ quotes Edward Everett as saying that Webber was a person of tradition and routine, and Story that he was modest, mild and quiet, but unconquerably reserved and staid, but Quincy is more eulogistic: "The urbanity and gentleness of his manners, the prudence, rectitude and firmness which characterized his service in the office of president, secured for his administration popularity and success, both with his pupils and with the public.§ There was certainly a retrocession in the interest in mathematics during these years. We have seen how much attention was paid to fluxions during the latter part of the eighteenth century, but Cajori‖ tells us that of 133 mathematical theses written between 1781 and 1807 only seven involved fluxional problems, though much attention was paid to astronomy. In 1803 John Farrar wrote a "*Calculation and Projection of a Solar Eclipse,*" quite an advance on Samuel Farrar who

* Lao Ginevra Simons, "Introduction of Algebra into American Schools in the Eighteenth Century," U.S. Bureau of Education, 1924. No. 18.

† Simons, p. 38.

‡ Cajori, "The Teaching and History of Mathematics in the United States," Bureau of Education Circular No. 3, 1891.

§ Quincy, *loc. cit.*, Vol. II, p. 299.

‖ Cajori, *cit.*, p. 59.

in 1793 offered *"A Prospective View of the Episcopal Church."*

This same John Farrar next claims our attention, for he succeeded to the Hollis Professorship. One can find differing opinions of him. Cajori* quotes the excellent Andrew P. Peabody as writing, "He delivered when I was in college a lecture every week to the junior class on natural philosophy and one to the senior class on astronomy. No student was willingly absent . . . he very soon rose from prosaic details to general laws which he seemed to ever approach with blended enthusiasm and reverence as if he were investigating and expounding divine mysteries. His face glowed with the inspiration of the theme." I quote also from Smith-Ginsburg:† "He was one of the best teachers of his time," but Cajori quotes Josiah Quincy as saying "We gained a miss from Farrar for the fourth time this term. This was much to the gratification of the class, who hate his branch, though like him." Whether Farrar was a good teacher or not, he knew good teaching when he saw it, for he published translations of excellent French texts by Bezout, Lacroix and others.

4. Benjamin Peirce. The first epoch in the history of mathematics at Harvard opened with the opening of the college, the second, nearly a hundred years later, with the appointment of Greenwood. The third epoch began about a century later still with the appointment of Benjamin Peirce in 1832. A long account of him will be found in Cajori‡ and *"Benjamin Peirce"* by R. C. Archibald.§ Here we have a man in quite a different class from any of his predecessors, and able to bear comparison with any who have succeeded him. He had a brilliant and very rapid mind, a profound interest in science, and a conception of a university as a place where mathematics should not only be pursued but advanced, both by teachers and pupils, which was quite different from anything in evidence in America before his time. As a scientist his most important work was in astronomy. In mathematics his outstanding work was his *"Linear Associative Algebra."* This highly original treatise has been sharply attacked, perhaps without sufficient reason. A careful discussion will be found in Archibald. His membership in the Royal Society of London, Royal Astronomical Society, Royal Society of Edinburgh, Royal Society of Göttingen and Honorary Fellowship in the University of Kiev show the esteem in which he was held outside his own country. A complete list of his honors is found on pages 10 and 11 of Archibald. In originality and power no contemporary American mathematician except Hill and Gibbs, if they deserve that classification, was comparable to him.

There is no unanimity of opinion of Peirce as a teacher. I record the fact that the late A. Lawrence Lowell who studied mathematics under Peirce, writing a graduation thesis on Quaternions, expressed the view that the most stimulating teachers he ever met were Henry Adams in history, and Peirce in mathe-

* Cajori, *cit.*, p. 127.

† "A History of Mathematics in America before 1900," by David Eugene Smith and Jekuthiel Ginsburg, Carus Monograph No. 5, Open Court Publishing Co., 1934, p. 97.

‡ Cajori, *cit.*, pp. 132–147.

§ This MONTHLY, January 1925, and separately Open Court Publishing Company.

matics. Now Lowell attended the Harvard Law School in the days of Langdell and Ames, of Gray and Thayer. Cajori on page 143 quotes Thomas Wentworth Higginson as saying, "We did not know whither he was going, but the huddle of new equations seemed like an outlet from the World, and a ladder to the stars. He gave a charm to the study of mathematics which, for me, has never failed." Many able astronomers studied with him, G. P. Bond, Asaph Hall, B. A. Gould, Simon Newcomb, and G. W. Hill. But there is little doubt that he was a sad failure as a teacher for ordinary students. In 1838 elaborate plans were made for freshman and sophomore mathematics at Harvard, but Cajori remarks on page 136, "This arrangement did not prove satisfactory. Professor Peirce's textbooks were found very difficult, and Peirce himself was not a good teacher except for boys of mathematical genius." In 1848 the Lawrence Scientific School was opened at Harvard, and Peirce presented a wonderful array of courses in mathematics and astronomy, reading the works of Cauchy, Lagrange, Laplace, Gauss and others. What a marvelous programme for the United States in 1848! But there were only two takers in 1849, and all was soon dropped. Repeated and loud complaints were made that mathematical teaching at Harvard was poor. Cajori quotes W. F. Allen of the class of 1851 as saying that only two members of his class of sixty elected mathematics, and they soon dropped it.

And what of Peirce's textbooks? Cajori says on page 441 that Peirce's *Analytical Mechanics* was acknowledged at the time, even in Germany, to be the best of its kind, but Archibald on page 14 quotes Newcomb as writing "The exposition of dynamical concepts in the first forty pages is pleasant reading for one already acquainted with the subject but that the student beginning the subject could understand it without clearer distinction of definitions, axioms and theorems, seems hardly possible." In 1848 a committee of the Overseers reported that Peirce's textbooks were abstruse and difficult and few could comprehend them without explanations, that his work was symmetrical and elegant, and could be pursued with pleasure by adult minds but that books for young students should be more simple. A minority report, submitted by Thomas Hill and J. Gill, disagreed totally, and commended the works for their terseness, simplicity of style, vigor and originality of thought.*

It is true that Peirce's books are brief and original. I have the impression that this conciseness was often attained by neglecting the real difficulties involved. Says Cajori on page 144, "As an instance of this we mention his assumption as self evident that a line which is contained upon a limited surface, but has neither beginning nor end on the surface, must be a curve reentering on itself. By this new axiom he reduces a demonstration which would occupy half a dozen pages, to a few lines!" This is certainly neat, but unfortunately the axiom, even if self-evident, is not always true. One has but to consider a double spiral which cuts at an angle, other than 90°, all the circles through two points. Turn to his *Curves, Functions and Processes*, Vol. I, page 176.

* Cajori, pp. 140–141.

To find the nth power of $1+i$, when i is infinitely small and n infinitely great so that $ni=a$

$$(1+i)^n = 1 + ni + \frac{n(n-1)}{1\cdot 2} i^2 + \frac{n(n-1)(n-2)}{1\cdot 2} i^3 + \cdots .$$

But n is infinite, therefore

$$n - 1 = n, \qquad n - 2 = n, \text{ and } \cdots .$$

$$(1+i)^n = 1 + ni + \frac{n^2 i^2}{1\cdot 2} + \frac{n^3 i^3}{1\cdot 2\cdot 3} = 1 + a + \frac{a^2}{1\cdot 2} + \cdots .$$

Put $a=1$

$$(1+i)^{1/i} = 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots$$

$$= (1+i)^{a/i} = \left[(1+i)^{1/i}\right]^a = e^a = 1 + a + \frac{a^2}{2} + \cdots .$$

$$1 + i = e^i; \qquad \log (1+i) = i.$$

If $a=mi$

$$(1+i)^m - 1 = a.$$

"All terms of the second member except the first being neglected because they are infinitely small."

If $e^m=b$

$$b^i - 1 = i \log b.$$

To differentiate a^x

$$da^x = a^x(a-1) = \log a \cdot a^x dx.$$

No wonder Peirce's work is condensed.

I have frequently wondered why it was that Peirce who was a close student of Cauchy and Gauss should be so completely oblivious of rigor. His pupil, Byerly, once told me that the master never bothered whether a series were convergent or not. The only explanation is that such things must have seemed to him unimportant and not worth caring about. Perhaps an explanation is also found in the following anecdote. I quote from Charles W. Eliot in Archibald, page 2, "One day in my senior year I ventured to say that what he had just been saying to us about functions and infinitesimal variables seemed to me to be theories and imaginations, rather than facts or realities. Peirce looked at me gravely and remarked gently, 'Eliot, your trouble is that your mind has a skeptical turn.' " Such a reproach was never addressed to Benjamin Peirce himself.

5. The period of retrocession. With the retirement of Benjamin Peirce in 1878 there came a great slump in the scientific activity of the Harvard mathematical department, although the teaching continued and improved. His son James Mills Peirce was a much better teacher, even though he lacked the spark of originality; his contributions to mathematics are negligible. Like his father he taught for forty-nine years, a most kindly and genial man. Like his father he was deeply interested in Hamilton's "Calculus of Quaternions", and lectured on the subject regularly until his death in 1905. He deserved well of the University, for his pioneer work in introducing the elective system, and building up the Graduate School.

Many Harvard men will recollect William Elwood Byerly of the class of 1871, who took the first Harvard degree of Ph.D. in 1873. He spent the next three years teaching at Cornell, and fell under the influence of Evan W. Evans who had lectured on modern methods in analytic geometry, broadly speaking, the sort of thing one finds in Salmon's Conic Sections. When Byerly came back to teach at Harvard in 1876, he offered a course dealing with these matters which became standard in Harvard teaching for the next sixty years. He was a remarkable elementary teacher, and his two textbooks on the "*Differential and the Integral Calculus*" marked a real advance. He had been brought up on Peirce's texts and when he encountered the best French books, notably the work of Bertrand, a great light broke on him, and he determined to publish something similar. It is hard to say what he might have accomplished had he been less burdened with teaching at Harvard College and administrative work at Radcliffe.

About contemporary with Byerly was the third Peirce, Benjamin Osgood, a singularly retiring and modest man, but a thorough scientist. He tied mathematics in with physics in a way that has been frequently lacking at Harvard especially from the time of his death in 1915 to the appointment of J. H. VanVleck in 1934.

6. The new movement. The renaissance of mathematics at Harvard came in the early nineties with the appointment of the "great twin brethren" William Fogg Osgood and Maxime Bôcher. It was the moment of the great awakening in American mathematics, when a number of able and enthusiastic young men, largely trained in Germany, set about raising the science as pursued in this country to the same plane on which it was pursued in Europe. The immediate means was the American Mathematical Society with its Bulletin and Transactions. It was the beginning of the era of Moore, Maschke and Bolza in Chicago, of Fiske and Cole at Columbia, of Fine at Princeton, Pierpont at Yale, VanVleck at Wisconsin and White at Vassar. Osgood and Bôcher, early presidents of the Society, were in the very middle of it. They introduced into Harvard new and advanced courses, largely dealing with what one might call "Göttingen mathematics," gathered around them graduate students whom they prepared for the doctorate, and treated their younger colleagues who came after them with

the greatest sympathy and kindness. Both were good teachers for curiously different reasons. Osgood was extremely conscientious. He felt it was his business to teach and teach well, to make his pupils understand. So he worked over a subject, turning it about in every way till it assumed a shape which he believed a beginner, whether a freshman or graduate student, could understand. Bôcher was different. His father was French and he had a limpid Gallic mind. He taught clearly because he thought clearly, that was the way his mind worked. He was cut off by death in 1918. Osgood continued until his retirement in 1933, the first member of the Department to retire quietly at the natural time since Benjamin Peirce withdrew fifty-five years before. As a sign of the recognition of these men in Europe it is worth noting that Bôcher wrote the article on "*Randwertheaufgaben*" in the Enzyklopädie, Osgood the even more important one on "*Analytische Funktionen.*" His "*Lehrbuch der Funktionentheorie*" is surely the most important of the eighty mathematical books which the members of the Department have published since Greenwood's appointment in 1726.

In the third and fourth decades of the present century the dominant figure in Harvard mathematics has been George David Birkhoff. His preference has been for substantial classical mathematics, but he has touched a variety of topics in turn. He was early interested in Poincaré's papers on celestial mechanics; the three body problem struck his fancy, as well as the stability of orbits. He was thus naturally led to the geometrical problem which Poincaré tried hard to solve and finally abandoned. Birkhoff solved it in 1912. His work in dynamical systems has been fundamental, and even more could be said of his solution of the ergodic problem. One is naturally tempted to compare Birkhoff with Benjamin Peirce, but there is no fair ground of comparison, for the conditions of American mathematics have changed radically in the interval. Moreover, when Peirce published, his were the only Harvard contributions to mathematical science. In Birkhoff's time scientific activity was very general throughout the Department. It would be a great mistake to pass over in silence the work of Morse and Stone, to mention but two of many.

7. The Lowell epoch. Important changes in the work of the Department came during the twenty-three years' presidency of Lowell, 1909–1932. There had been for a number of years two kinds of introductory mathematical instruction, one in the College, the other in the Lawrence Scientific School. The distinguished pupil of Benjamin Peirce could not recognize the existence of two different kinds of mathematics; then why have two kinds of instruction? The Department held the same opinion, but insisted that in taking on an additional burden of elementary teaching they should not be handicapped by the addition of new permanent members who were capable only in the elementary field. The University authorities accepted this view. E. V. Huntington came over with the Scientific School courses; the appointments of Birkhoff and Dunham Jackson were more or less connected with these changes.

In 1913 the visiting committee of the Board of Overseers found that the freshman sections were too large, and recommended that smaller sections would

be better for the majority of beginners. They did not suggest that the burden of elementary instruction falling on the permanent members of the staff be correspondingly raised. The solution seemed to be to give temporary half-time appointments to able young men who were deemed capable of handling sections of fifteen or twenty men. The solution was not ideal. It was not always easy to find a sufficient number of mathematical aspirants who had not only the mathematical knowledge, but the skill and personality. Their work was closely supervised by the permanent members, and the system has, all things considered, worked well. What better plan could be found for teaching four or five hundred freshmen yet maintaining standards in the higher branches?

A fundamental change in the Harvard requirements for the A.B. degree was introduced gradually during the Lowell regime. This was the introduction of a final general examination in a subject, not a course. The preparation for this involved the appointment of a body of tutors. The Department of Mathematics took some time to fall in with this plan, but finally accepted it in 1926. They insisted, however, that tutoring should be considered as one of the functions to be assumed by all of the staff, and not the specialty of a separate and somewhat nondescript body of teachers. Each permanent member should give a definite part of his time to tutoring; this usually came to the responsibility for five pupils. Once again they insisted, however, that they should only undertake this if they were strengthened accordingly. The authorities agreed. Marston Morse and Marshall Stone were appointed.

The members of the Department have taken part in various mathematical activities outside of the University. They have held many offices in the Mathematical Society and the Mathematical Association. Many have done editorial work on mathematical journals. For a number of years beginning in 1899 they took over from the University of Virginia the responsibility for publishing the Annals of Mathematics. The subsidy to this from the University was subsequently withdrawn, and the editorial responsibility for publishing passed to Princeton. The Harvard Department felt that they could do more for mathematical science by giving their time and strength to their own research and training their own pupils than by editing the writings of others. This may have been narrow and shortsighted. The mathematical future was not easy to predict at that time.

8. Fundamental principles. In the last seventy-five years the Harvard Department has evolved a definite doctrine of its duties and responsibilities. These fall under four heads:

1. To inject the elements of mathematical knowledge into a large number of frequently ill informed pupils, the numbers running up to 500 each year. Mathematical knowledge for these people has come to mean more and more the calculus.

2. To provide a large body of instruction in the standard topics for a College degree in mathematics. In practice this is the one of the four which it is hardest to maintain.

3. To prepare a number of really advanced students to take the doctor's degree, and become university teachers and productive scholars. The number of these men slowly increased from one in two or three years, to three or four a year.

4. To contribute fruitfully to mathematical science by individual research.

To fulfill these tasks the Department gradually evolved one sacrosanct and fundamental rule; all permanent members must have the same duties and responsibilities. I think that this conception goes back to J. M. Peirce. It seems somehow characteristic of his generous nature. Perhaps the oldest members receive favored treatment in the choice of hours, otherwise there is no distinction. Each must do his share of tutoring, of elementary teaching, of standard teaching, of dull administrative detail. Each must be ready to help any qualified advanced pupil who wishes to work with him. Of course, each must do his share of research, not only do it, but want to do it. It can not be maintained that all are equally good teachers of elementary courses. But the Department has always been skeptical about the man who is merely a teacher, especially an elementary teacher, and nothing else. Such a man, freed from competition and lacking the stimulation of original work, is often inclined to become an unprogressive and stereotyped teacher. As for research, no one imagines that the work of all is equally important scientifically, but it is morally. There is a diversity of gifts, but the same spirit.

The fine fruit of all of this has been the feeling, frequently shown, that what counted was the success of the combined undertaking, not the triumphs of an individual. On several occasions a senior man, who thoroughly deserved any advancement the University could give, has welcomed the promotion over his head of a junior man when that seemed for the advantage of the whole body. I do not think that any set of teachers, no matter how brilliant, who were essentially career men, could have accomplished what has been done by the Harvard Department of Mathematics.

The Derivatives of Composite Functions

Since the publication of my note on "The derivatives of composite functions" (p. 9 of the present volume of this MONTHLY), my attention has been called to a number of papers on this and related topics. Among these I should mention a paper by L. S. Dederick, "Successive derivatives of a function of several functions," Annals of Mathematics, vol. 27, 1925–26, p. 385 and one by I. Opatowski, "Combinatoric interpretation of a formula for the nth derivative of a function of a function," Bulletin of the Am. Math. Soc., vol. 45, 1937, p. 944. Each of these gives references to further interesting papers.

ARNOLD DRESDEN

MATHEMATICS, 600 B.C.–400 B.C.*

MAX DEHN, Illinois Institute of Technology

1. Isolated arithmetical and geometrical facts were, without doubt, known in prehistoric times much as such facts are now known among the most primitive tribes. Rather advanced mathematical knowledge appears in ancient Egyptian papyri (for instance in the Rhind Papyrus of the 14th century B.C.) and on numerous Babylonian cuneiform texts dating from 2000 B.C. onwards. Certainly the Greeks learned many of the algebraic methods and the techniques of geometric measurements from these ancient peoples through the lively commerce of the Eastern Mediterranean. Our reports begin with Greek mathematics after 600 B.C.

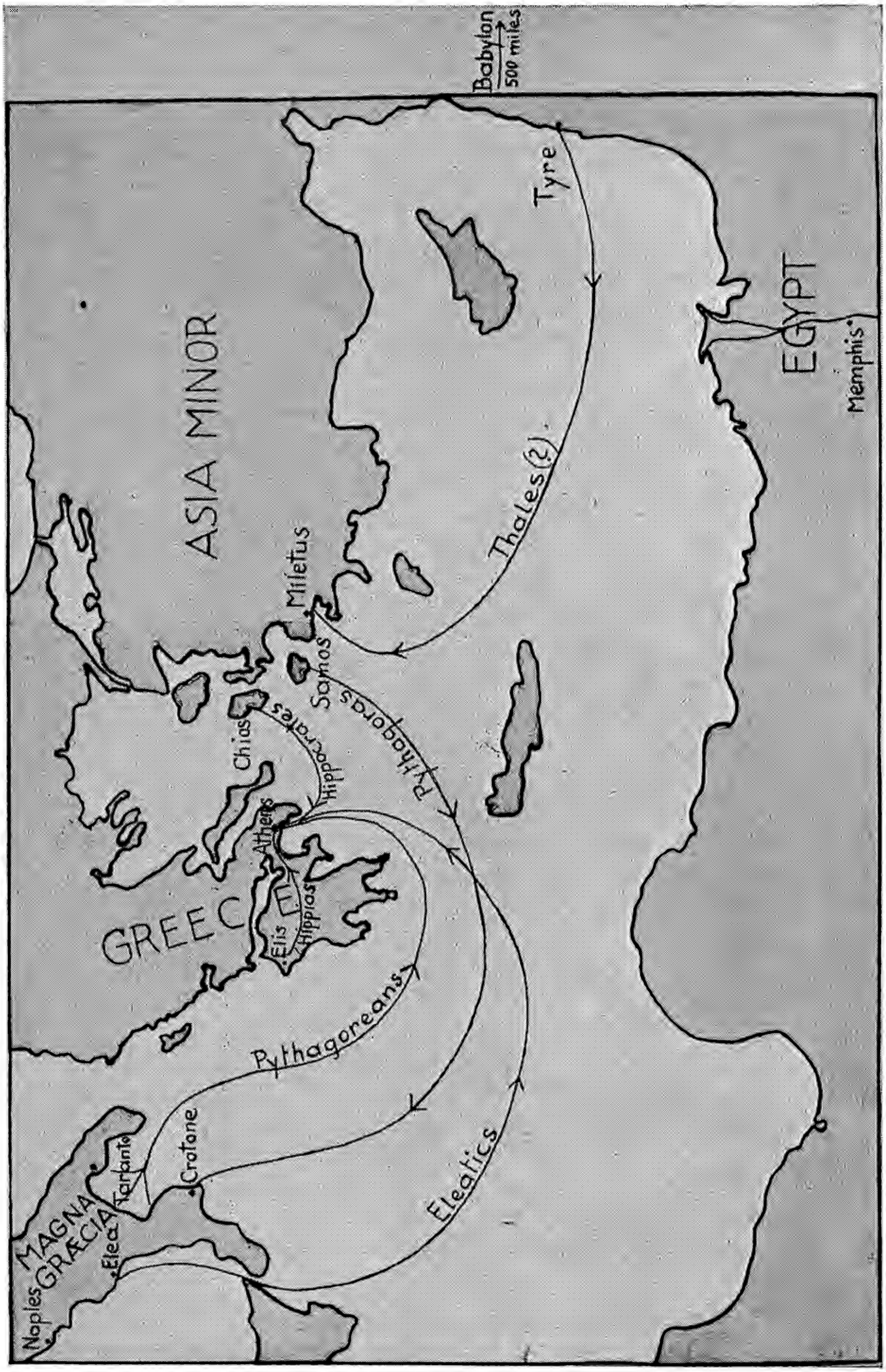
2. **Sources.** The Sources for the history of Mathematics in Greece during the period from 600 B.C. to 400 B.C. are very scarce and unreliable. We have a fragment of mathematical history by *Eudemos* (ca. 320 B.C.) in an excerpt of the sixth century A.D. This fragment itself is in a bad state, corrupted by later changes. There are, however, scattered among the works of Greek authors, enough passages concerned with the mathematics and mathematicians of Ancient Greece, for us to derive a fairly clear idea of this early period.

3. **Early Greeks.** While there is no mathematician known from ancient Egypt or Babylon, we do know the names of famous Greek mathematicians. *Thales* (ca. 600 B.C.) of Miletus (see map), who was probably of Phoenician origin, is known as the father of Greek mathematics. He had many disciples. It may be that there is a direct connection between him and *Pythagoras* (ca. 550 B.C.) from Samos. The latter, who was the head of an aristocratic brotherhood, a school of wisdom and science, was a political and philosophical leader in Southern Italy. He emphasized the importance of Mathematics in the higher or liberal education, and for many centuries his name invoked an aura of mysticism. After his death, his school flourished for more than a hundred years, and numbered several famous mathematicians among its members. They will be mentioned in the second report.

Hippocrates of Chios (ca. 450 B.C.) was probably not connected with the Pythagoreans. He taught mathematics at Athens. We have a fragment of his mathematical work transmitted by Eudemos. This is the first *published* mathematical investigation known. Hippocrates is probably also the author of the first manual of geometry.

Hippias of Elis (ca. 430 B.C.) was a famous Sophist, a man with vast knowledge in mathematics and astronomy. An outstanding teacher, he was paid for his courses, which he gave mainly at Athens, where teaching and research in

* This is the first of a series of articles by Professor Dehn which, it is hoped, will cover the whole history of mathematics in compact form. An important feature of the series is the collection of maps, which are the work of Mrs. Dehn. Professor M. H. Heins, of Illinois Institute of Technology, is doing some of the editorial work connected with the articles.


Babylon
500 miles
ASIA MINOR
Tyre
EGYPT
Memphis
Thales (?)
Miletus
Samos
Pythagoras
Chios
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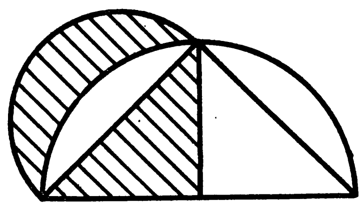
Athens
GREECE
Elis
Hippias
Pythagoreans
Eleatics
Crotone
Taranto
MAGNA GRÆCIA
Elea
Naples

mathematics were concentrated at the end of the period with which the present report is concerned.

Even in this early period we begin to see many of the features of modern scientific activity: authors famous for their achievements, ambitious to find new results; renowned teachers; pupils eager to learn; books where results are collected, digested, and presented in such a way that the reader understands the facts and proofs, and is inspired to do research himself.

4. Achievements. What is left to us of Babylonian and Egyptian mathematics shows only prescriptions for computations or for solutions of particular problems. But we find in the old Greek mathematics, proofs of the given solutions of problems and of the various theorems; we find convincing explanations. Great problems are proposed and treated. Problems of construction are solved with the help of ruler and compass. Among such problems are the conversion of areas into each other, the most important case being the squaring of the rectangle; and the construction of the regular pentagon by means of the golden ratio.



Also propounded at this period were three classical problems of construction: the squaring of the circle, the trisection of an angle, and the duplication of the cube. The first two problems stimulated the *construction of the first curve* apart from the "naturally" given circle. This curve, which was invented by Hippias, is the quadratrix, whose equation in rectangular coordinates is $y=x \cot \pi x/2r$. The construction of points on this curve, approaching the y-axis at the level $y=2r/\pi$, corresponds to the Archimedian computation of the perimeters of regular polygons approaching the circumference of a circle. In the construction of Hippias, the limiting process is visualized by the continuous curve approaching the y-axis.

The problems of trisecting an angle and of duplicating the cube led to new mechanical devices other than the compass, and finally, at the beginning of the next period, resulted in the discovery of the conics.

5. Theorems. Probably the first theorems, found by the Greeks, were propositions about angles. The *Pythagorean theorem*, as a relation between the lengths of the sides of a right triangle, was in all likelihood already known to the Babylonians; as a theorem about areas it is perhaps a Greek achievement. At all events, the knowledge of this theorem which was always attributed to Pythagoras himself, was a matter of great moment to all educated Greeks.

Endeavoring to square the circle, Hippocrates discovered areas bounded by two circular arcs which could be constructed by compass and rule. The simplest

case is indicated in the preceding figure, in which the two shaded areas are equal.

To this period belongs the discovery and the construction of the *five regular solids*.

The greatest achievement of this epoch was the discovery and proof of the existence of *irrational ratios* in the incommensurability of side and diagonal of a square. Whether the original proof was given by arithmetical or geometrical methods is unknown. This was the first example of a mathematical truth contrary to naïvely simplifying intuition. The necessity for a strict proof became apparent, and this influenced the whole development of mathematics in the direction of rigor.

Further, we have the discovery of the projection of the infinite process of counting into arithmetical, geometrical, and kinematic ideas. These phenomena were found and discussed by the *Eleates* (Elea, a city of Southern Italy). The finite sum of an infinite geometric progression, the indefinite subdivision of a finite line or of a finite movement were all in contradiction to naïve intuition and provoked profound problems as well as new constructions in Philosophy.

WHAT IS A MATRIX?

C. C. MACDUFFEE, Hunter College of the City of New York

1. History. J. J. Sylvester, who had a penchant for startling nomenclature, introduced the term *matrix* into mathematics [1]. He applied it to a rectangular array of numbers "out of which determinants can be formed." Two such arrays were considered to be equal only if corresponding elements were equal. Arthur Cayley [2] used the term in the same sense, as any array of coefficients, with the insistence that "the idea of matrix precedes that of determinant."

The branch of mathematics which is now called the algebra of matrices had four sources. W. R. Hamilton [3] first presented it under the title of "Linear and vector functions." Cayley [4] considered a matrix as a single quantity, defined sum, product and scalar product, and stated that "square matrices are of greater importance than rectangular." E. Laguerre [5] also laid the foundations of matric algebra, and remarked that "the calculus of matrices gives a simple interpretation of ordinary complex numbers, of quaternions, of the algebraic clefs of Cauchy, of the imaginaries of Galois." G. Frobenius [6] encountered matric algebra in the guise of bilinear form theory. In the awkward notation of the composition of forms, he discovered many of the fundamental theorems of matric theory. The abstract point of view of Cayley, and Sylvester's term *matrix*, gradually became standard.

2. The total matric algebra. If the universe consisted of a single particle of matter, the phenomenon of gravitation could not exist. In fact, the particle would be practically without properties, for the physical properties of matter can for the most part be described only in terms of reactions among two or more

particles. A single particle of matter would be a forlorn and uninteresting thing.

Similarly a mathematical entity, such as number, point or element, is a name and nothing more until its properties are defined; and its properties involve its relations with the other entities of the mathematical system to which it belongs. For instance, it is futile to attempt to define a complex number in any other way than as an element of the complex field. The complex field can be defined by giving enough of its properties to characterize it, whereupon a complex number is set apart from every other kind of number by its properties which arise from its membership in the complex field.

To define a matrix as a rectangular array of numbers is as inadequate as to define a single particle of matter or a single complex number. *A matrix is a number of a total matric algebra.*

A total matric algebra of order n over a ring R[7] may be defined as follows: Consider the set of all $n \times n$ arrays of elements of R such as

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdot & \cdot \cdot \cdot \cdot \cdot \cdot \\ a_{n1} & a_{n2} \cdots a_{nn} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} \cdots b_{1n} \\ b_{21} & b_{22} \cdots b_{2n} \\ \cdot & \cdot \cdot \cdot \cdot \cdot \cdot \\ b_{n1} & b_{n2} \cdots b_{nn} \end{bmatrix}, \cdots.$$

More briefly we may write $A = (a_{rs})$ where r is the row index and s the column index of the element a_{rs}. We shall understand that $A = B$ means

$$a_{rs} = b_{rs} \qquad r, s = 1, 2, \cdots, n.$$

Three operations, addition $(+)$, scalar multiplication $(\cdot)$ and multiplication $(\times)$ are defined by the identities

$$A + B = (a_{rs} + b_{rs}), \qquad k \cdot A = (ka_{rs}),$$

$$A \times B = \left(\sum_{i=1}^{n} a_{ri} b_{is} \right)$$

where k is any number of R. It may readily be verified that addition is associative and commutative, and that the array O all of whose elements are 0 is such that

$$A + O = O + A = A$$

for every A. The array $-A = (-a_{rs})$ is such that

$$A + -A = O.$$

Thus the arrays of M form a commutative group with respect to addition. Multiplication is associative, and the following distributive laws hold:

$$A \times (B + C) = A \times B + A \times C, \qquad k \cdot (A + B) = k \cdot A + k \cdot B,$$

$$(k + l) \cdot A = k \cdot A + l \cdot A.$$

The mathematical system M so defined is called the *total matric algebra* of order n^2 over R, and the arrays $A, B, \cdots$ are called *matrices* of order n over R.

Since R is a ring, M is a ring. If R has a unit element 1, M has the unit element

$$I = (\delta_{rs}), \qquad \delta_{rr} = 1, \ \delta_{rs} = 0 \text{ for } r \neq s,$$

and the scalar multiples of I constitute a subalgebra of M isomorphic with R. But the assumption of the commutativity of R does not imply the commutativity of M for $n > 1$.

Let R be a ring with unit element 1, and let ϵ_{ij} be the matrix whose element in row i and column j is 1, and all of whose other elements are 0. Then

$$A = \sum_{i,j=1}^{n} a_{ij}\epsilon_{ij}$$

so that M may be thought of as a linear algebra over R with n^2 basis numbers ϵ_{ij} having the multiplication table

$$\epsilon_{ij}\epsilon_{jk} = \epsilon_{ik}, \qquad \epsilon_{ij}\epsilon_{lk} = 0 \qquad j \neq l.$$

In fact, the total matric algebra may be defined as such a linear algebra, and it may be proved that such an algebra can be represented by $n \times n$ arrays of numbers of R with the operations of addition, scalar multiplication and multiplication as we have defined them.

The total matric algebra M may contain a subalgebra of order less than n^2. Thus all linear combinations of the matrices

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

with coefficients in the real field form a subalgebra of order 2 of the total matric algebra of order 2^2. This algebra is isomorphic with the complex field under the correspondence

$$a + bi \leftrightarrow aI + bJ = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

This is an instance of the very general theory of matric representation which was glimpsed by Laguerre [5]. Subsequently it was proved by H. Poincaré [8] that every associative algebra of finite order n over a ring R with unit element can be represented as an algebra of $n \times n$ matrices.

3. Endomorphisms. It was remarked in the first section that Frobenius encountered matric algebra by way of bilinear form theory, and his early work demonstrates that matrices may be introduced in this way in a logically satisfactory manner. The approach is not altogether satisfactory psychologically, however, for the composition of bilinear forms, as Frobenius defined it, has something of the appearance of artificiality.

Another approach to the theory of matrices which has recently been gaining favor [**9, 10**] is somewhat sophisticated but sound both logically and psychologically. Not only is it obvious how the sum and product of two matrices must be defined, but their laws of combination are almost self evident.

A vector space [**11**] is a commutative group with an operator domain F which is a field. That is, the space V consists of elements $\alpha, \beta, \cdots$, which form a commutative group with respect to addition, and this group admits an operation, called scalar multiplication, such that for all operators (numbers of F) k and l

$$k\cdot(\alpha+\beta) = k\cdot\alpha + k\cdot\beta, \qquad (k+l)\cdot\alpha = k\cdot\alpha + l\cdot\alpha,$$
$$(kl)\cdot\alpha = k\cdot(l\cdot\alpha), \qquad 1\cdot\alpha = \alpha.$$

An endomorphism $\mathfrak{a}$ of a vector space V is a correspondence

$$\alpha \rightarrow \alpha', \ \beta \rightarrow \beta', \cdots$$

of the elements of V such that

$$\alpha + \beta \rightarrow \alpha' + \beta', \qquad k\cdot\alpha \rightarrow k\cdot\alpha', \qquad k \text{ in } F.$$

While every vector α corresponds to a unique vector α', it is not necessary that every vector α' be the correspondent of a vector α. Thus the correspondence

$$\alpha \rightarrow 0, \ \beta \rightarrow 0, \cdots$$

is an endomorphism, called the zero endomorphism.

The endomorphism $\mathfrak{a}$ may be thought of as being brought about by an operator, and from this point of view we write

$$\mathfrak{a}\alpha = \alpha', \ \mathfrak{a}\beta = \beta', \cdots.$$

If $\mathfrak{a}$ and $\mathfrak{b}$ are two such operators, the correspondence

$$\alpha \rightarrow \mathfrak{a}\alpha + \mathfrak{b}\alpha$$

may readily be shown to be an endomorphism, which we define to be the sum of the endomorphisms $\mathfrak{a}$ and $\mathfrak{b}$. Thus by definition

$$(\mathfrak{a} + \mathfrak{b})\alpha = \mathfrak{a}\alpha + \mathfrak{b}\alpha.$$

The set of all endomorphisms of V form a commutative group with respect to addition, the zero endomorphism being the identity.

If $\mathfrak{a}$ and $\mathfrak{b}$ are two endomorphisms, the correspondence

$$\alpha \rightarrow \mathfrak{a}(\mathfrak{b}\alpha)$$

may be shown to be an endomorphism. This is called the product of the two endomorphisms $\mathfrak{a}$ and $\mathfrak{b}$, so that by definition

$$(\mathfrak{a}\mathfrak{b})\alpha = \mathfrak{a}(\mathfrak{b}\alpha).$$

The identity endomorphism

$$\alpha \rightarrow \alpha,\ \beta \rightarrow \beta, \cdots$$

is the identity element of multiplication.

It can be proved very easily and naturally that multiplication of endomorphisms is associative, and distributive with respect to the addition of endomorphisms. Thus, let α be any vector of V, and $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ any three endomorphisms. Then by the definition of product

$$[\mathfrak{a}(\mathfrak{b}\mathfrak{c})]\alpha = \mathfrak{a}[(\mathfrak{b}\mathfrak{c})\alpha] = \mathfrak{a}[\mathfrak{b}(\mathfrak{c}\alpha)],$$
$$[(\mathfrak{a}\mathfrak{b})\mathfrak{c}]\alpha = (\mathfrak{a}\mathfrak{b})(\mathfrak{c}\alpha) = \mathfrak{a}[\mathfrak{b}(\mathfrak{c}\alpha)].$$

Since the endomorphisms $\mathfrak{a}(\mathfrak{b}\mathfrak{c})$ and $(\mathfrak{a}\mathfrak{b})\mathfrak{c}$ have the same effect on every vector α of V, they are equal.

It is now clear that the set of all endomorphisms of V is a ring with unit element. *A matrix over a field F may be defined as an endomorphism of a vector space V over F or, alternatively, as the operator which produces this endomorphism.*

For matrices of finite order, the equivalence of this definition with that of Section 2 is not hard to establish. Every vector space V of order n over the field F can be represented by n-tuples

$$\alpha = (a_1, a_2, \cdots, a_n), \qquad \beta = (b_1, b_2, \cdots, b_n), \cdots$$

of numbers of F. The relation $\alpha = \beta$ means that

$$a_i = b_i \qquad i = 1, 2, \cdots, n.$$

Addition and scalar multiplication are represented as follows:

$$\alpha + \beta = (a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n),$$
$$k \cdot \alpha = (ka_1, ka_2, \cdots, ka_n) \qquad k \text{ in } F.$$

The vectors

$$\epsilon_1 = (1, 0, \cdots, 0),\ \epsilon_2 = (0, 1, \cdots, 0), \cdots, \epsilon_n = (0, 0, \cdots, 1)$$

form a basis for V so that every vector α can be uniquely written

$$\alpha = a_1\epsilon_1 + a_2\epsilon_2 + \cdots + a_n\epsilon_n.$$

An endomorphism $\mathfrak{a}$ of V is uniquely determined by its effect upon the basis vectors. Let

$$\mathfrak{a}\epsilon_i = \sum_{j=1}^{n} a_{ij}\epsilon_j \qquad j = 1, 2, \cdots, n,$$

and consider the correspondence

$$\mathfrak{a} \rightarrow (a_{ij}).$$

If $\mathfrak{b}$ is another endomorphism, and if $\mathfrak{b} \rightarrow (b_{ij})$, then $\mathfrak{a} = \mathfrak{b}$ if and only if

$$a_{ij} = b_{ij}, \qquad i, j = 1, 2, \cdots, n.$$

If a_{ij} are any n^2 elements of F, (a_{ij}) determines an endomorphism, so that the correspondence $\mathfrak{a}\leftrightarrow(a_{ij})$ is biunique.

It is almost immediate that

$$\mathfrak{a}+\mathfrak{b}\leftrightarrow(a_{rs}+b_{rs}),\qquad \mathfrak{a}\times\mathfrak{b}\leftrightarrow(\sum b_{ri}a_{is})$$

so that, if we define

$$(a_{rs})+(b_{rs})=(a_{rs}+b_{rs}),\qquad (a_{rs})\times(b_{rs})=(\sum a_{ri}b_{is}),$$

the correspondence $\mathfrak{a}\leftrightarrow(a_{rs})$ is an isomorphism.

It has been remarked that the laws of combination of endomorphisms, such as the associative and distributive laws, are very easily and naturally provable. Since matrices (in the sense of Section 2) are isomorphic with endomorphisms, these same laws of combination must therefore hold for matrices.

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THE HEAVISIDE OPERATIONAL CALCULUS

H. B. CURRY, Frankford Arsenal*

1. Introduction. Operational methods for dealing with certain types of problems form a part of most college courses on formal solutions of differential equations. These methods go far back in the history of analysis; certain germs of ideas on this subject are found in Leibniz; and they were particularly interesting to the mathematicians of the middle of the nineteenth century.† Toward the close of that century Oliver Heaviside, in the course of his engineering investigations, extended these methods in a bold and independent manner. He did not always stop to justify his conclusions from the point of view of mathematical rigor; and indeed for him, since the answer to a practical problem can usually be checked by other means, such a justification was doubtless a second-

* This paper is the substance of an address delivered, by invitation, before the Philadelphia Section of the Association on November 28, 1942.

† For the history of the subject see Davis and Gardner (works cited in the bibliography at the end of the paper).

ary matter. The result is that Heaviside handed to mathematicians a problem, *viz.*, the theoretical systematization of his methods, so that they become not only an engine of empirical discovery, but an instrument of mathematical proof.

This problem has been essentially solved for at least ten years. In fact a distinguished colleague has stated, in conversation, that the Heaviside operational calculus is no longer of any mathematical interest; indeed further, that the best engineers have, even in their practice, completely replaced the Heaviside calculus with the theory of Laplace or Fourier transforms. Be that as it may, I know from the experience of the last six months that there are engineers who find the theory of integral transforms involves difficulties which they would prefer to avoid. Moreover there are serious engineering problems, *viz.*, those having to do with discrete mechanisms or networks, for which the difficulties of the integral transforms appear to be irrelevant. There is, consequently, some interest in a theory of the Heaviside calculus of a more elementary character.

Accordingly, the calculus is here discussed from a point of view which is essentially algebraic rather than analytical. This, of course, implies a restriction on the scope of the treatment, because it is limited to the rational aspects such as arise from ordinary linear differential equations with constant coefficients. For the more general cases of partial differential equations, fractional operators, *etc.*, the theory of integral transforms is doubtless unavoidable. However the treatment is sufficient for the engineering applications just alluded to. Moreover the theory treats with ease those cases of discrete systems which are commonly called "impulsive,"* such as electrical networks with a pure capacitance in some member; in these cases the theory may conceivably overcome difficulties which are not met by the theory of integral transforms. Presumably the two methods can be combined; but that question is not gone into here.

The discussion is, at least from one point of view, more elementary than that requiring infinite integrals. On the other hand, since the paper is addressed to mathematicians, I have not hesitated to use some of the simpler concepts of modern algebra. These give the work a certain abstractness and generality. Thus, although this subject has a bearing on the elementary teaching of differential equations, an exposition intelligible to college juniors is not attempted. However, the presentation is doubtless intelligible for college teachers; and, except for the concepts mentioned and the omission of certain proofs, it is intended to be self contained.

2. Generalities concerning linear operators. We consider a certain set $\mathfrak{L}$ of operators over a space $\mathfrak{A}$. *I.e.*, if L is in $\mathfrak{L}$ and A is in $\mathfrak{A}$, then $L[A]$, the result of applying the operator L to A, is a unique element of $\mathfrak{A}$. In the sequel the argument of an operator will be enclosed in square brackets; when this operand is itself a function, the argument of this function will be indicated by a subscript, thus: $L[f(t)]_t$. When the operator itself is designated by a complex expression, that expression will be enclosed in braces ("$\{$," "$\}$"); thus $\{L_1+L_2\}[A]$ is the

* The term "impulsive" in this connection is taken from Bush (see the bibliography).

result of applying the operator L_1+L_2 to A.* However braces or brackets will be omitted in certain cases where their explicit appearance seems superfluous.

Concerning the space $\mathfrak{A}$ we shall suppose that it is a linear vector space; *i.e.*, that it is an Abelian group with respect to addition, and that it admits multiplication by scalars (complex numbers), satisfying the usual postulates.

Under these circumstances the usual combinations of operators may be defined thus (the derived operators being defined in terms of their effects on an unspecified operand A):

$$\{L_1 + L_2\}[A] = L_1[A] + L_2[A]. \tag{1}$$

$$\{L_1L_2\}[A] = L_1[L_2[A]]. \tag{2}$$

We can also define multiplication of operators by scalars; but this is unnecessary if we suppose that our set $\mathfrak{L}$ contains, for every scalar c, the operation of multiplying by c. This operation will be denoted by "$\{c\}$," so that we have

$$\{c\}[A] = cA. \tag{3}$$

Ordinarily the braces will be omitted, so that the same symbol is to be interpreted as an operation or a scalar according to the context.

A *linear* operator is one which satisfies the following:

$$L[A_1 + A_2] = L[A_1] + L[A_2], \tag{4}$$

and for any scalar c

$$L[cA] = cL[A]. \tag{5}$$

THEOREM 1. *If $\mathfrak{L}$ is a set of linear operators which is closed under addition and multiplication, then $\mathfrak{L}$ is a ring.*

THEOREM 2. *If $\mathfrak{L}_0$ is a set of permutable linear operators, and $\mathfrak{L}$ is the ring generated by $\mathfrak{L}_0$,† then $\mathfrak{L}$ is a commutative ring.*

* The notation is a modification of that used by A. Church in his earlier work on mathematical logic (see "Sets of postulates for the foundation of logic," Annals of Mathematics, vol. 33, 1932, pp. 346–366). The notation is also useful in connection with ordinary functions symbolized by complex symbols, the argument in this case being enclosed in parentheses and the function in braces. Thus $\{L[f(t)]_t\}(t+h)$ means the result of substituting $t+h$ for t in $L[f(t)]_t$. This is, in general, distinct from $L[f(t+h)]_t$, which is the result of operating on $f(t+h)$ regarded as a function of t. In fact if

$$Lf(t) = \frac{f(t) - f(0)}{t} \quad \text{for} \quad t \neq 0$$
$$= f'(0) \qquad \text{for} \quad t = 0$$

and $f(t) = t^2$, then $\{L[f(t)]_t\}(t+h) = t+h$, $L[f(t+h)]_t = t+2h$. A considerable amount of work on symbolic operators would probably be made clearer by the systematic adoption of some such notation. The other notation of Church, whereby, if M is any expression containing t, $\lambda t.M$ is that function which assigns to any given t the value M, is also convenient; but it is not used here on account of its too great divergence from ordinary notations. For the present paper the notation explained in the text suffices.

† *I.e.*, the smallest ring which includes $\mathfrak{L}_0$.

The proofs of these theorems will not be given here. We may note, however, that in Theorem 1 the only law which requires that the operators be linear is the distributive law.

The interesting case of Theorem 2 is where $\mathfrak{A}$ is the set of infinitely differentiable functions of a real variable and $\mathfrak{L}_0$ consists of D (differentiation) together with $\{c\}$ for every scalar c. In this case $\mathfrak{L}$ is the ring of polynomials in D with scalar coefficients. The assertion of Theorem 2, then, is that these polynomials obey the rules of manipulation of ordinary algebra. This justifies the applications of symbolic methods made à la Boole in elementary courses in differential equations; likewise it has been shown that most of the algebraic part of the Heaviside theory, in particular his "expansion theorem," can be justified by this means.*

The essential content of the Heaviside theory, however, is that $\mathfrak{A}$ and $\mathfrak{L}$ can be so modified that the resulting set of operators forms a field. This has somewhat the same advantages as the introduction of negative numbers into algebra. I shall not elaborate on these advantages here, but shall proceed at once to derive sufficient conditions that a set of operators may constitute a field.

THEOREM 3. *If L is a linear operator such that for every B in $\mathfrak{A}$, the equation*

$$L[X] = B$$

is satisfied by one and only one X in $\mathfrak{A}$; then there exists a linear operator L^{-1} called the inverse of L such that the equations

$$B = L[A]$$
$$A = L^{-1}[B]$$

are equivalent.

Proof. Let $L^{-1}[B]$ be the unique solution of the equation

$$L[X] = B.$$

Then $L^{-1}[B]$ is defined for every B in $\mathfrak{A}$. To prove L^{-1} is linear, let $B=B_1+B_2$, $A_1=L^{-1}[B_1]$, $A_2=L^{-1}[B_2]$. Then

$$B_1 = L[A_1], \qquad B_2 = L[A_2],$$
$$B = L[A_1] + L[A_2] = L[A_1 + A_2],$$
$$A_1 + A_2 = L^{-1}[B_1] + L^{-1}[B_2] = L^{-1}[B_1 + B_2].$$

Again, let $B=cB_1$, $A_1=L^{-1}[B_1]$, $A=cA_1$. Then, since L is linear,

$$L[A] = cL[A_1] = cB_1 = B.$$

Therefore

$$A = L^{-1}[cB_1] = cL^{-1}[B_1], \text{ q.e.d.}$$

* For references see footnote[214] in Doetsch.

COROLLARY. *The inverse has the following properties:*

$$LL^{-1} = L^{-1}L = I, \tag{6}$$

where I is the identity operator defined by

$$I[A] = A; \tag{7}$$

also

$$(L^{-1})^{-1} = L. \tag{8}$$

Proof. The first of these is clear. The proof of the theorem shows that L^{-1} is linear; further it has a unique inverse, since the equation $L^{-1}[B]=A$ is equivalent to $L[A]=B$. The rest follows by the symmetry of conditions on L and L^{-1}.

THEOREM 4. *If L_1 and L_2 are operators with unique inverses, then L_1L_2 has a unique inverse, viz.,*

$$(L_1L_2)^{-1} = L_2^{-1}L_1^{-1}; \tag{10}$$

furthermore, if L_1, L_2 are permutable, then any two of the four operators L_1, L_2, L_1^{-1}, L_2^{-1} are permutable.

Proof. By the usual arguments we can show that if L_1 and L_2 have unique inverses, then L_1L_2 has a unique inverse, *viz.*, $L_2^{-1}L_1^{-1}$. Hence, if L_1 and L_2 are permutable,

$$L_1^{-1}L_2^{-1} = (L_2L_1)^{-1} = (L_1L_2)^{-1} = L_2^{-1}L_1^{-1}.$$

To prove the theorem it remains to show that

$$L_1L_2^{-1} = L_2^{-1}L_1, \qquad L_2L_1^{-1} = L_1^{-1}L_2.$$

From symmetry it is sufficient to prove the first of these. This we do as follows:

$$L_1L_2^{-1} = L_2^{-1}L_2L_1L_2^{-1} = L_2^{-1}L_1L_2L_2^{-1} = L_2^{-1}L_1.$$

THEOREM 5. *If p is a linear operator such that for every scalar c the operator $\{p-c\}$ has a unique inverse; then the set $\mathfrak{R}_0$, consisting of $\{p-c\}$ and $\{p-c\}^{-1}$ for all scalars c, generates a field.**

Proof. It is clear that for any scalars c_1 and c_2 the operators $\{p-c_1\}$ and $\{p-c_2\}$ are permutable. Hence, by theorems 2 and 4, $\mathfrak{R}_0$ generates a commutative ring $\mathfrak{R}$. This ring evidently contains a unit element, *viz.*, the identity operator pp^{-1}. It remains to show that every operator in $\mathfrak{R}$ has also an inverse in $\mathfrak{R}$.

If $\phi(p)$ is any polynomial (over the field of complex numbers) $\phi(p)$ can be factored into linear factors thus

$$\phi(p) = c_0(p - c_1)(p - c_2) \cdots (\phi - c_n).$$

Hence $\phi(p)$ has an inverse by Theorem 4. We show by induction that every operator in $\mathfrak{R}$ is of the form $\{\phi(p)\}\{\psi(p)\}^{-1}$ and has the inverse $\{\psi(p)\}\{\phi(p)\}^{-1}$.

* *I.e.*, the ring generated by $\mathfrak{R}_0$ is a field.

3. The system $\mathfrak{H}$. We proceed to specialize the space $\mathfrak{A}$ and the operator p.

The first specialization suggested is that where $\mathfrak{A}$ is the class of functions $f(t)$ of class C^∞ for all real t, and p is D (*i.e.*, differentiation). In this case, however, p does not satisfy the conditions on $\mathfrak{R}$ in Theorem 3 unless we restrict $\mathfrak{A}$ so that, for example, $f(t)$ is in $\mathfrak{A}$ only when $f(0)=0$.* But when this is done p is, in general, no longer an operator on $\mathfrak{A}$ to $\mathfrak{A}$. In fact we do not get an interpretation of the theorems of §2 in the way suggested unless we restrict $\mathfrak{A}$ so that every $f(t)$ in $\mathfrak{A}$ vanishes at 0 together with all its derivatives.†

The specialization adopted here is the following: Let $\mathfrak{F}$ be the class of functions $f(t)$ of class C^∞. Let $\eta_0, \eta_1, \cdots$ be an infinite set of indeterminates like the units of a hypercomplex algebra. Let $\mathfrak{A}$ be the set of all finite sums of the form

$$A = a(t)\eta_0 + a_1\eta_1 + \cdots + a_m\eta_m, \tag{11}$$

where $a(t)$ is in $\mathfrak{F}$ and $a_1, a_2, \cdots, a_m$ are scalars. The number m may vary with A.‡ Finally let p be defined by the requirement that, for A as in (11),

$$p[A] = a'(t)\eta_0 + a_0\eta_1 + \cdots + a_m\eta_{m+1}, \tag{12}$$

where $a_0 = a(0)$.

In this case it is clear that $\mathfrak{A}$ is a linear space and p an operator on $\mathfrak{A}$ to $\mathfrak{A}$. That the hypotheses of Theorem 5 are satisfied will be shown as follows:

---

* The equation $D[x(t)]_t = a(t)$ always has the solution

$$x(t) = \int_0^t a(\tau)d\tau;$$

but it also has other solutions unless these are excluded from $\mathfrak{A}$ by some such restriction as that indicated.

† Hazeltine (see the bibliography) proposes what amounts to considering the set of functions having these properties at $t=-\infty$. However, for physical applications we are interested in functions with discontinuities at $t=0$, and for these the formalism considered below has advantages.

‡ Since it is not convenient to impose the condition $a_m \neq 0$, m is not even uniquely determined by A.

The formulation in the text is, of course, equivalent to the following: Let $\mathfrak{A}$ be the class of all infinite sequences A of the form

$$A = [a(t), a_1, a_2, \cdots],$$

where $a(t)$ is in $\mathfrak{F}$, the a_i are constants, and not more than a finite number of the a_i are different from 0. Sums and products by scalars are defined thus:

$$[a(t), a_1, a_2, \cdots] + [b(t), b_1, b_2, \cdots] = [a(t) + b(t), a_1 + b_1, a_2 + b_2, \cdots]$$
$$c[a(t), a_1, a_2, \cdots] = [ca(t), ca_1, ca_2, \cdots].$$

Then the indeterminates η_i are defined thus

$$\eta_0 = [1, 0, 0, \cdots]$$
$$\eta_1 = [0, 1, 0, 0, \cdots]$$
$$\eta_2 = [0, 0, 1, 0, 0 \cdots], \quad \text{etc.},$$

and it is a theorem that every A and $\mathfrak{A}$ has the form (11) if we add the special definition

$$a(t)\eta_0 = [a(t), 0, 0, \cdots].$$

(A more general definition of $b(t)A$, for $b(t)$ in $\mathfrak{F}$ and A in $\mathfrak{H}$, is not needed in this paper.)

THEOREM 6. *If A and p are defined as above, then for every scalar c the equation*

$$\{p - c\}[X] = A \tag{13}$$

has a unique solution.

Proof. Let A be given as above and suppose that

$$X = x(t)\eta_0 + x_1\eta_1 + \cdots + x_n\eta_n.$$

Then

$$\begin{aligned} p[X] &= x'(t)\eta_0 + x(0)\eta_1 + x_1\eta_2 + \cdots + x_n\eta_{n+1}, \\ \{p - c\}[X] &= (x'(t) - cx(t))\eta_0 + (x(0) - cx_1)\eta_1 + (x_1 - cx_2)\eta_2 + \cdots \\ &\quad + (x_{n-1} - cx_n)\eta_n + x_n\eta_{n+1}. \end{aligned}$$

We may suppose without loss of generality that $n > m$ (since we have not assumed $x_n \neq 0$). Then in order to satisfy (13) it is necessary and sufficient that

$$\begin{aligned} x'(t) - cx(t) &= a(t), \\ x(0) - cx_1 &= a_1, \\ x_1 - cx_2 &= a_2, \\ \cdots&\cdots \\ x_{m-1} - cx_m &= a_m, \\ x_m - cx_{m+1} &= 0, \\ \cdots&\cdots \\ x_{n-1} - cx_n &= 0, \\ x_n &= 0. \end{aligned}$$

These equations have a unique solution. Working in inverse order we have

$$\begin{aligned} x_n &= 0, \\ x_{n-1} &= 0, \\ \cdots& \\ x_m &= 0, \\ x_{m-1} &= a_m, \\ x_{m-2} &= a_{m-1} + ca_m, \\ \cdots&\cdots \\ x_{m-k} &= a_{m-k+1} + ca_{m-k} + c^2a_{m-k-1} + \cdots + c^{k-1}a_m, \\ x(0) &= a_1 + ca_2 + \cdots + c^{m-1}a_m, \\ x(t) &= x(0)e^{ct} + \int_0^t e^{c(t-\tau)}a(\tau)d\tau. \end{aligned} \tag{14}$$

This is evidently a solution, and the only one. For $c=0$ we have

$$\frac{1}{p}[A] = \left(a_1 + \int_0^t a(\tau)d\tau\right)\eta_0 + a_2\eta_1 + \cdots + a_m\eta_{m-1}.$$

This completes the proof of Theorem 6. It then follows, by the argument of §2, that the set $\mathfrak{H}$ of operators of the form $\{\phi(p)\}\{\psi(p)\}^{-1}$ constitutes a field.

In interpretation η_0 may be thought of as the Heaviside unit step function, η_1 as the unit impulse, and $\eta_2, \eta_3, \cdots$ as impulses of higher order. However other interpretations are possible; in particular it is not necessary that $a(t)\eta_0$ be thought of as a function which is 0 for all $t<0$.

Let us now consider two theorems concerning the system $\mathfrak{H}$.

The first of these has to do with the concept of *rank*, which is defined as follows: Let A be given by (11) and let

$$a_0 = a(0),$$
$$a_{-k} = a^{(k)}(0), \qquad k = 1, 2, 3, \cdots;$$

then the rank of A is the greatest r such that $a_r \neq 0$.* We then have

THEOREM 7. *If A is of rank r, then for any scalar c, $\{p-c\}[A]$ is of rank $r+1$.*

The final theorem concerns a connection between the system $\mathfrak{H}$ and ordinary differential operators. Let us say that an element A of $\mathfrak{A}$ is *congruent* to a function $f(t)$ of $\mathfrak{F}$, in symbols

$$A \cong f(t), \tag{15}$$

if and only if

$$a(t) = f(t),$$

where A is given by (11). Then we have the following:

THEOREM 8. *If $\phi(p)$ is a polynomial in p, and*

$$A \cong f(t);$$

then

$$\{\phi(p)\}[A] \cong \phi(D)\}[f(t)]_t. \tag{16}$$

The proof of this is merely a matter of formal manipulation.

4. Application to differential equations. In the following bold face letters will be used to denote matrices of n rows and either n columns or one column according to the context. The presentation of the applications will be informal, and proofs will generally be omitted.

* Note that the rank can be any integer positive, negative, or zero. If no r satisfying the conditions exists the rank is not defined.

Let us consider a system of differential equations of the form

$$\{\mathbf{Z}(D)\}[\mathbf{x}(t)]_t = \mathbf{g}(t), \tag{17}$$

where $\mathbf{Z}(D)$ is a matrix polynomial in D of degree q with constant coefficients. If $\mathbf{x}(t)$ is any solution of (17) and if

$$\mathbf{X} = \mathbf{x}(t)\eta_0 + \mathbf{x}_1\eta_1 + \cdots + \mathbf{x}_r\eta_r; \tag{18}$$

then by Theorem 8

$$\{\mathbf{Z}(p)\}[\mathbf{X}] \cong \mathbf{g}(t),$$

and hence $\mathbf{X}$ is a solution of the equation

$$\{\mathbf{Z}(p)\}[\mathbf{X}] = \mathbf{G}, \tag{19}$$

where

$$\mathbf{G} = \mathbf{g}(t)\eta_0 + \mathbf{g}_1\eta_1 + \cdots + \mathbf{g}_s\eta_s. \tag{20}$$

$\mathbf{g}_1, \mathbf{g}_2, \cdots$ have values determined by $\mathbf{x}$.* Conversely suppose $\mathbf{g}_1, \mathbf{g}_2, \cdots$ are given arbitrary values and that $\mathbf{X}$ is a solution of (19); then $\mathbf{x}(t)$ is a solution of (17). However the equation (19) always has a unique solution provided only that the determinant of $\mathbf{Z}(p)$ is not zero; this solution is given by

$$\mathbf{X} = \{(\mathbf{Z}(p))^{-1}\}[\mathbf{G}],$$

where $(\mathbf{Z}(p))^{-1}$ is the inverse matrix of $\mathbf{Z}(p)$. It should be noted that this argument gives a rigorous proof of the existence of a solution of (17) under these circumstances.

Let us suppose now that

$$Z(D) = \mathbf{z}_0 + \mathbf{z}_1 D + \mathbf{z}_2 D^2 + \cdots + \mathbf{z}_q D^q \tag{21}$$

and that the determinant of $\mathbf{z}_q$ does not vanish. Such systems will be called *non-impulsive.*† In that case every element of the matrix $(\mathbf{Z}(p))^{-1}$ will be a rational function of p whose denominator is of degree nq and whose numerator is at most of degree $(n-1)q$. It follows, by reference to the argument of Theorem 7, that $(\mathbf{Z}(p))^{-1}$ decreases the rank of its argument by at least q units. Hence if $\mathbf{g}_1 = \mathbf{g}_2 = \cdots = 0$, $\mathbf{G}$ is at most of rank 0 and $\mathbf{X}$ is at most of rank $-q$. Therefore, the $\mathbf{x}(t)$ determined in this way is precisely the solution of (17) such that

$$\mathbf{x}(0) = \mathbf{x}'(0) = \cdots = \mathbf{x}^{(q-1)}(0) = \mathbf{0}. \tag{22}$$

It may be shown that the solution such that

$$\mathbf{x}(0) = \mathbf{c}_0,\ \mathbf{x}'(0) = \mathbf{c}_1, \cdots, \mathbf{x}^{(q-1)}(0) = \mathbf{c}_{q-1} \tag{23}$$

is obtained by putting

* The explicit formulas are (38), given in the appendix.

† In this case the equations may be solved for the highest derivatives.

$$
\begin{aligned}
\mathbf{g}_1 &= \mathbf{z}_1\mathbf{c}_0 + \mathbf{z}_2\mathbf{c}_1 + \cdots + \mathbf{z}_q\mathbf{c}_{q-1},\\
\mathbf{g}_2 &= \mathbf{z}_2\mathbf{c}_0 + \mathbf{z}_3\mathbf{c}_1 + \cdots + \mathbf{z}_q\mathbf{c}_{q-2},\\
&\cdots\cdots\cdots\cdots\cdots\\
\mathbf{g}_{q-1} &= \mathbf{z}_{q-1}\mathbf{c}_0 + \mathbf{z}_q\mathbf{c}_1,\\
\mathbf{g}_q &= \mathbf{z}_q\mathbf{c}_0.
\end{aligned}
\tag{24}
$$

The effect of adding additional impulsive terms to $\mathbf{G}$ is merely to add impulsive terms to $\mathbf{X}$.*

In the impulsive case, where the determinant of $\mathbf{z}_q$ is zero, the situation is more complicated. Since certain of the derivatives can be eliminated, the initial conditions (23) cannot be given arbitrarily. In extreme cases $\{\mathbf{Z}(p)\}^{-1}$ may actually increase the rank of its argument, so that the output ($\mathbf{X}$) may be impulsive (rank >0) even when the input ($\mathbf{G}$) is not. Such a phenomenon occurs in the case of a condenser, where

$$
\begin{aligned}
\dot{q}(t) - i(t) &= 0\\
q(t) &= ce(t).\dagger
\end{aligned}
\tag{25}
$$

Nevertheless in all these cases the argument of the second preceding paragraph shows that if (17) has a solution at all that solution is obtainable from (19). Moreover the interpretation of the η_i, given in §3, suggests that the solution for $\mathbf{g}_1=\mathbf{g}_2=\cdots=0$ gives the output when the input is suddenly applied to a system initially undisturbed (there being in that case certain discontinuities at $t=0$).‡

The practical determination of the solution of (19) can be accomplished by the use of (12) and (14) if the determinant of $\mathbf{Z}(p)$ can be factored into linear factors.§ This process may be shortened by developing the elements of the matrix $(\mathbf{Z}(p))^{-1}$ in partial fractions. The "Heaviside expansion theorem" is

---

* For a somewhat fuller statement see the appendix below.

† These equations form an example of the preceding theory for the following specializations of $\mathbf{x}(t)$, $\mathbf{Z}(D)$, $\mathbf{g}(H)$:

$$
\mathbf{x}(t) = \begin{bmatrix} q(t) \\ i(t) \end{bmatrix}
$$

$$
\mathbf{Z}(D) = \begin{bmatrix} D & -1 \\ 1 & 0 \end{bmatrix}
$$

$$
\mathbf{g}(t) = \begin{bmatrix} 0 \\ ce(t) \end{bmatrix}.
$$

Of course the example is trivial in that the second equation is already solved for $q(t)$ and the first gives by inspection

$$
i(t) = \dot{q}(t) = c\dot{e}(t)
$$

‡ For further discussion of the impulsive case see the appendix.

§ There are, of course, other methods, such as expansions in series, which are not treated here.

merely a formula expressing the result of the process in a rather special case.*

5. The superposition theorem. Up to the present we have made no assumption in regard to the input. In the usual treatment of the Heaviside calculus it is traditional to solve the problem first for a special case, obtaining the so called "indicial admittance," and then to express the general solution by means of an integral. Although it is not clear that this detour is always advantageous, yet there is sufficient interest in it to justify considering the form which this theorem takes in the most general impulsive case. On this as a basis one can then proceed to establish contact with the Carson integral equation and so to infinite integral forms of the Heaviside theory.

In the present matrix notation the *indicial admittance* is to be defined as the matrix $\mathbf{A}$ such that

$$\{\mathbf{Z}(p)\}[\mathbf{A}] = \mathbf{i}\eta_0$$

where $\mathbf{i}$ is the unit matrix. If this indicial admittance is given by

$$\mathbf{A} = \mathbf{a}(t)\eta_0 + \mathbf{a}_1\eta_1 + \cdots + \mathbf{a}_m\eta_m,$$

then it may be shown that

$$(\mathbf{Z}(p))^{-1} = \mathbf{B}_0(p) + \mathbf{a}_0 + \mathbf{a}_1 p + \cdots + \mathbf{a}_m p^m, \tag{26}$$

where $\mathbf{a}_0 = \mathbf{a}(0)$ and $\mathbf{B}_0(p)$ is an operation which decreases the rank. Again, if $\mathbf{X}$ is given by (19), it may also be shown that

$$\begin{aligned} \mathbf{X} = &\left(\mathbf{a}(t)\mathbf{g}(0) + \int_0^t \mathbf{a}(t-\tau)\mathbf{g}'(\tau)d\tau\right)\eta_0 \\ &+ \mathbf{a}_1 p[\mathbf{g}(t)\eta_0] + \mathbf{a}_2 p^2[\mathbf{g}(t)\eta_0] + \cdots \\ &+ \mathbf{a}_m p^m[\mathbf{g}(t)\eta_0] + p[\mathbf{A}]\mathbf{g}_1 + p^2[\mathbf{A}]\mathbf{g}_2 + \cdots + p^s[\mathbf{A}]\mathbf{g}_s. \end{aligned} \tag{27}$$

This is the principle of superposition for the most general impulsive system. The result is valid if $\mathbf{Z}(p)$ is any matrix rational in p.

---

* The theory of the general linear homogeneous differential system with constant coefficients, *viz.*,

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{a}\mathbf{x}(t),$$

where $\mathbf{a}$ is a constant matrix, can be based on the foregoing discussion for the case $\mathbf{G} = \mathbf{c}\eta_1$. Such a treatment has many advantages over the orthodox methods. In fact the latter involve, for the treatment of the general case, formal difficulties as advanced as those of the theory of elementary divisors. But the most general case, including all existence and uniqueness theorems, can be handled by the techniques mentioned in the text together with the formula

$$\{p - c\}^{-k}[\eta_1] = \frac{t^{k-1}}{(k-1)!}e^{ct}\eta_0,$$

which follows from (14). This supports the contention that the Heaviside calculus can be a tool of theoretical, as well as practical, investigation.

To proceed from here it is advisable to transform the integral by integration by parts and change of the variable from τ to $t-\tau$. Since the impulsive terms will not further concern us, we may express the result in the form of a congruence, thus

$$\mathbf{X} \cong \int_0^t \mathbf{a}'(\tau)\mathbf{g}(t-\tau)d\tau + \sum_{i=0}^{m} \mathbf{a}_i \mathbf{g}^{(i)}(t) + \sum_{j=1}^{s} \mathbf{a}^{(j)}(t)\mathbf{g}_j. \tag{28}$$

Next we take the special case

$$G = \mathbf{g}_0 e^{\lambda t}\eta_0. \tag{29}$$

Then the above formula becomes

$$\mathbf{X} \cong \left(\int_0^t \mathbf{a}'(\tau)e^{-\lambda\tau}d\tau + \sum_{i=0}^{m} \mathbf{a}_i\lambda^i\right)\mathbf{g}_0 e^{\lambda t}. \tag{30}$$

If now there exists a constant a_∞ and a real λ_0 such that, as $t\to\infty$

$$\mathbf{a}(t) - \mathbf{a}_\infty \in O(e^{\lambda_0 t}) \tag{31}$$

then for the real part of $\lambda > \lambda_0$, $\mathbf{x}(t)$ is asymptotic as $t\to\infty$ to

$$\mathbf{y}(\lambda)\mathbf{g}_0 e^{\lambda t}$$

where

$$\begin{aligned} \mathbf{y}(\lambda) &= \int_0^\infty \mathbf{a}'(\tau)e^{-\lambda\tau}d\tau + \sum_{t=0}^{m} \mathbf{a}_i\lambda^i \\ &= \lambda\int_0^\infty (\mathbf{a}(\tau) - \mathbf{a}_\infty)e^{-\lambda\tau}d\tau + \mathbf{a}_\infty + \sum_{i=1}^{m} \mathbf{a}_i\lambda^i. \end{aligned} \tag{32}$$

This situation will be expressed by the notation

$$\mathbf{X} \sim \mathbf{y}(\lambda)\mathbf{g}_0 e^{\lambda t}, \tag{33}$$

where "$\sim$" is to be read "is asymptotically congruent to."

To get another asymptotic formula we must use a generalization of Theorem 8, which, since it concerns the underlying system $\mathfrak{H}$, will be stated as follows:

THEOREM 9. *If $\phi(p)$, $\psi(p)$ are polynomials in p, A is in $\mathfrak{A}$ and is such that*

$$A \cong g(t),$$

and if further $f(t)$ is such that

$$\{\phi(D)\}[g(t)]_t = \psi(D)[f(t)]_t;$$

then

$$\frac{\phi(p)}{\psi(p)}[A] \cong f(t) + h(t),$$

where

$$\{\psi(D)\}[h(t)]_t = 0.$$

The proof of this theorem is again a matter of formal manipulation. In the case where $g(t)=g_0e^{\lambda t}$, where $\psi(\lambda)\neq 0$, the hypotheses are fulfilled for $f(t)=g_0e^{\lambda t}\phi(\lambda)/\psi(\lambda)$. If furthermore the roots of $\phi(\lambda)$ all have negative real parts, $h(t)\to 0$ as $t\to\infty$ and

$$\frac{\phi(p)}{\psi(p)}[A] \sim \frac{\phi(\lambda)}{\psi(\lambda)} g_0e^{\lambda t}.$$

The preceding argument holds when $\phi(p)$, A, $g(t)$, $f(t)$ (but not $\psi(p)$) are matrices. It can therefore be applied in the case where $\phi(p)/\psi(p)$ is $\{\mathbf{Z}(p)\}^{-1}$ and A is the $\mathbf{G}$ of (19). The restriction on the roots of $\psi(\lambda)$ then corresponds to the requirement that the system (17) be stable. In that case we have, if $\psi(\lambda)\neq 0$,

$$\mathbf{X} \sim [\mathbf{Z}(\lambda)]^{-1}\mathbf{g}_0e^{\lambda t}. \tag{34}$$

If, now, we compare (33) and (34) we have

$$\lim_{t\to\infty} (\mathbf{y}(\lambda) - [\mathbf{Z}(\lambda)]^{-1})e^{\lambda t} = 0;$$

and hence, if the real part of λ is non-negative,

$$\mathbf{y}(\lambda) = [\mathbf{Z}(\lambda)]^{-1}. \tag{35}$$

In view of (32) this is the analogue of the Carson integral equation. If $\mathbf{Z}(p)$ is a polynomial in p then (31) is a consequence of the other by hypotheses made. Since $\mathbf{y}(0)=(\mathbf{Z}(0))^{-1}=\mathbf{a}_\infty$ by (32), the result may be written

$$[\mathbf{Z}(\lambda)]^{-1} - [\mathbf{Z}(0)]^{-1} = \lambda\int_0^\infty (\mathbf{a}(\tau) - \mathbf{a}_\infty)e^{-\lambda\tau}d\tau + \sum_{i=1}^{m} \mathbf{a}_i\lambda^i. \tag{36}$$

Appendix*

Note on the impulsive case. Inasmuch as Bromwich was unable to solve the general impulsive case, it may be worth while to state some additional facts concerning it. In this statement it is assumed that $\mathbf{Z}(D)$ is a matrix polynomial, as given by (21). Furthermore it will be assumed, without further mention, that we are dealing with solutions of (19) with preassigned $\mathbf{g}(t)$ and that a "solution" is such as to satisfy (17).

First we shall review some of the facts considered in §4. If we define

$$\begin{aligned} \mathbf{x}_0 &= \mathbf{x}(0) \\ \mathbf{x}_{-k} &= \mathbf{x}^{(k)}(0), \qquad k = 1, 2, \cdots, q-1, \\ \mathbf{z}_k &= \mathbf{0} \quad \text{for} \quad k < 0; \end{aligned} \tag{37}$$

then by direct calculation from (19) we obtain

$$\mathbf{g}_j = \sum_{k=j-r}^{q} \mathbf{z}_k\mathbf{x}_{j-k}. \tag{38}$$

* Not in the address as delivered.

Let us call the right side of (38) $\mathbf{g}_j(\mathbf{x})$, where $\mathbf{x}$ may be thought of as the three dimensional matrix whose plane sections are the $\mathbf{x}_i$. Then a necessary condition that

$$\mathbf{x}_i = \mathbf{c}_i, \qquad i = 1 - q,\ 2 - q, \cdots \tag{39}$$

is that

$$\mathbf{g}_j(\mathbf{x}) = \mathbf{g}_j(\mathbf{c}). \tag{40}$$

In the non-impulsive case this is also sufficient, because we can show by induction, starting with the largest j, that (40) has the unique solution (39). Since the solution of (1) is unique this sufficiency holds also in the general case, provided only that a solution of (19) satisfying (39) exists. It remains to find what condition must be imposed on the $\mathbf{c}_i$ in order that this should be so.

To investigate this we can suppose that $q=1$, since the general case can be reduced to this one by increasing n. Then by eliminating certain components of $p\mathbf{X}$ it can be shown that (19) is equivalent to a system consisting of ν equations $(0 \leqq \nu < n)$ forming a non impulsive sub-system in ν unknowns (which we shall take to be the first ν), together with $n-\nu$ algebraic equations determining the remaining unknowns uniquely in terms of the first ν and the inputs. It can be shown further that a necessary and sufficient condition that a solution of (19) satisfying (39) exist is that the last $n-\nu$ components of $\mathbf{c}_0$ have the values determined by putting $t=0$ in the last $n-\nu$ equations; the $\mathbf{c}_i$ for $i>0$ and the first ν components of $\mathbf{c}_0$ being arbitrary.

It should be noted that the solution for all $\mathbf{g}_j=0$ $(j>0)$ does not in general correspond to the case where all the arbitrary components of the $\mathbf{c}_i$ are zero. The explanation of this is that the above elimination may require operating on the right sides of the equations with operators containing p, and hence the new inputs may have impulsive terms when there were none in the originals and vice versa.

The following example illustrates the phenomenon: Consider the system

$$\begin{aligned} \dot{x}_1(t) - \dot{x}_2(t) &= g_1(t), \\ x_2(t) - \dot{x}_3(t) &= g_2(t). \\ x_3(t) &= g_3(t). \end{aligned}$$

The corresponding Heaviside system

$$\begin{aligned} pX_1 - pX_2 &= G_1, \\ X_2 - pX_3 &= G_2, \\ X_3 &= G_3. \end{aligned}$$

has the solution

$$X_1 = \frac{1}{p}G_1 + G_2 + pG_3,$$

$$X_2 = G_2 + pG_3,$$
$$X_3 = G_3.$$

If $G_i = g_i(t)\eta_0 + g_{i1}\eta_1$, then

$$X_1 = \left(\int_0^t g_1(\tau)d\tau + g_2(t) + \dot{g}_3(t) + g_{11}\right)\eta_0 + (g_3(0) + g_{21})\eta_1 + g_{31}\eta_2,$$
$$X_2 = (g_2(t) + \dot{g}_3(t))\eta_0 + (g_{21} + g_3(0))\eta_1 + g_{31}\eta_2,$$
$$X_3 = g_3(t)\eta_0 + g_{31}\eta_1.$$

If now $g_{11} = g_{21} = g_{31} = 0$, X_1 and X_2 have the impulsive term $g_3(0)\eta_1$; if we take $g_{11} = g_2(0) - \dot{g}_3(0)$, $g_{21} = -g_3(0)$, $g_{31} = 0$, then X is nonimpulsive and $x_1(0) = 0$. Evidently $x_2(0) = g_2(0) + \dot{g}_3(0)$ in any case.

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The following contains a list of items consulted in connection with the above paper. It is not intended to be exhaustive. Several important papers were not available, on account of war conditions, during the writing of this article (see the bibliographies in works by von Karman and Pipes). A few works not bearing directly on the Heaviside calculus are cited in the footnotes, but are not listed here.

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DISCUSSIONS AND NOTES

Edited by MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The department of Discussions and Notes is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ANALOGS

A. K. WALTZ, Clarkson College

As has been the case with many discoveries, Fourier series originated, indirectly, from the study of another topic. Viewed directly from a mathematical standpoint, one would expect series of the form,

$$f(x) = \sum_{n=0}^{\infty} a_n(\sin x)^n, \tag{1}$$

$$f(x) = \sum_{n=0}^{\infty} b_n(\cos x)^n, \tag{2}$$

to find their way into text books before the Fourier series since they are the analogs, in trigonometric form, of the Maclaurin expansion. Furthermore, the determination of the coefficients is not difficult, although somewhat tedious.

The coefficients a_n in (1) depend upon the successive derivatives of $(\sin x)^n$, $m > n-1$. These derivative may take any of the following forms:

$$D^m(\sin x)^n = n(\sin x)^{n-m}\phi(\cos x), \qquad m < n,$$

$$D^m(\sin x)^n = n\phi(\cos x), \qquad (m = n + 2p,\ p = 0, 1, 2, \cdots),$$

$$D^m(\sin x)^n = n \sin x\phi(\cos x), \qquad (m = n + 2p + 1,\ p = 0, 1, 2, \cdots),$$

where $\phi(\cos x)$ is a polynomial in $\cos x$ of degree, respectively, m, n, and $n-1$.

When expanded about the origin, $x=0$, the first non-vanishing derivative is then seen to be of order $m=n$. For $m>n$ the derivatives vanish alternately, beginning with $m=n+1$. As a consequence of these statements the following recursion formulae may be written:

$$a_{2k}D^{2k}(\sin x)_0^{2k} = f^{(2k)}(0) - \sum_{p=0}^{k-2} a_{2p+2}D^{2k}(\sin x)_0^{2p+2}, \qquad n \text{ even},$$

$$a_{2k+1}D^{2k+1}(\sin x)_0^{2k+1} = f^{(2k+1)}(0) - \sum_{p=0}^{k-1} a_{2p+1}D^{2k+1}(\sin x)_0^{2p+1}, \qquad n \text{ odd}.$$

Except for a_0, which is determined from the original equation, these formulae determine the a_n. To complete the expansion the usual methods of convergence must be applied.

Replacing $\sin x$ by $\cos x$ and expanding about $x=\pi/2$, similar formulae may be written to determine b_n in (2).

When n is even the recursion formula for expansion (1) may be replaced by

$$a_{2k}(2k)! = b_2 f''(0) + b_4 f^{\mathrm{IV}}(0) + \cdots + b_{2k-2} f^{(2k-2)}(0) + f^{(2k)}(0),$$

where the b_{2i} are integers satisfying the equations:

$$\begin{aligned}
&-2^2 b_2 + 2^4 b_4 - 2^6 b_6 + \cdots + (-1)^{k-1} 2^{2k-2} b_{2k-2} + (-1)^k 2^{2k} = 0,\\
&-4^2 b_2 + 4^4 b_4 - 4^6 b_6 + \cdots + (-1)^{k-1} 4^{2k-2} b_{2k-2} + (-1)^k 4^{2k} = 0,\\
&-6^2 b_2 + 6^4 b_4 - 6^6 b_6 + \cdots + (-1)^{k-1} 6^{2k-2} b_{2k-2} + (-1)^k 6^{2k} = 0,\\
&\quad\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots\\
&-(2k-2)^2 b_2 + (2k-2)^4 b_4 + \cdots + (-1)^{k-1}(2k-2)^{2k-2} b_{2k-2}\\
&\qquad\qquad\qquad\qquad + (-1)^k (2k-2)^{2k} = 0.
\end{aligned}$$

These equations are due to the fact that a_{2k} vanishes for each of the expansions, $\cos(2k-2)x$, $\cos(2k-4)x$, $\cdots$, $\cos 2x$, into even powers of $\sin x$.

Likewise when n is odd

$$a_{2k+1}(2k+1)! = c_1 f'(0) + c_3 f'''(0) + \cdots + c_{2k-1} f^{(2k-1)}(0) + f^{(2k+1)}(0),$$

where the c_{2i-1} are integers satisfying similar equations with even base integers and even exponents replaced by odd base integers and odd exponents.

The coefficients as far as a_8 are:

$$\begin{aligned}
&a_0 = f(0), \qquad a_1 = f'(0), \qquad 2!a_2 = f''(0), \qquad 3!a_3 = f'(0) + f'''(0),\\
&4!a_4 = 4f''(0) + f^{\mathrm{IV}}(0), \qquad 5!a_5 = 9f'(0) + 10f'''(0) + f^{\mathrm{V}}(0),\\
&6!a_6 = 64f''(0) + 20f^{\mathrm{IV}}(0) + f^{\mathrm{VI}}(0),\\
&7!a_7 = 225f'(0) + 259f'''(0) + 35f^{\mathrm{V}}(0) + f^{\mathrm{VII}}(0),\\
&8!a_8 = 2304f''(0) + 784f^{\mathrm{IV}}(0) + 56f^{\mathrm{VI}}(0) + f^{\mathrm{VIII}}(0).
\end{aligned}$$

Applications of these series, as outstanding as those for the Fourier series, remain to be found. However, there are some important cases where they could be used instead of other methods. If in (1) and (2), respectively, we take $f(x)=\sin kx$ and $f(x)=\cos kx$, k an odd integer, the expansions terminate as do the expansions of powers of polynomials by the Maclaurin expansion. Thus,

$$f(x) = \sin 5x = 5\sin x - 20\sin^3 x + 16\sin^5 x,$$
$$f(x) = \cos 5x = 5\cos x - 20\cos^3 x + 16\cos^5 x.$$

Again, if

$$f(x) = (1 - h^2\sin^2 x)^{1/2} \quad\text{or}\quad f(x) = (1 - h^2\sin^2 x)^{-1/2}, \qquad h < 1,$$

equation (1) gives the series upon which are built tables of elliptic integrals. Other cases may probably suggest themselves to the reader.

A CERTAIN TYPE OF INTEGRAL

L. A. AROIAN, Hunter College

In a study of R. A. Fisher's z distribution [1] we found the semi-invariants and moments of z. From the mathematical point of view these statistical results may be used to evaluate

$$
(1)\qquad I(r, n_1, n_2) = \int_{-\infty}^{\infty} x^r y_x dx, \qquad y_x = \frac{e^{n_1 x}}{(n_1 e^{2x} + n_2)^{(n_1+n_2)/2}}, \quad r, n_1, n_2, \text{ positive integers.}
$$

We first give the semi-invariants of z, next find the moments of z about the origin from the semi-invariants, and finally show how (1) is related to the moments of z about the origin.

The ordinates of the z distribution are given by

$$
(2)\qquad y_z = k y_x, \qquad k = \frac{2 n_1^{n_1/2} n_2^{n_2/2}}{\mathrm{B}(n_1/2, n_2/2)},
$$

y_x as defined above, for $-\infty < z < \infty$. The semi-invariants of z are known [1]:

$$
(3)\qquad
\begin{aligned}
\lambda_{1:x} &= \frac{1}{2}\log\frac{n_2}{n_1} - \frac{1}{2}\left\{\sum_{k=1}^{n_2/2-1}\frac{1}{k} - \sum_{k=1}^{n_1/2-1}\frac{1}{k}\right\}, && n_1 \text{ and } n_2 \text{ even,}\\
\lambda_{1:x} &= \frac{1}{2}\log\frac{n_2}{n_1} - \left\{\sum_{k=0}^{(n_2-3)/2}\frac{1}{2k+1} - \sum_{k=0}^{(n_1-3)/2}\frac{1}{2k+1}\right\}, && n_1 \text{ and } n_2 \text{ odd,}\\
\lambda_{1:x} &= \frac{1}{2}\log\frac{n_2}{n_1} + \frac{1}{2}\sum_{k=1}^{n_1/2-1}\frac{1}{k} - \sum_{k=0}^{(n_2-3)/2}\frac{1}{2k+1} + \sigma, && n_1 \text{ even, } n_2 \text{ odd,}\\
\lambda_{1:x} &= \frac{1}{2}\log\frac{n_2}{n_1} + \sum_{k=0}^{(n_1-3)/2}\frac{1}{2k+1} - \frac{1}{2}\sum_{k=1}^{n_2/2-1}\frac{1}{k} - \sigma, && n_1 \text{ odd, } n_2 \text{ even,}\\
\lambda_{r:x} &= \sum_{k=0}^{\infty}(r-1)!\left\{\frac{(-1)^r}{(n_1+2k)^r} + \frac{1}{(n_2+2k)^r}\right\}, && r \geqq 2.\\
\sigma &= \log 2 = .693147181.
\end{aligned}
$$

The moments about the origin of any distribution, μ_r', are related to the semi-invariants, λ_r, by the well-known formulas [2], [3]:

$$
(4)\qquad
\begin{aligned}
\mu_1' &= \lambda_1\\
\mu_2' &= \lambda_2 + \lambda_1^2\\
\mu_3' &= \lambda_3 + 3\lambda_2\lambda_1 + \lambda_1^3\\
\mu_4' &= \lambda_4 + 4\lambda_3\lambda_1 + 3\lambda_2^2 + 6\lambda_2\lambda_1^2 + \lambda_1^4
\end{aligned}
$$

and in general

$$\mu_r' = \sum \sum \cdots \sum \frac{r!}{(a!)^u (b!)^v (c!)^w \cdots} \frac{\lambda_a^u \lambda_b^v \lambda_c^w \cdots}{u!v!w! \cdots},$$

$$a > b > c > \cdots, \quad au + bv + cw + \cdots = r.$$

Since

$$\mu_r' = \int_{-\infty}^{\infty} z^r y_z dz = k \int_{-\infty}^{\infty} x^r y_x dx = kI(r, n_1, n_2),$$

we obtain our final result

$$I(r, n_1, n_2) = \frac{\mu_r'}{k}. \tag{5}$$

For μ_r' we substitute the results given by (4) and (3). For n_1 and n_2 large, approximate values of $\lambda_{r:z}$ may be used [1]. Special cases of (1) for particular values of n_1, n_2, and r may be found in Bierens de Haan [4, pp. 125, 127].

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RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

Plane Trigonometry with Tables. By E. R. Heineman. New York, McGraw-Hill, 1942. 12+167+3+75 pages. $2.00.

The first sentence in the preface to this text states, "The principal objective of the author in writing this trigonometry textbook has been teachability." The reviewer feels that the author has clearly attained his objective.

There are many evidences throughout the text of close familiarity with the usual stumbling blocks that trip students of trigonometry and every effort is made to remove such obstructions. This is evidenced by such points as emphasis on the difference between cofunctions and reciprocal functions and warnings

against the use of the symbol "sin" without an angle written after it. Other details of development that make a favorable impression are (1) generous lists of selected problems including the true-false type, a total of over 100 problems on identities, (2) an approach to logarithms that emphasizes their meaning as well as their use in computation, (3) numerous illustrative examples giving model forms for solutions of problems. Although the author is unusually careful to explain the meaning of symbols and to give references, on page 17 the symbol for "less than" is employed in several problems without explanation.

In the general organization of material the text is to be commended for introducing the functions of general angles first and obtaining the functions of an acute angle as a special case. Reduction procedures are handled neatly and efficiently by a geometric discussion. The chapter on graphing the trigonometric functions is adequate but could easily be more complete by giving a discussion of the period, amplitude and sketch of functions of the type $y=A\cdot\sin(Bx+C)$. It is a question in the reviewer's mind whether the order of arrangement is advisable in presenting trigonometric equations before inverse functions. Furthermore there is no discussion of the general solution of trigonometric equations or of the general values of inverse functions. There is no treatment in the text itself of the use of small angles but a page in the tables gives a method of interpolating the value of the logarithm of the function of a small angle accurately to five places.

The fact remains that this is an extremely teachable textbook in plane trigonometry.

D. W. Western

Mathematics in Daily Use. By W. W. Hart, Cottell Gregory and Veryl Schult. Boston, D. C. Heath and Company, 1942. 7+376 pages. $1.32.

This text is designed for students in secondary schools who for various reasons will not study algebra or for students who will study algebra later. It is not intended to replace first-year algebra.

The material covered is rather complete for such a text. It includes integers, fractions, measurements, percentage, graphs, family income, community activities, business transactions, and an introduction to algebra. A student who masters the material included in this text will have a good working knowledge of problems encountered in everyday life.

Fractions, both common and decimal, and percentage are fully treated. Illustrative problems are good, and examples are many and varied. Insurance, taxes, installment buying, savings, interest, margin, profit—all topics of everyday business—are explained.

Many students will enjoy the chapter on *Leisure Time Problems*, especially the puzzles and games of chance.

Short tests on each chapter are included, and there is an ample supply of examples and problems for class use and homework.

Sara L. Nelson

Principles of College Algebra. By M. S. Knebleman and T. Y. Thomas. New York, Prentice-Hall, 1942. 10+380 pages. $2.50.

The authors have as one of their purposes that of teaching the student that algebra is based on reason and not on magic. To this end they here develop the subject from a set of postulates and definitions which they insist that the student thoroughly understand. Lest the beginners think that they are being hoodwinked at the start, the Dedekind Cut Postulate is carefully explained in the first chapter where the real number system is built up. The fifteen chapters give a careful treatment of the usual topics of college algebra including partial fractions, logarithms, matrices and determinants, the theory of equations, permutations and combinations, probability, and infinite sequences and series.

The exercises are numerous and well graded. Problems of the "story" type are almost entirely lacking "since these involve non-algebraic or superficial difficulties that tend to disappear with general mental growth and increased familiarity with the subject." Besides the exercises after each topic there are cumulative exercises at the end of each chapter. These have been selected, for the most part, from recent college examinations. Careful references to the sources are given so that the students will be fully aware, for example, that the question "Define: (a) irrational number, (b) real number" (page 29) was taken from an examination of Arizona University. The book concludes with ten complete final examinations given by a number of unspecified colleges and universities of wide geographical distribution. Answers are given to most of the problems in the exercises, but not to the problems on the final examinations.

D. W. Hall

NEW BOOKS RECEIVED

Analytic Geometry. By E. S. Smith, M. Salkover, and H. K. Justice. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1943. 12+298 pages. $2.50.

Basic Mathematics for Pilots and Flight Crews. By C. V. Newsom and H. D. Larsen. New York, Prentice-Hall, Inc., 1943. 6+153 pages. $2.00.

Elementary Navigation. By L. E. Moore. Boston, D. C. Heath & Co., 1943. 7+222 pages. $1.60.

Arithmetic for the Emergency. By G. M. Ruch, F. B. Knight, and J. W. Studebaker. Chicago, Scott, Foresman and Company, 1942. 170 pages. $0.64.

Mathematics for the Emergency. By C. J. Lapp, F. B. Knight, and H. L. Rietz. Chicago, Scott, Foresman & Company, 1942. 158 pages $1.80.

Formalization of Logic. By Rudolf Carnap. Cambridge, Harvard University Press; London, Humphrey Milford, 1943. 18+159 pages. $3.00.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

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PROBLEMS FOR SOLUTION

E 576. *Proposed by M. A. Dernham, San Francisco*

What well known type of sequence may be defined as one in which each term but the first may be obtained by taking the preceding term and a common constant and dividing their product by their sum?

E 577. *Proposed by V. Thébault, San Sebastián, Spain*

Given an "isosceles" tetrahedron $A_1A_2A_3A_4$ (so that every two opposite edges are equal), let perpendiculars be drawn to the faces $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, $A_1A_2A_3$ at their circumcenters O_1, O_2, O_3, O_4, to meet the hemispheres described exteriorly (or interiorly) on the respective circumcircles in P_1, P_2, P_3, P_4. Show that the tetrahedra $O_1O_2O_3O_4$ and $P_1P_2P_3P_4$ are isosceles, and that they have the same centroid as $A_1A_2A_3A_4$.

E 578. *Proposed by R. V. Heath, Wall St., New York City*

Find a perfect cube whose digits form a permutation of consecutive digits. (Cf. E 538.)

E 579. *Proposed by Howard Eves, Syracuse University*

Show that a matrix of m rows and n columns contains $(2^m-1)(2^n-1)$ submatrices.

E 580. *Proposed by Albert Furman, Infantry School, Fort Benning*

A closed tank containing V gallons of water at temperature T has an inlet pipe which supplies water at temperature t. Assuming an ideal situation where there is no loss of heat and an instantaneous diffusion in the mixture, show that the temperature of v gallons of water drawn from the tank into an open container is

$$t + (T - t)(1 - e^{-v/V})V/v.$$

SOLUTIONS

Three Similar Polygons

E 451 [1942, 613]. *Proposed by Joseph Rosenbaum, Bloomfield, Conn.*

Given a regular polygon of n sides, $n>4$, design a quadrilateral, Q, such that (1) it shall be possible to fit $2n$ of the Q's to the polygon to form a new regular polygon of n sides, and (2) it shall be possible to fit $2n$ additional Q's to the new polygon to form a still larger third regular polygon of n sides.

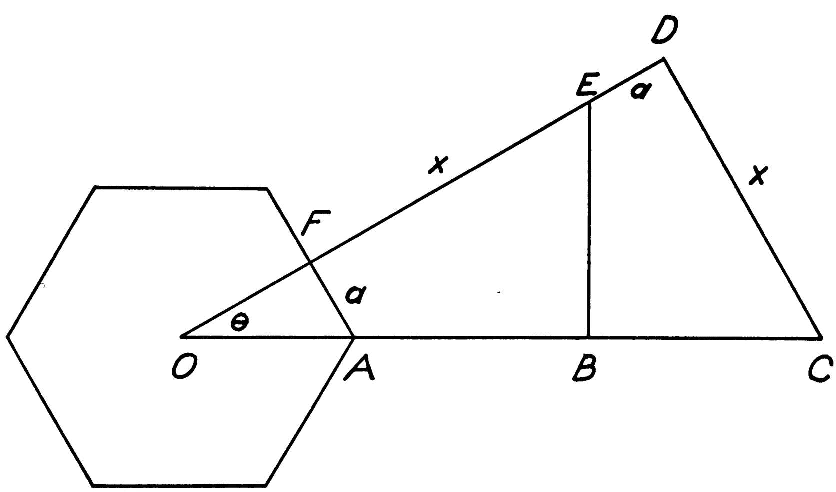
Solution by Howard Eves, Syracuse University. Let $2a$ be the side of the given polygon, and let $\theta=180°/n$. Then the problem will be solved if we can construct two congruent quadrilaterals $FEBA$ and $DCBE$ (see figure). Set $FE=DC=x$. Then we must have

$$x = DC = DO \tan\theta = (a + x + a\cot\theta)\tan\theta = x\tan\theta + a(1+\tan\theta),$$

whence

$$x = a(1+\tan\theta)/(1-\tan\theta).$$

This will always be possible if $\tan\theta<1$, *i.e.*, if $\theta<45°$, or $n>4$.



Also solved by the proposer.

Isogonal Conjugates

E 537 [1942, 546]. *Proposed by V. Thébault, San Sebastián, Spain*

Let L, M, N and L', M', N' be the orthogonal projections of a point P on the sides and corresponding altitudes of a given triangle. Show that the lines LL', MM', NN' are in general concurrent, and find the locus of P when they are parallel.

Solution by J. H. Butchart, Grinnell College. Let AD, BE, CF be the altitudes of the given triangle ABC, and let U, V, W be the midpoints of PD, PE, PF.

Then U is the center of the rectangle $PLDL'$. The line through D parallel to LL', being the image of DP by reflection in AD, is the isogonal conjugate of DP with respect to the orthic triangle DEF. This, and the two analogous lines through E and F, meet in the isogonal conjugate point to P with respect to DEF. Hence the lines LL', MM', NN' for the homothetic figure UVW are likewise concurrent.

The isogonal conjugate lines for PD, PE, PF are parallel only when P lies on the circumcircle of DEF, which is the nine-point circle of ABC. This, therefore, is the locus of P when LL', MM', NN' are parallel.

Also solved by W. B. Carver, Howard Eves, L. M. Kelly, and the proposer.

Eves generalizes this to an *affine* problem, replacing the altitudes by any three Cevians HA, HB, HC. He projects P parallel to the Cevians onto the sides, and *vice versa* (so that Butchart's rectangles become parallelograms). Instead of the circle we now have a "nine-point conic": the locus of centers of conics through the four points A, B, C, H. (When the quadrangle $ABCH$ is convex, this locus is a hyperbola whose asymptotes are diameters of the two circumscribed parabolas. See H. F. Baker, *Principles of Geometry*, Cambridge, Eng., 1930, p. 41: "The eleven-point conic.")

The proposer remarks that (in the original problem) the lines LL', MM', NN' are diameters of a conic inscribed in the triangle DEF. The foci of this conic are P and the isogonal conjugate point. When the diameters are parallel, the conic must be a parabola, and its focus P lies on the circumcircle of DEF.

The Centers of Three Circles

E 539 [1942, 546]. *Proposed by Howard Eves, Syracuse University*

Give a ruler construction for finding the centers of three given linearly independent circles, no two of which are intersecting, tangent, or concentric.

Elucidation by the Proposer. We define a ruler construction as any construction carried out by the straight-edge alone under Euclid's first two postulates:

I. We may connect any two given points by a straight line.

II. We may extend any given straight line indefinitely in either direction.

The Poncelet-Steiner Theorem states that in the presence of a circle *and its center* we may carry out any Euclidean construction with ruler alone. This naturally suggests the question: How many circles do we need in the plane in order to find their centers by a ruler construction? In this connection we have the following results:

(a) If we are given one circle alone, the ruler is *not* sufficient to find its center.

(b) If we are given two circles which intersect, are tangent, or are concentric, the ruler *is* sufficient to find their centers.

(c) If the two circles are non-intersecting, non-tangent, and non-concentric, then the ruler is *not* sufficient to find their centers, but we need a third circle.

For (a) and part of (b), see, *e.g.*, H. Steinhaus, *Mathematical Snapshots*, New York, 1938, pp. 44, 39. Result (c) is established by Cauer, Schur, and Mieren-

dorff, *Mathematische Annalen*, vol. 73 (1912), pp. 90–94 and vol. 74 (1913), pp. 462–464. However, no neat and direct construction is offered for the case of three circles.

Editorial Note. Contributions on this subject will still be welcomed.

The Sixteen-Point Sphere

E 540 [1942, 546]. *Proposed by L. M. Kelly, U. S. Coast Guard Academy*

Can the radius of the sixteen-point sphere ever be one-half the circumradius of the tetrahedron? (The sixteen-point sphere passes through the circumcenters of the faces.)

Solution by Howard Eves, Syracuse University. We answer the question in the affirmative and back the assertion by considerations of continuity, as follows. Let R be the circumradius, and r the radius of the sixteen-point sphere. If the tetrahedron is regular, then $r=R/3$; and if it is trirectangular, then $r=\infty$ with R finite. Since we may continuously deform a regular tetrahedron into a trirectangular one, it follows that at some intermediate stage we must have $r=R/2$. (Incidentally, it must likewise be possible to have $r=R$.)

A Square with Four Equal Digits

E 542 [1942, 613]. *Proposed by V. Thébault, San Sebastián, Spain*

In what scale of notation (with radix less than a hundred) will the four-digit number 58 58 58 58 be a perfect square?

Solution by Irving Kaplansky, Harvard University. Let the radix be r, so that $58<r<100$. For $n=58(r+1)(r^2+1)$ to be a square, r must be odd, as otherwise n would be twice an odd number. Also either $r+1$ or r^2+1 must be divisible by 29. Since the solution of

$$r^2 \equiv -1 \pmod{29}$$

is $r\equiv \pm 12$, we have $r\equiv -1$ or $\pm 12 \pmod{29}$. These conditions are fulfilled only by $r=75$ or 99. The former is rejected by trial, or perhaps more neatly by observing that with $r=75$ we would have $n\equiv 3 \pmod 5$, a non-residue. With $r=99$ we have

$$\begin{aligned}\sqrt{n} &= 7540 \text{ (denary)}\\ &= \overline{76}\ \overline{16} \text{ (in the scale of 99).}\end{aligned}$$

Also solved by D. H. Browne, R. C. Buck, W. E. Buker, E. P. Starke, and the proposer.

Perpendicular Polar Planes

E 543 [1942, 613]. *Proposed by N. A. Court, University of Oklahoma*

Find a point whose polar planes for three given spheres (with non-collinear centers) are mutually perpendicular. Show that the problem may have two solutions. When will they be real?

Solution by R. C. Buck, Cambridge, Mass. Let P be the desired point, and P_1, P_2, P_3 the centers of the spheres. Then it is necessary and sufficient that the lines PP_1, PP_2, PP_3 be mutually perpendicular. The locus of points such that PP_i is perpendicular to PP_j is the sphere with diameter P_iP_j. By considering the possible intersections of three such spheres, we see that there are two real points P, or none, according as the triangle $P_1P_2P_3$ is acute or obtuse. In the intermediate case of a right triangle, there is just one point P, but one of its polar planes is the plane at infinity.

Also solved by Howard Eves and the proposer.

A Commensurable Tetrahedron

E 544 [1942, 613]. *Proposed by E. P. Starke, Rutgers University*

Show that it is possible to construct a tetrahedron such that the length of every edge, the area of every face, and the volume all are integers.

Solution by the Proposer. Such a tetrahedron (probably not the simplest) has one edge 896, the opposite edge 990, and the remaining four each 1073. The areas of the faces are 436800 and 471240 (two each), and the volume is 62092800.

To determine this example, a rhombus $PQRS$ was taken with each side equal to c, and diagonals $PR=2a$ and $QS=2b$, whence

$$a^2 + b^2 = c^2. \tag{1}$$

The triangle PRS was rotated about PR until the distance QS became $2a'$. It is required that a', a, b, c be integers, and also that there exist integers b' and k such that

$$a'^2 + b'^2 = c^2, \qquad k^2 + a'^2 = b^2. \tag{2}$$

The tetrahedron $PQRS$ will then have edges $2a$, $2a'$, c, c, c, c; face areas ab, ab, $a'b'$, $a'b'$; and volume $2aa'k/3$.

To solve the equations (1) and (2), we note that c^2 is a sum of two squares in two ways. Thus c is a product of two integers each of which is a sum of two squares. This suggests the equations

$$u^2 = r^2 + s^2, \qquad u'^2 = r'^2 + s'^2, \qquad c = uu',$$
$$a = |rr' - ss'|, \qquad b = rs' + r's, \qquad a' = |r's - rs'|, \qquad b' = rr' + ss',$$
$$k^2 = 4rr'ss'.$$

By convassing the simplest solutions of the Pythagorean equation $u^2=r^2+s^2$, the following two are found for which $rr'ss'$ is a square:

$$20^2 + 21^2 = 29^2, \qquad 35^2 + 12^2 = 37^2.$$

The values $r=20$, $r'=35$, $s=21$, $s'=12$, $u=29$, $v=37$, thus suggested, determine the tetrahedron given above.

Concyclic Points on the Sides of a Triangle

E 545 [1942, 614]. *Proposed by A. H. Stone, Institute for Advanced Study*

Starting with a point P on the side BC of a triangle ABC, mark Q on AB with $BQ=BP$, R on CA with $AR=AQ$, P' on BC with $CP'=CR$, Q' on AB with $BQ'=BP'$, and so on. Prove that the construction closes, *i.e.*, that $CP=CR'$, and that the six points P, Q, R, P', Q', R' are concyclic. (Some obvious restrictions must be placed on the directions on the sides of the triangle in which the intervals BQ, *etc.*, are taken.)

Solution by Howard Eves, Syracuse University. Let P_0, Q_0, R_0 be the points of contact of the incircle. Then we see that

$$PP_0 = QQ_0 = RR_0 = P'P_0 = Q'Q_0 = R'R_0 = P''P_0.$$

This shows that the construction closes, and that P, Q, R, P', Q,$'$ R' lie on a circle concentric with the incircle.

Note. This proof generalizes to any odd polygon possessing an incircle.

Also solved by R. K. Allen, W. B. Clarke, Joseph Rosenbaum, Alan Wayne, M. W. White, and the proposer. Concerning the generalization to any odd polygon, Rosenbaum remarks that the construction may close even if there is no incircle. For, given a polygon, it may be possible to make the sides touch a circle, without altering their lengths, by suitably changing the angles. This process will not affect the positions of the points P, Q, $\cdots$ on their respective sides.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4086. *Proposed by P. Erdös, Princeton, N. J.*

Let A_1, A_2, $\cdots$, A_{2n+1} be the vertices of a regular polygon, and O any point in its interior. Show that at least one of the angles A_iOA_j satisfies the relation.

$$\pi\left(1 - \frac{1}{2n+1}\right) \leqq A_iOA_j < \pi.$$

4087. *Proposed by Betty Dick and B. M. Stewart, Michigan State College*

Let P be a plane polygon with vertices A_1, A_2, $\cdots$, A_n, and consider $A_{n+k}=A_k$. Determine points B_1, B_2, $\cdots$, B_n such that B_i is on the side A_iA_{i+1}

with $A_iA_{i+1}=k\cdot A_iB_i$, where k is a fixed real number. Let the lines A_iB_{i+1} and $A_{i+1}B_{i+2}$ intersect in the point C_i, thus determining the polygon Q with the vertices $C_1, C_2, \cdots, C_n$. Let $R(k)$ be the ratio of the area of Q to the area of P, wherein not to restrict the type of polygon we use Klein's definition of area, *Elementary Mathematics from an Advanced Standpoint, Geometry*, p. 10. (1) Show that k and $R(k)$ are invariants under affine transformations. (2) As a corollary to (1) show that for any triangle we have $R(k)=(k-2)^2/(k^2-k+1)$. (3) For any parallelogram we have $R(k)=(k-1)^2/(k^2+1)$. (4) Show that $R(k)$ does not have the same value for all quadrilaterals.

This problem is a generalization of the so-called Problem of Steinhaus which asserts for any triangle $R(3)=1/7$. An early reference, suggestive of this problem, is problem 276 in *Mathesis*.

4088. *Proposed by V. Thébault, San Sebastián, Spain*

If three circles $A(\rho)$, $B(\rho)$, $C(\rho)$ with the same radius ρ are described in the triangle ABC, and then the circles with centers A, B, C orthogonal respectively to $C(\rho)$, $A(\rho)$, $B(\rho)$; these three circles have the same radical center M_1 whatever the value of ρ. The same is true of three circles with centers A, B, C orthogonal respectively to $B(\rho)$, $C(\rho)$, $A(\rho)$, the radical center being M_2. If O is the circumcenter of ABC, show also that: (1) The triangles ABC and OM_1M_2 have the same centroid; (2) The straight line M_1M_2 is perpendicular to the straight line through the centroid and the Lemoine point; (3) if M_1' and M_2' are the symmetrics of M_1 and M_2 with respect to O, then M_1M_1' and M_2M_2' are parallel to the Euler line.

SOLUTIONS

Ellipses Inscribed in a Square

4029 [1942, 202]. *Proposed by H. L. Dorwart, Washington and Jefferson College*

Let d_1, d_2, d_3, d_4 be the distances in order from the sides of a square of length k units to any interior point P. Then

$$(\sqrt{d_1d_2} \pm \sqrt{d_3d_4})/k \quad \text{and} \quad (\sqrt{d_1d_4} \pm \sqrt{d_2d_3})/k$$

represent the sines and cosines of two angles θ_1 and θ_2, since the sum of the squares of these expressions equals

$$(d_1d_2 + d_3d_4 + d_1d_4 + d_2d_3)/k^2 = (d_1 + d_3)(d_2 + d_4)/k^2 = 1.$$

Give a geometric interpretation for the angles θ_1 and θ_2.

Solution by the Proposer. To be specific, let

$$(1) \qquad \begin{cases} \sin\theta_1 = (\sqrt{d_1d_2} + \sqrt{d_3d_4})/k, & \cos\theta_1 = (\sqrt{d_1d_4} - \sqrt{d_2d_3})/k \\ \sin\theta_2 = (\sqrt{d_1d_2} - \sqrt{d_3d_4})/k, & \cos\theta_2 = (\sqrt{d_1d_4} + \sqrt{d_2d_3})/k, \end{cases}$$

and let the vertices of the square be $A(K, O)$, $B(O, K)$, $C(-K, O)$, $D(O, -K)$, where $\sqrt{2}\cdot K=k$. Then consider the family of conics

$$\frac{x^2}{(1-\lambda^2)K^2} + \frac{y^2}{\lambda^2 K^2} = 1, \tag{2}$$

which for $\lambda^2 < 1$ are ellipses inscribed in the square.

Writing (2) as a quadratic equation in λ^2,

$$K^2\lambda^4 + (x^2 - y^2 - K^2)\lambda^2 + y^2 = 0,$$

we obtain

$$\begin{aligned} 4K^2 &= (y + K + x)(y + K - x) + (y - K + x)(y - K - x) \\ &\quad \pm 2\sqrt{(x - y + K)(x + y - K)(x - y - K)(x + y + K)}. \end{aligned}$$

This can be condensed if we put

$$\begin{aligned} l_1 &= -x - y + K, & l_3 &= x + y + K, \\ l_2 &= x - y + K, & l_4 &= -x + y + K, \end{aligned}$$

whence

$$4K^2\lambda^2 = l_1l_2 + l_3l_4 \pm 2\sqrt{l_1l_2l_3l_4}$$

and

$$\pm 2K\lambda = \sqrt{l_1l_2} \pm \sqrt{l_3l_4}.$$

also

$$4K^2(1 - \lambda^2) = 4K^2 - (l_1l_2 + l_3l_4) \mp 2\sqrt{l_1l_2l_3l_4}.$$

But $(l_1+l_3)(l_2+l_4) = 4K^2$, hence

$$4K^2(1 - \lambda^2) = l_1l_4 + l_2l_3 \mp 2\sqrt{l_1l_2l_3l_4}$$

and

$$\pm 2K\sqrt{1 - \lambda^2} = l_1l_4 \mp l_2l_3.$$

Evidently $l_1 = \sqrt{2}d_1$, $l_2 = \sqrt{2}d_2$, $l_3 = \sqrt{2}d_3$, $l_4 = \sqrt{2}d_4$, so that

$$\pm \lambda = (\sqrt{d_1d_2} \pm \sqrt{d_3d_4})/k$$

and

$$\pm \sqrt{1 - \lambda^2} = (\sqrt{d_1d_4} \mp \sqrt{d_2d_3})/k.$$

Using (1), it is now evident that

$$\pm \frac{\lambda}{\sqrt{1 - \lambda^2}} = \tan\theta_1 \text{ or } \tan\theta_2.$$

In general, through any point P inside the square, there will pass two members of (2), and if for the ellipse $x^2/a^2 + y^2/b^2 = 1$ we call the lines $y = \pm bx/a$ the *diagonals* of the ellipse, we can now interpret θ_1 and θ_2 as the inclinations of the

diagonals of the two members of the inscribed family of ellipses passing through any general point P of the square $ABCD$.

Editorial Note. The only other conics tangent to the sides of the square are two types of hyperbolas whose focal axes are respectively on the diagonals AC and BD; and the only real intersections P of these hyperbolas are those for two of the same type. Hence P must lie within the vertical angles at the vertices, for example one such point P is within the angle between the extensions of BA and DA. If P is within this angle and the distances $d_i > 0$ are defined as above, we may write

$$(1)\qquad \begin{aligned} k\cosh\theta_1 &= \sqrt{d_2d_3} + \sqrt{d_4d_1}, & k\sinh\theta_1 &= \sqrt{d_1d_2} + \sqrt{d_3d_4},\\ k\cosh\theta_2 &= \sqrt{d_2d_3} - \sqrt{d_4d_1}, & k\sinh\theta_2 &= \sqrt{d_1d_2} - \sqrt{d_3d_4}, \end{aligned}$$

and from these we obtain

$$(2)\qquad \begin{aligned} d_1 &= k\sinh^2\left(\frac{\theta_1+\theta_2}{2}\right), & d_2 &= k\cosh^2\left(\frac{\theta_1-\theta_2}{2}\right),\\ d_3 &= k\cosh^2\left(\frac{\theta_1+\theta_2}{2}\right), & d_4 &= k\sinh^2\left(\frac{\theta_1-\theta_2}{2}\right). \end{aligned}$$

It then follows from the equations of the sides that the coordinates of P are

$$(3)\qquad P:\quad (k/\sqrt{2})\cosh\theta_1\cosh\theta_2,\qquad (k/\sqrt{2})\sinh\theta_1\sinh\theta_2.$$

This suggests at once the equations of two hyperbolas of the same type

$$(4)\quad H_i:\quad x = (k/\sqrt{2})\cosh\theta_i\cosh\theta,\qquad y = (k/\sqrt{2})\sinh\theta_i\sinh\theta,\qquad i = 1, 2.$$

If $\theta=\theta_2$ in H_1 we see that P lies on H_1; if $\theta=\theta_1$ in H_2 we see that it lies also on H_2. If in H_1 we set $\theta=-\theta_1$ we easily find that the equation of AB, $x+y-k/\sqrt{2}=0$ is satisfied. The slope of H_1 at the point θ is $\tanh\theta_1/\tanh\theta$, and hence H_1 is tangent to AB at the point for $\theta=-\theta_1$; similarly for H_2 at $\theta=-\theta_2$. Thus the points of contact T_{1i} of H_i with AB have the coordinates

$$(4)\qquad T_{1i}:\quad (k/\sqrt{2})\cosh^2\theta_i,\qquad -(k/\sqrt{2})\sinh^2\theta_i;$$

and the straight lines $y=\pm\tanh^2\theta_i x$ meet the sides of the square in the four points of contact of H_i with the sides. If the two distinct points T_{11} and T_{12} are given on the extension of BA as points of contact of H_1 and H_2 then the corresponding point P of intersection has coordinates which are the geometric means of the coordinates of the two given points on BA using suitable signs for the y's. The two hyperbolas have slopes at each of their four intersections whose product is unity.

For the case of the ellipses where P is inside the square we may write

$$\begin{aligned} k\cos\theta_1 &= \sqrt{d_2d_3} - \sqrt{d_4d_1}, & k\sin\theta_1 &= \sqrt{d_3d_4} + \sqrt{d_1d_2},\\ k\cos\theta_2 &= \sqrt{d_2d_3} + \sqrt{d_4d_1} & k\sin\theta_2 &= \sqrt{d_3d_4} - \sqrt{d_1d_2}, \end{aligned}$$

so that if P is within the triangle OAB the four functions are each positive. It then follows that

$$d_1 = k\sin^2\left(\frac{\theta_1-\theta_2}{2}\right), \qquad d_2 = k\cos^2\left(\frac{\theta_1+\theta_2}{2}\right),$$

$$d_3 = k\cos^2\left(\frac{\theta_1-\theta_2}{2}\right), \qquad d_4 = k\sin^2\left(\frac{\theta_1+\theta_2}{2}\right);$$

and P has the coordinates $(k/\sqrt{2})\cos\theta_1\cos\theta_2$, $(k/\sqrt{2})\sin\theta_1\sin\theta_2$. We then obtain similar results in a similar manner.

Arithmetic Progressions in Positive Integers

4031 [1942, 262]. *Proposed by S. H. Gould, Victoria University, Toronto*

Let m be any fixed positive integer, $k=1, 2, 3, \cdots$, and $r=0, 1, 2, \cdots, m(k-1)$. Take unity for first term when $k=1$, construct inductively the arithmetic progression of order $m(k-1)$, the first term of whose rth difference series is the $(r+1)$st term of the A.P. of order $m(k-2)$. Prove that its $(mk+2)$nd term is k^{mk}.

Solution by the Proposer. Letting A_k^i denote the $(i+1)$st term of the A.P. of order $m(k-1)$, it is clear that we must prove that, if

$$A_1^i = 1 \qquad\qquad i = 0, 1, 2, \cdots$$

$$(1) \qquad A_k^i = \sum_{s=0}^{\min\{i,\, m(k-1)\}} \binom{i}{s} A_{k-1}^s \qquad\qquad k = 2, 3, \cdots$$

then

$$A_k^{mk+1} = k^{mk}.$$

Writing $(mk+1)=s_1$ and $s=s_2$, we have

$$A_k^{mk+1} = \sum_{s_2=0}^{s_1}{}' \binom{s_1}{s_2}\cdot A_{k-1}^{s_2} = \sum_{s_2=0}^{s_1}{}'\cdot \sum_{s_3=0}^{s_2}{}' \binom{s_1}{s_2}\cdot\binom{s_2}{s_3}\cdot A_{k-2}^{s_3}$$

$$(2) \qquad = \cdots = \sum_{s_2=0}^{s_1}{}' \sum_{s_3=0}^{s_2}{}' \cdots \sum_{s_k=0}^{s_{k-1}}{}' \binom{s_1}{s_2}\binom{s_2}{s_3}\cdots\binom{s_{k-1}}{s_k},$$

$$= \sum_{s_1 s_2\cdots s_k}{}' \frac{s_1!}{(s_1-s_2)!(s_2-s_3)!\cdots s_k!} = \sum_{\lambda_1\cdots\lambda_k}{}' \frac{(mk+1)!}{\lambda_1!\lambda_2!\cdots\lambda_k!},$$

where $\lambda_1=(s_1-s_2)$ *etc.*, and the primes indicate that summations are to be made only for:

$$(\mathrm{B}) \qquad s_t = \sum_{j=t}^{k}\lambda_j \leqq m(k+1-t) \qquad\qquad t = 2, 3, \cdots, k.$$

By the polynomial theorem, $k^{mk+1}=(1)$ without the restriction (B), and (see

lemma), among the k cyclic permutations of any set $(\lambda_1, \lambda_2, \cdots, \lambda_k)$, there is exactly one which satisfies (B).

Thus $A_k^{mk+1} = k^{mk}$.

LEMMA. *For a given ordered set* $\lambda_1, \lambda_2, \cdots, \lambda_k$, $k \geqq 2$, *and its cyclic permutations, of non-negative integers whose sum is* $mk+1$, *where* m *is a positive integer, there is among these* k *permutations one and only one set (i.e., there is exactly one value of* a*) for which*

$$\sum_{j=a+1}^{a+q} \lambda_j \leqq mq, \qquad q = 1, 2, \cdots, k-1. \tag{3}$$

This is equivalent to (B).

Proof. There cannot be two sets satisfying (3). For, if there are two such distinct sets, say for $a=0$ for one and $1 \leqq a < k$ for the other; then $\lambda_1+\lambda_2+ \cdots +\lambda_a \leqq ma$ and $\lambda_{a+1}+\lambda_{a+2}+ \cdots +\lambda_k \leqq m(k-a)$, and the sum gives the contradiction $mk+1 \leqq mk$.

Suppose now that no set satisfies (3); we shall deduce in this case the contradiction that the sum of the k integers λ_j is greater than $mk+1$. To do this, we let $v_1, v_2, \cdots, v_i$ be the smallest positive integers such that

$$\sum_{j=1}^{v_1} \lambda_j > mv_1, \quad \sum_{j=v_1+1}^{v_1+v_2} \lambda_j > mv_2, \cdots, \quad \sum_{j=v_1+v_2+\cdots+v_{i-1}+1}^{v_1+v_2+\cdots+v_i} \lambda_j > mv_i, \tag{4}$$

with $v_1+v_2+ \cdots +v_{i-1} < k \leqq v_1+v_2+ \cdots +v_i = k+l$. This means that, if $l>0$, the last element λ_l of the subset (v_i) in (4) falls in a prior subset (v_r), $r \geqq 1$, and we first suppose that λ_l does not fall at the end of (v_r). Then we have from the inequalities in (4) for (v_r)

$$\lambda_{v_1+v_2+\cdots+v_{r-1}+1} + \cdots + \lambda_l + \lambda_{l+1} + \cdots + \lambda_{v_1+v_2+\cdots+v_r} > mv_r,$$

and from the minimal choice of v_r

$$\lambda_{v_1+v_2+\cdots+v_{r-1}+1} + \cdots + \lambda_l \leqq m[l - (v_1 + v_2 + \cdots + v_{r-1})],$$

from which it follows that

$$\sum_{j=l+1}^{v_1+v_2+\cdots+v_r} \lambda_j > m(v_1 + v_2 + \cdots + v_r - l). \tag{5}$$

Also by addition of the inequalities in (4) for $(v_{r+1}), (v_{r+2}), \cdots, (v_{i-1})$, we have

$$\sum_{j=v_1+v_3+\cdots+v_r+1}^{v_1+v_2+\cdots+v_{i-1}} \lambda_j > m(v_{r+1} + v_{r+2} + \cdots + v_{i-1}). \tag{6}$$

The upper limit for the sum of (v_i) in (4) may be written $k+l$, and by addition of this inequality with (5) and (6) we have

$$(7)\qquad \sum_{j=l+1}^{l+k} \lambda_j > m(v_1 + v_2 + \cdots + v_r - l + v_{r+1} + \cdots + v_i) + 2 = mk + 2.$$

If λ_l falls at the end of (v_r), then (5) drops out and we have again a contradiction. If $l=0$, the sum of the inequalities in (4) gives at once the contradiction that the sum of the k integers λ_j is $\geqq mk+2$, $i \geqq 2$.

Editorial Note. This interesting problem and its interesting solution were received from R. D. James with the following account of the origin of the problem. In an article by Heilbronn, Landau, and Sherk in the *Journal Tchécoslovaque de Mathématique et de Physique*, 65, 1935–36, pp. 117–140, there is a lemma (*Satz* 8) equivalent to the following: Given the numbers A_k^i defined as in (1) above but for $m=2$, then $A_k^{2k+1} \leqq k^{2k}$. After a study of this it seemed to James that the result should be an actual equality, but he could not find a proof and suggested the problem to J. S. Vigder. The latter considered the more general problem using the positive integer m in place of 2, and saw that a proof involved the polynomial theorem as above, but he was unable to complete the proof and passed the matter on to the proposer. The proposer formulated the problem differently and came through with a solution resulting from his proof of his lemma (3).

An Oval and its Normal Expansion

4036 [1942, 340]. *Proposed by L. A. Santaló, Rosario, Argentina*

Let C_1 be an oval with a continuously varying radius of curvature R; at each point of C_1 a normal of length R is drawn exteriorly giving points of a second curve C_2 (which may not be convex); and let A be the area enclosed between the two curves. From a chosen fixed point a vector is drawn parallel to the normal at a point of C_1 and of length R for that point, thus giving as the point varies on C_1 a curve C_3 having the area A_3 and length L_3. If L_2 is the length of C_2 and A_1 is the area of C_1, show that

$$\text{(a)}\quad A = 3A_3; \qquad \text{(b)}\quad L_2L_3 \geqq 8\pi A_1;$$

the equality in (b) is true only when C_1 is a circle.

I. *Solution by Fritz John, University of Kentucky.* Let $p(\alpha)$ denote the "function of support" of C_1, *i.e.*, $p(\alpha)$ shall be the distance of that tangent of C_1 from the origin, whose normal forms the angle α with the x-axis (See Courant: *Calculus*, II, p. 213). Then

$$x = p\cos\alpha - p'\sin\alpha, \qquad y = p\sin\alpha + p'\cos\alpha$$

is a parametric representation for C_1. The radius of curvature of C_1 is given by $R=p+p''$, the enclosed area by

$$A_1 = \frac{1}{2}\int_0^{2\pi}(xy' - yx')d\alpha = \frac{1}{2}\int_0^{2\pi}(p^2 + pp'')d\alpha = \frac{1}{2}\int_0^{2\pi}(p^2 - p'^2)d\alpha.$$

Similarly the parametric representations of C_2 and C_3 are respectively

$$x = (2p + p'') \cos \alpha - p' \sin \alpha, \qquad y = (2p + p'') \sin \alpha + p' \cos \alpha,$$

and

$$x = (p + p'') \cos \alpha, \qquad y = (p + p'') \sin \alpha;$$

hence the areas enclosed by C_2 and C_3 are easily found to be

$$A_2 = \frac{1}{2}\int_0^{2\pi} (4p^2 - 7p'^2 + 3p''^2)d\alpha$$

$$A_3 = \frac{1}{2}\int_0^{2\pi} (p^2 - 2p'^2 + p''^2)d\alpha.$$

Consequently $A = A_2 - A_1 = 3A_3$, which is the first statement.

Now "Wirtinger's inequality" states, that for a function $f(\alpha)$ of class C' with $\int_0^{2\pi} f(\alpha)d\alpha = 0$

$$\int_0^{2\pi} f'^2(\alpha)d\alpha > \int_0^{2\pi} f^2(\alpha)d\alpha,$$

unless f is of the form $f(\alpha) = a \cos \alpha + b \sin \alpha$; (see Hardy-Littlewood-Polya: *Inequalities*, pp. 185–187). For $f = p'$ it follows that $\int_0^{2\pi} p''^2(\alpha)d\alpha > \int_0^{2\pi} p'^2(\alpha)d\alpha$, and hence $A_3 > A_1$, unless $p' = a \cos \alpha + b \sin \alpha$; in the latter case $p = c + a \sin \alpha - b \cos \alpha$, and C_1 is a circle of radius c. The isoperimetric inequality (which may be based on Wirtinger's inequality), yields

$$L_2^2 \geqq 4\pi A_2, \qquad L_3^2 \geqq 4\pi A_3;$$

hence

$$L_2 L_3 \geqq 4\pi\sqrt{A_2 A_3} = 4\pi\sqrt{(3A_3 + A_1)A_3} > 4\pi\sqrt{4A_1^2} = 8\pi A_1$$

unless C_1 is a circle.

In the case where C_1 is a circle of radius c, C_2 is a circle of radius $2c$, and C_3 a circle of radius c, so that $L_2L_3 = 8\pi A_1$.

II. *Solution by the Proposer*. We consider two normals to C_1 corresponding to the directions ϕ and $\phi + d\phi$; a point on a normal whose distance to C_1 is a constant equal to λ will describe a curve whose arc s^* satisfies

$$ds^* = (R + \lambda)d\phi.$$

The area A will be then

$$A = \iint ds^* d\lambda = \int_0^{2\pi} d\phi \int_0^{R} (R + \lambda)d\lambda = \frac{3}{2}\int_0^{2\pi} R^2 d\phi = 3A_3$$

which proves (a).

We have also, if s_2 is the arc of C_2,

$$(1) \qquad ds_2 = \sqrt{4R^2 d\phi^2 + dR^2} = \sqrt{4R^2 + R'^2}\, d\phi$$

where R' represents the derivative with respect to ϕ. We have also

$$(2) \qquad ds_3 = \sqrt{R^2 + R'^2}\, d\phi.$$

From (1) and (2) we deduce, representing by s_1 the arc of C_1

$$ds_2 \geqq 2Rd\phi = 2ds_1 \quad \text{and} \quad L_2 \geqq 2L_1$$
$$ds_3 \geqq Rd\phi = ds_1 \quad \text{and} \quad L_3 \geqq L_1.$$

This gives us

$$(3) \qquad L_2 L_3 \geqq 2L_1^2$$

But it is known that for every plane closed curve we have $L_1^2 - 4\pi A_1 \geqq 0$; so this inequality and (3) proves the last part (b).

The equality in (b) is valid only if $R' = 0$, and then the radius of curvature is constant and the closed curve must be a circle.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Dr. H. F. Bright of San Angelo College has been appointed to an assistant professorship at Denison University.

Dr. Jesse Douglas has been appointed to an assistant professorship at Brooklyn College.

Associate Professor R. C. Hildner of Mt. Union College has been appointed to an assistant professorship at the College of Wooster.

Professor E. J. Moulton of Northwestern University is on leave of absence and Professor H. T. Davis is acting head of the department of mathematics.

Assistant Professor W. H. Myers has been appointed acting head of the mathematics department at San José State College.

Mr. N. D. Nelson of the University of Wisconsin has been appointed to an assistant professorship at Amherst College.

Dr. E. A. Nordhaus of the University of Wisconsin has been appointed to an assistant professorship at Michigan State College.

Assistant Professor C. V. L. Smith of Lafayette College is now a lieutenant,

E-V (RS), U.S.N.R. and is officer-in-charge of the Naval Training School, Elementary Electricity and Radio Matériel at the University of Houston, Texas.

Associate Professor V. H. Wells of Williams College has been promoted to a full professorship.

Rev. C. R. Wheeler, professor of mathematics at St. Bonaventure College, has been commissioned a first lieutenant in the Army Chaplain Corps.

Professor W. E. Wilson of the Colorado School of Mines has been appointed research engineer with the Armour Research Foundation of Chicago.

Dr. P. M. Young of Miami University is now teaching navigation at the U.S.N.R. Midshipmen's School in New York City.

The following appointments to instructorships are announced:
Allegheny College: R. E. Smith
The University of Chicago: H. L. Meyer
Cornell University: E. J. Scott (part-time)
Eau Claire State Teachers College (Wisconsin): Bjorne R. Ullsvik
Illinois Institute of Technology: Dr. Edward Helly
The University of Michigan: P. S. Jones
The University of North Carolina: Dr. C. M. Smith
Pennsylvania State College: Dr. Helen B. Owens
Tuskegee Institute: Dr. J. E. Wilkins
The University of Wisconsin: Dr. F. A. Butter, Jr.

Assistant Professor L. I. Neikirk of the University of Washington died on December 10, 1942. He was a charter member of the Mathematical Association.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

NATIONAL COMMITTEE ON PHYSICISTS AND MATHEMATICIANS

Recent action by the National Headquarters of the Selective Service System recognizes a critical shortage in the field of mathematics and sets up a new procedure whereby a national point of view may be brought to bear on the decisions of local boards in the cases of registrants who are occupied as mathematicians or are students of mathematics. This recent action of National Headquarters is best understood by reference to the historical background.

On November 7, 1942, National Headquarters announced a policy under which the Director of Selective Service might authorize the appointment of a

National Committee in a critical scientific or specialized field, to assist the Selective Service System by reviewing affidavits for occupational classification; at the same time the National Committee on Physicists was established. On April 5, 1943, this policy was amended so as to apply to mathematicians as well as to physicists; *the National Committee on Physicists was replaced by the National Committee on Physicists and Mathematicians.* In addition to the two physicists who, with representatives of the National Roster of Scientific and Specialized Personnel and of the public, constituted the earlier Committee, the new National Committee on Physicists and Mathematicians will include two mathematicians.

This procedure involves no change in the fundamental principles underlying the granting of occupational deferment, which remains, in the first instance, a matter for the decision of each registrant's local board. The creation of the National Committee on Physicists and Mathematicians does, however, provide a means whereby the individual local boards will be advised concerning the national as well as the local needs in the critical fields of both physics and mathematics.

In the AMERICAN MATHEMATICAL MONTHLY, March, 1943, attention was called to amendments made on December 14, 1942, to Occupational Bulletins No. 10 and No. 23. *The amended Bulletins remain in force and should be consulted in preparing requests for occupational classification of teachers or students of mathematics.**

Mathematicians should note the changes in procedure made necessary by the establishment of the National Committee on Physicists and Mathematicians. *The principal one is that the employer is now to file papers with the National Committee as well as with the employee's local board.* (See 3(b) below.) The following detailed outline, it is hoped, will be of value to mathematicians and their employers as the new procedure goes into effect.

1. Although full responsibility for presenting occupational deferment requests lies with employers,† the mathematician-registrant will be wise to see that his employer is fully informed of the new procedure established by Activity and Occupational Bulletin No. 35 as amended on April 5, 1943.

2. The employer must first decide whether he regards the registrant as a "necessary" man for whom he wishes to request occupational deferment (or continued occupational deferment). In the case of men with dependents, even if deferred at present, it is desirable to file information with the local boards establishing their occupational status as further ground for deferred classification.

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* The substance of the amended Bulletins No. 10 and No. 23 is to be found also in Activity and Occupation Bulletins No. 33-5 and No. 33-6, issued on March 1, 1943.

† Throughout this discussion the term "employer" means that person within an organization—industrial, educational, or governmental—who has been assigned the responsibility for handling deferment problems. In educational institutions, the head of the mathematics department must be an active adviser in matters involving both staff and students.

3. For every registrant for whom occupational deferment (or continued deferment) is to be requested, the employer will:

a) Fill out *completely* Selective Service Form 42 or 42A in triplicate. File a copy with the local board with a brief letter stating that the original has been sent to the National Committee on Physicists and Mathematicians as authorized in Selective Service Activity and Occupation Bulletin No. 35, issued March 1, 1943, and amended April 5, 1943.

b) Send the original and one copy of Form 42 or 42A (see also (c) below) to the:

NATIONAL COMMITTEE ON PHYSICISTS AND MATHEMATICIANS
National Roster of Scientific and Specialized Personnel
Washington, D. C.

c) Send in addition to Form 42 or 42A, a "Report of Status" for the National Committee on Physicists and Mathematicians. Copies of these "Report of Status" blanks may be secured from the National Committee on Physicists and Mathematicians.

d) For those *not now deferred*, this procedure should be followed at once. For those *now deferred*, this procedure should be followed *at least thirty days before the expiration of the deferment period.*

4. For every registrant for whom deferment is not to be requested by his employer or college, a "Report of Status" in duplicate should be filled out and sent to the National Committee on Physicists and Mathematicians at once so that efforts may be made to place him where his talents and training will be of best service to the war effort.

The plan under which deferment applications for mathematicians are now to be made is announced in Activity and Occupation Bulletin No. 35, as amended on April 5, 1943. Extracts giving the essential features of this plan are published herewith:

"*Filing Affidavit—Occupational Classification.* When such a National Committee has been appointed for a scientific or specialized field, the employer or recognized university or college desiring occupational classification for a registrant who possesses the training, qualification, or skill in that field, or is in training or preparation therefor, may prepare Affidavit—Occupational Classification (Form 42 or 42A) in duplicate, file the copy with the local board in the usual manner, and forward the original to the National Committee. (Part I, C, 3)

"*Action by the National Committee.* When the National Committee receives a Form 42 or 42A, it will investigate the registrant. If, in its opinion, the registrant possesses the training, qualification, or skill and is a necessary man in an essential activity or a necessary man in training or preparation therefor, the National Committee is authorized to place a stamped endorsement prescribed by the Director of Selective Service on the original Form 42 or 42A and to file the form with the registrant's local board. (Part I, C, 4)

"*Consideration by local board.* When the original Form 42 or 42A, stamped by the National Committee, is received by the local board, it shall be considered as showing that the registrant has been investigated, that in the opinion of the National Committee he possesses the training, qualification, or skill required, and that, as the case may be, he is a necessary man in an essential activity or a necessary man in training or preparation in such scientific and specialized field. (Part I, C, 5)

"*Classification Advice* (*Form 59*). When a Form 42 or 42A, stamped by a National Committee, has been received for a registrant and the local board, nevertheless, classified the registrant as available for military service or for assignment to work of national importance, the local board shall notify the National Committee, as well as the employer or university or college, by mailing to the National Committee and to the employer or university or college Classification Advice (Form 59). In such case the National Committee, as well as the employer or university or college, may appeal to the board of appeal from classification of the registrant. (Part I, C, 6)

"*National Committees.** The following National Committees for scientific and specialized fields presently are authorized by the Director of Selective Service:

PHYSICISTS AND MATHEMATICIANS

The National Committee on Physicists and Mathematicians, National Roster of Scientific and Specialized Personnel, Washington, D. C.

ENGINEERS AND CHEMISTS

The National Committee on Engineers and Chemists, National Roster of Scientific and Specialized Personnel, Washington, D. C." (Part I, C, 7)

J. R. KLINE,

May 11, 1943. *Secretary of the War Policy Committee*

THE CHICAGO INSTITUTE OF MILITARY STUDIES

The Institute of Military Studies at the University of Chicago was established to advance civilian knowledge of military practice, theory, and history. Classes are conducted for men anticipating military service or for those who expect to train others for service.

The Institute has published under its own imprint three different manuals, namely, *The Organization of the Army*, *A Review of Arithmetic*, prepared by Mr. Zens Smith, and *Some Military Applications of Elementary Mathematics*, published with the permission of The Department of Mathematics, United States Military Academy at West Point. Over nine thousand copies of the last pamphlet have already been distributed, and it is being widely used in public schools, in colleges, and in military and naval training centers.

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* As amended 4-5-43.

Individual copies of *A Review of Arithmetic* will be supplied at twenty-five cents. Copies of *Some Military Applications of Elementary Mathematics* are available at the cost price of fifteen cents each. Requests for these pamphlets should be addressed to: The Institute of Military Studies, The University of Chicago, Chicago, Illinois.

Two other texts sponsored by the Institute are now in press and shortly will be available to the public. The first of these will be of especial interest to mathematicians, and is entitled *Manual for Instruction in Military Maps and Aerial Photography*, prepared by Dean Norman F. Maclean, Acting Director of the Institute, and by Mr. Everett C. Olson, Assistant Professor of Geology at the University of Chicago. This manual differs from other manuals on cartography in that, as the title indicates, it is prepared for instructors. In its organization, it recognizes the limited time allotted to map instruction in the training programs of the Army, and it assumes no more than the average background of the inductees who will receive that training; also, consideration is given to the limited facilities and materials available for civilian instructors.

BIBLIOGRAPHY ON ARTILLERY

Professor J. M. Thomas was invited to submit a brief list of titles which might be used for classroom reference upon the mathematics of artillery fire. With the exception of the last title, the following short bibliography is due to him.

H. G. Brohop, *Elements of modern field artillery*, Banta, Menasha, Wisconsin, 1917.

T. J. Hayes, *Elements of ordnance*, Wiley, New York, 1938.

T. J. Hayes, *Exterior ballistics*, Wiley, New York, 1938, $1.00.

J. Haag, *Application au tir*, Gauthier-Villars, Paris, 1926.

F. R. Moulton, *New Methods in exterior ballistics*, University of Chicago Press, Chicago, Ill., 1926.

K. Popoff, *Das Hauptproblem der äusseren Ballistik in Lichte der modernen Mathematik*, Leipzig, Akademische Verlagsgesellschaft, 1932.

W. H. Tschappat, *Ordnance and gunnery*, New York, 1917.

Sophia H. Levy, *Introductory artillery mathematics and antiaircraft mathematics*, University of California Press, Berkeley, Calif., 1943.

Coast artillery, Military Service Publishing Co., Harrisburg, Pa., 1942, $6.00.

Antiaircraft defense, Military Service Publishing Co., Harrisburg, Pa., 1943, $2.00.

Coast artillery field manual, FM 4–15, Government Printing Office, Washington, D. C., 1940, $.50.

Coast artillery field manual, FM 4–10, Government Printing Office, Washington, D. C., 1940, $.25.

Field artillery field manual, FM 6–40, Government Printing Office, Washington, D. C., 1939, $.25.

Orientation, The Coast Artillery School, Fort Monroe, Va., 1942, $.90

Mathematics for the coast artillery officer, The Coast Artillery School, Fort Monroe, Va., 1943, $.25.

Field artillery, gunnery, Book 161, Field Artillery School, Fort Sill, Oklahoma.

J. M. Thomas, *Elementary mathematics in artillery fire*, McGraw-Hill, New York, 1942.

THE MATHEMATICAL ASSOCIATION OF AMERICA

NEW MEMBERS

The following thirty-eight persons have been elected to membership on applications duly certified:

E. W. ANDERSON, Ph.D. (Iowa State) Asst. Prof., Iowa State Coll., Ames, Iowa

J. F. ARENA, A.M. (Michigan) Teacher, Community High School, Hardin, Ill.

H. C. BOARDMAN, B.S., C.E. (Illinois) Research Engr., Chicago Bridge and Iron Co., Chicago, Ill.

A. C. COHEN, JR., Ph.D. (Michigan) Chief, Quality Control, Picatinny Arsenal, Dover, N. J.

MARGARET R. DAVIS, A.M. (Columbia) Instr., Antioch Coll., Yellow Springs, Ohio

JESSE DOUGLAS, Ph.D. (Columbia) Asst. Prof., Brooklyn Coll., Brooklyn, N. Y.

B. B. DRESSLER, A.M. (Illinois) Instr., Alabama Poly. Inst., Auburn, Ala.

SISTER EDWARD JOSEPH, M.S. (Notre Dame) Instr., St. Mary's Coll., Notre Dame, Ind.

S. A. FRANCIS, A.M. (California) Chm. of Dept., Math. and Eng., San Mateo Jr. Coll., San Mateo, Calif.

W. A. FRASER, Ph.D. (Iowa State) Physicist, in charge, Glass Plant Research, Bausch and Lomb Optical Co., Rochester, N. Y.

J. R. HANNA, M.S. (Kans. State T. C., Emporia) Instr., Univ. of Wichita, Wichita, Kansas

CHARLES HATFIELD, JR., A.M. (Kentucky) Part-time Instr., Cornell Univ., Ithaca, N. Y.

G. E. HAY, Ph.D. (Toronto) Asst. Prof., Univ. of Michigan, Ann Arbor, Mich.

W. J. HORN, B.S. in Archt. (A. and M. Coll. of Texas) Structural Engr., Truscon Steel Co., Houston, Texas

FREE JAMISON, Ph.D. (Pittsburgh) U.S.N., Air Navigation School, Hollywood, Fla.

W. J. LEVEGUE. Student, Univ. of Colorado, Boulder, Colo.

R. T. LUGINBUHL, B.S. (Ursinus) Asst. Instr., Univ. of Pennsylvania, Philadelphia, Pa.

G. H. LUNDBERG, A.M. (Colo. State Coll. Educ.), A.M. (Vanderbilt) Instr., Vanderbilt Univ., Nashville, Tenn.

JOHN MANDEL, M.S. (Brussels) Chemist, Foster D. Snell, Inc., Brooklyn, N. Y.

ELLA MARTH, M.S. (St. Louis Univ.) Teacher, Eliot School, St. Louis, Mo.

W. T. MARTIN, Ph.D. (Illinois) Asst. Prof., Massachusetts Inst. of Tech., Cambridge, Mass.

W. H. MILLS, A.B. (Swarthmore) Junior Physicist, Ballistic Research Lab., Aberdeen Proving Ground, Md.

ALFONSO NÁPOLES, Ph.D. (National Univ., Mexico) Prof., Head of Dept., National Univ., Mexico, D. F.

W. V. NEVINS, III, A.M. (Columbia) Instr., Alfred Univ., Alfred, N. Y.

MAX PETERS, M.S. (C.C.N.Y.) Teacher, New Utrecht High School, Brooklyn, N. Y.

GEORGE POLYA, Ph.D. (Budapest) Asso. Prof., Stanford Univ., Stanford University, Calif.

H. A. POPPEN, A.M. (Northwestern Univ.) Asst. Prof., Illinois State Normal Univ., Normal, Ill.

H. L. RICE, M.S. (Iowa) Instr., Univ. of Omaha, Omaha, Nebr.

CHARLES SALTZER, A.B. (Western Reserve) Grad. Asst., Univ. of Nebraska, Lincoln, Nebr.

O. T. SCHULTZ, M.S. (Chicago) Asst. Project Engr., Sperry Gyroscope Co., Inc., Garden City, N. Y.

E. J. SCOTT, A.M. (Vanderbilt) Prof., Lawrence Inst. of Tech., Detroit, Mich.

JACK SILBER, B.S. (Chicago) Instr., Illinois Inst. of Tech., Chicago, Ill.

C. F. TAYLOR, B.S. (Northeastern Univ.) Elec. Engr., General Electric Co., West Lynn Works, Lynn, Mass.

V. C. THROCKMORTON, A.M. (Southern California) Instr., Univ. of Southern California, Los Angeles, Calif.

H. S. TONEY, A.M. (Ohio State) Prof., Wilberforce Univ., Wilberforce, Ohio

L. A. WALKER, A.M. (Stanford) Teacher, Math. and Physics, High School, San Mateo, Calif.

J. T. WIEGAND, B.E. (Chicago T.C.) Instr., Illinois Inst. of Tech., Chicago, Ill.

R. S. WOLFE, A.M. (Washington) Instr., Northwestern Univ., Evanston, Ill.

W. B. CARVER, *Secretary-Treasurer*

SPRING MEETING OF THE MICHIGAN SECTION

The nineteenth annual meeting of the Michigan Section of the Mathematical Association of America was held at the University of Michigan, Ann Arbor, Michigan, on Saturday, March 14, 1942. Morning and afternoon sessions and a business-luncheon meeting were held, at all of which the chairman of the Section, Professor T. R. Running, presided.

The meeting was attended by fifty-six persons, including the following thirty-four members of the Association: H. M. Ackley, N. H. Anning, J. W. Baldwin, W. D. Baten, F. A. Beeler, W. M. Borgman, J. B. Brandeberry, R. V. Churchill, C. J. Coe, C. C. Craig, Wayne Dancer, C. M. Erikson, K. W. Folley, J. F. Heyda, T. H. Hildebrandt, Frances C. Hinds, L. A. Hopkins. L. S. Johnston, P. S. Jones, Wilfred Kaplan, W. C. Krathwohl, Theodore Lindquist, Sister Mary Paula, E. D. McCarthy, A. L. Nelson, J. K. Peterson, H. H. Pixley, G. Y. Rainich, E. D. Rainville, L. J. Rouse, T. R. Running, E. R. Stabler, A. G. Swanson, Fern Welker.

At the business meeting Professor Wayne Dancer of the University of Toledo was elected chairman for 1942–43, and Professor C. J. Coe of the University of Michigan was reelected secretary-treasurer.

At the morning session the following program of seven papers was presented, but due to the dangerously icy condition of the roads, Professors Stewart and Sleight were unable to be present and their papers were read by title.

1. *Boolean algebra as an introduction to postulational methods* by Professor E. R. Stabler, University of Michigan.

Professor Stabler showed how a set of postulates and theorems for Boolean algebra may be used for the study of consistency, independence, and categoricalness of a set of postulates. The proposed postulates were those listed as Set III, pp. 20–28, this MONTHLY, January, 1941, together with the requirement of a unit element. The speaker also indicated that the system could conveniently be used to study such concepts as the relative truth of theorems and the equivalence of different sets of postulates.

2. *Discontinuous rates of change* by Professor E. D. McCarthy, University of Detroit.

The speaker started with the assumption that time varies discontinuously, taking on only integral values. Thus velocity and distance in rectilinear motion are given by $V=\sum a$ and $S=\sum V$ respectively. Several simple cases were in-

vestigated, and results were obtained similar to those for the continuous case. In particular, a study of the case $a=kS$ led to the difference equation

$$V_n - (k+2)V_{n-1} + V_{n-2} = 0,$$

of which the solution is $V=c_1L^t+c_2L^{-t}$, where L is a root of $L^2-(k+2)L+1=0$. The exponential and trigonometric cases were discussed.

3. *A pair of linear transformations* by Professor G. Y. Rainich, University of Michigan.

Professor Rainich discussed a pair of linear transformations in the plane. He showed that in addition to satisfying the Hamilton-Cayley equations individually, they also satisfy a third equation involving both of them. A condition for the existence of a common invariant direction was obtained, and expressed in terms of the coefficients of the specified equations.

4. *A maximum problem* by Dr. B. M. Stewart, Michigan State College.

Dr. Stewart's paper was published in this MONTHLY vol. 49, 1942, pp. 454–456.

5. *Arithmetic according to Cocker* by Professor E. R. Sleight, Albion College.

This paper gave a historical account of Cocker himself, and presented a review of the arithmetics which were purported to have been written by him. Cocker's arithmetics were compared with those which preceded them, and it was pointed out that his predecessors seemed to know when to stop, but that Cocker continued indefinitely.

6. *Solution of difference equations by the Mellin inversion theory* by Dr. E. W. Paxson, Wayne University, introduced by Professor Nelson.

Dr. Paxson discussed the solution of linear difference equations with constant coefficients. He reduced the problem to that of solving certain integral equations. The integral equations were solved by means of the Mellin inversion theory.

7. *Demonstration of mathematical principles and applications with moving apparatus* by P. S. Jones, Edison Institute.

Mr. Jones presented a demonstration of still and moving mathematical models constructed by his students at the Edison Institute. The models displayed a professional finish due to the mechanical expertness of his classes, but the speaker pointed out that the construction of some of the simpler ones was within the ability of the average class, and could form a valuable teaching aid. The following models were demonstrated:

(1) Pin and string construction of the ellipse; a pair of rolling ellipses; four bar linkage for the same motion.

(2) Elliptic trammel for blackboard construction.

(3) Elliptic billiard table.

(4) String models of the conics.

(5) Free hanging and loaded cables.

(6) Rolling wheel tracing a cycloid.

(7) Cycloidal and linear tracks showing the cycloid as tautochrone and brachistochrone.

(8) Involute of a circle and involute gear.

(9) Spiral of Archimedes and uniform motion cam.

(10) Limaçon and simple harmonic motion.

(11) Trisectrix of Archimedes.

At the afternoon session the regional governor, Professor W. C. Krathwohl of the Illinois Institute of Technology, presented an invited address on "Aptitude tests as aids to administrators and teachers of mathematics." Professor Krathwohl illustrated his remarks with tests given at the Illinois Institute of Technology. He showed how these tests are analyzed and used in the guidance of students at the Institute, and in placing them in positions later. He made it clear that such tests, when properly devised and administered, can be of great value to the educational institution, to the student, and to the employer.

At the luncheon meeting Professor Krathwohl discussed the general affairs of the Association, and reported on the September meeting of the Board of Governors.

C. J. Coe, *Secretary*

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-sixth Summer Meeting, New Brunswick, N. J., September 11–13, 1943.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

Allegheny Mountain, Pittsburgh, Pa., April, 1944
Illinois
Indiana
Iowa
Kansas
Kentucky
Louisiana-Mississippi, Ruston, La., 1943
Maryland-District of Columbia-Virginia
Metropolitan New York
Michigan
Minnesota
Missouri
Nebraska
Northern California, Berkeley, Jan. 29, 1944
Ohio
Oklahoma
Philadelphia, Philadelphia, Nov. 27, 1943
Rocky Mountain
Southeastern
Southern California
Southwestern
Texas
Upper New York State
Wisconsin

The Chauvenet Prize

In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICA established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association. Through two subsequent gifts the prize is now awarded every three years. The last award was made in December 1941 to Professor Saunders Mac Lane for his two papers in the *American Mathematical Monthly:* "Modular fields," volume 47, 1940, pp. 259-274 and "Some recent advances in algebra," volume 46, 1939, pp. 3-19.

As determined more recently by the Trustees, the prize is to be fifty dollars and is to be awarded for a noteworthy expository paper such as will come within the range of profitable reading of members of the Association. The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included; they carry their own reward.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

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AUGUST-SEPTEMBER 1943

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EARLE RAYMOND HEDRICK

W. B. FORD, University of Michigan

The announcement in a recent number of this MONTHLY of the death of Earle Raymond Hedrick brought a distinct shock and keen sense of loss to the entire fraternity of American mathematicians. He was an outstanding figure among us for more than a generation, his manifold activities and prodigious energy being in fact the marvel of his contemporaries. In particular, our Mathematical Association is deeply indebted to his efforts as he was prominently identified with the pioneer movements leading to its organization in 1915 and he served as its first president. Moreover, his counsel was ever afterward sought in whatever concerned its further development. But he was equally identified with many other interests affecting mathematics, these ranging over research, editorship, education in its various phases and the applications of mathematics, especially to engineering problems. And in recent years he became an administrative officer carrying great responsibilities in the conduct of one of America's leading universities. Such an exceptional career commands our admiration and respect and merits thoughtful consideration.

Earle Raymond Hedrick was born at Union City, Indiana, September 27, 1876 and was descended from early Pennsylvania settlers dating around the year 1670. His secondary education centered about the public schools in Ann Arbor, Michigan, from which he passed into the neighboring University of Michigan, graduating with degree A.B. in 1896. A year of high school teaching at Cheboygan, Michigan, immediately followed after which he proceeded to Harvard University for advanced study. Here his talents were soon recognized by both faculty and students. It was my fortune to have been one of these fellow students at the time and I can testify that Hedrick's keen mind was quite the envy of the aspiring young group of that period. While most of us followed the customary practice of taking notes during a lecture and felt obliged to work over them ponderously later, Hedrick seemed to grasp all instantly. He would ply the lecturer from time to time with inquiries indicating fine discriminations of thought such as arise from possible exceptional cases or bearings in related fields. To me at least his performance was little less then phenomenal.

I am sure that none of us was surprised when at the end of his second year he was awarded the Parker fellowship for study abroad and proceeded to what then seemed to be the world citadel of mathematics: Göttingen, with Klein and Hilbert. Here, as I learned in later years, he made much the same impression upon his associates as he had done at Harvard. Two years sufficed at Göttingen for him to finish his thesis, which was in the field of differential equations, and to receive the doctoral degree. Some months of the following year (1901) he spent at Paris attending lectures by Picard, Goursat, Hadamard and others. Then came his appointment as instructor of mathematics at the Yale Scientific School, a post he held for the following three years. Characteristic of the usual

EARLE RAYMOND HEDRICK

impression made by him upon others, one of his colleagues of that period told me in later years that to him Hedrick then seemed a "ball of fire."

In 1903 Hedrick was appointed professor of mathematics at the University of Missouri, a post which he held for twenty-one years except for an absence of six months in 1919 when he accompanied the American expeditionary force to France as director of the mathematical educational corps. Long before the end of this twenty-one year period his energy and talents had made a wide impression and met with generous recognition. This was evidenced by his election to numerous boards and administrative positions directing the mathematical activities of the country, also by his continued contributions to research, his editorship of an extended series of mathematical texts and his active participation in the general problems and discussions surrounding the teaching of mathematics. To this period belongs also his translation, in collaboration with Otto Dunkel, of the first two volumes of *Goursat's Cours d'Analyse*, widely used thereafter in all college circles. It was during this time that he took over the arduous duties of Editor in Chief of the Bulletin of the American Mathematical Society, an office to which, despite the time and labor involved, he gave unstinted effort for no less than seventeen years.

In 1924 Hedrick accepted the position of Professor and Head of the Department of Mathematics at the University of California at Los Angeles and in this capacity he continued until 1937. This period brought steady continuation of his ardent labors in the interest of mathematics and an ever increasing country-wide recognition of his ability and leadership. It brought, among other marks of distinction, his election to the presidency of the American Mathematical Society and to that of Vice President and Chairman of Section A of the American Association for the Advancement of Science. These years were characterized by much travel, largely transcontinental, taken by him in behalf of the many scientific organizations in which he was interested. Seldom was an important meeting of mathematicians held, however distant from California, at which it was not possible to find Hedrick's genial presence and valued counsel.

The year 1937 marked a signal change in Hedrick's career. From the position of Professor of Mathematics at the University of California at Los Angeles he became one of the two Vice Presidents and Provosts of the University of California. The merits involved in such a change speak for themselves. Thus he became the highest administrative officer of the University at Los Angeles. The creditable manner in which he discharged the duties of this high office is perhaps best evidenced by the following extract taken from resolutions recently adopted by the faculty of that University:

> "As a high administrative officer, he directed his full energies to actions which would enhance the reputation of the University in all its fields of activity. By performing manifold duties on the Los Angeles campus with wisdom and foresight, Dr. Hedrick exercised a great influence throughout the University and this influence extended to other universities throughout the country through his expert analysis of complex administrative matters."

Notwithstanding the duties attendant to his administrative office, Hedrick's underlying interest in mathematics continued and he was to be found scarcely less often than before at mathematical meetings. As the time approached for relinquishing active university service, friends sometimes asked what he intended to do after retirement. His reply was "Return to mathematics." Retirement came in due course in 1942 and the "return" came directly afterward through acceptance of a position as Visiting Professor of Mathematics at Brown University. It was anticipated that he would be instrumental in developing the Program of Advanced Instruction and Research in Mechanics upon which this institution had embarked a year earlier. In particular, he was to lend his talents in the inauguration of the new *Quarterly of Applied Mathematics* which is sponsored in connection with that Program. But scarcely had these duties been undertaken when illness intervened. Months of confinement followed, hopeful but doubtful, until the end came on February 3 of the present year.

Those who knew Hedrick best will miss most his genial companionship and sincere spirit of friendship. To all he leaves behind an enviable record of accomplishment.

MATHEMATICS, 400 B.C.–300 B.C.

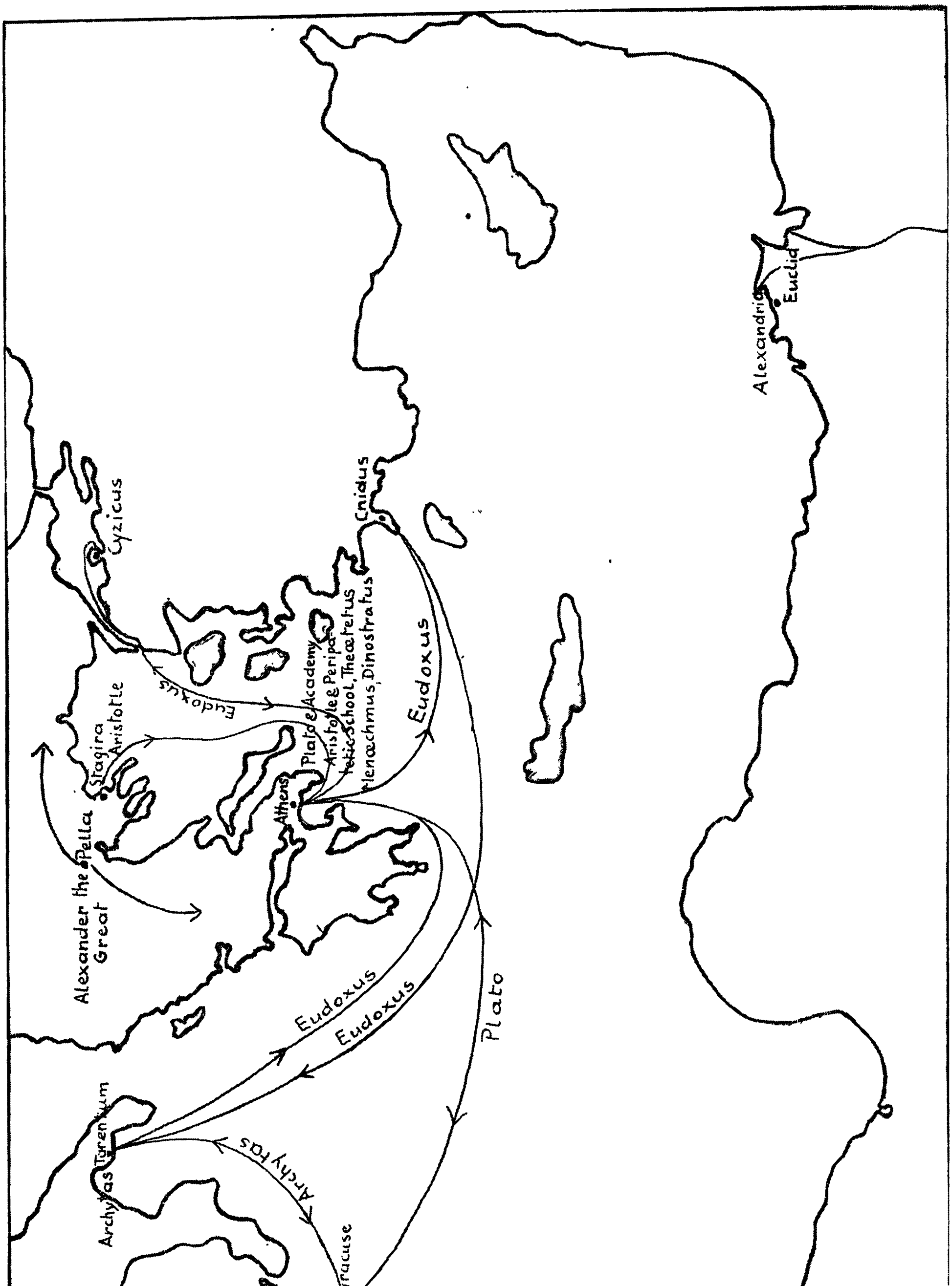
MAX DEHN, St. John's College

1. Survey of the century. The most important men of this period are Plato and Aristotle. They clarified the aims and methods of scientific work. Not only did they dominate the spiritual life of this era, but they have remained to the present day—at different times one more than the other—the leaders of all people struggling to find the truth and to order the world of phenomena.

We owe to this period the outstanding systematic work on mathematics by Euclid (ca. 300 B.C.). It was used as a textbook soon after it was written, superseding all other textbooks written before it. Euclid's work was the only textbook for the elements of mathematics everywhere until about one hundred and fifty years ago and is even used in some countries today (for example, England).

A little older than Euclid's *Elements* is the oldest mathematical treatise preserved in its original form—a work of Autolycus belonging to the domain of applied mathematics which describes the simplest phenomena of the movement of the stars as phenomena in the geometry of the sphere.

The Academy founded by Plato about 380 B.C. at Athens favored the study of mathematics. Important progress was made at the Academy in both mathematical method and mathematical knowledge. Typical scholarly work was done in the Peripatetic School founded by Aristotle about 350 B.C. at Athens. Eudemus, a member of this school, was the author of the first history of mathematics. For the larger part of this century Athens was the center of mathematics. However, at the end of the period, in line with political developments,



Alexandria
Euclid
Cnidus
Cyzicus
Eudoxus
Stagira
Aristotle
Alexander the Great
Pella
Athens
Plato's Academy
Aristotle & Peripatetic School, Theaetetus
Menaechmus, Dinostratus
Eudoxus
Eudoxus
Eudoxus
Plato
Archytas Tarentum
Archytas
Syracuse

Alexandria, the city founded in Egypt 331 B.C. by Alexander the Great, became an important cultural and especially mathematical center. Euclid taught here.

2. Mathematical reasoning. The philosophical discussions of this period led the mathematician to a higher level of consciousness of what he was doing. He became aware that the objects of his geometric research had no existence in the outer reality appearing to our senses. They are something between this reality and the realm of ideas to which such concepts as that of the integers belongs.

Further he was taught that it was his duty to formulate the foundations of his deductions, the definitions as well as the basic suppositions (the axioms). Also the form of the deduction itself was strongly influenced by the philosophical discussions.

The analytical method was introduced. This method starts with the assumption that the required construction has been carried out, and so leads to simpler figures easier to construct. Thereafter, one returns to the original problem of construction.

Connected with this method is the method of indirect proofs of theorems, which was probably already used in the first period, but now was in a certain sense the fashion in mathematical works. Aristotle laid the logical foundations upon which this method rests, the axiom of contradiction and the axiom of the excluded third. Both devices have turned out to be of great importance in modern discussions of mathematical methods.

A special case of the indirect proof appears in the method of exhaustion, indispensable for the proofs of theorems concerning areas not bounded by straight lines or volumes of general polyhedra (for example, pyramids).

3. Various achievements. A great achievement was the invention of a sound method of handling irrational ratios: this was accomplished by embedding them in the set of the rational ratios. This method is developed to a high degree of perfection in the Fifth Book of Euclid's *Elements*. The two mathematicians Theatetus of Athens and Eudoxus of Cnidos, both intimately connected with Plato, certainly contributed a great deal to this development.

To this time belongs, so far as we know, the discovery of the simplest properties of the conics. Menaechmus, also a follower of Plato, is believed to be the discoverer of all three types of conics, regarding them as the loci consisting of the intersection of a cone with a plane perpendicular to the generating line of the cone. Menaechmus used the hyperbola and the parabola simultaneously for the solution of the problem of doubling the cube, that is to construct $\sqrt[3]{2}$.

A little older than Menaechmus, Archytas of Tarentum used a three-dimensional construction for the duplication of the cube. He must have been a man of extraordinary scientific renown. The Roman poet Horace wrote a poem about him, but unfortunately associated his name incorrectly with the achievements of Archimedes.

Menaechmus' brother, Dinostratus, in squaring the circle by Hippias' quadratrix proved

$$\lim x \cot \frac{x\pi}{2r} = \frac{2r}{\pi}.$$

His proof is exact in the modern sense under the assumption that the quadratrix is a continuous curve.

4. Euclid's *Elements*. It is to the end of this period that we must assign the *Elements* of Euclid, probably written in Alexandria. We find there the greatest part of what nowadays is called elementary geometry. Some important elementary theorems, such as those concerning the intersection of the medians or the altitudes of a triangle, are not to be found there.

Euclid's final aim was obviously the metric theory of the regular solids. In this theory occur various irrational ratios. This gave Euclid the opportunity to build on a broad basis the theory of the domain of irrationals which contains as special cases the irrationals associated with the regular solids. Thus he first developed the theory of the whole numbers. Here is found the process for determining the greatest common divisor of two whole numbers. This process, called the Euclidean algorithm, dominates under many different guises both the elementary and the advanced theories of arithmetic and algebra. He developed also other theories of purely arithmetical interest. We cite as an example the theory of perfect numbers, which are defined by the property that each is equal to the sum of its divisors (*e.g.* 6, 28, 496). This theory has not made much progress since the time of Euclid.

Then follows the comprehensive theory of those irrationals which are generated by using, apart from addition and multiplication, the single or double extraction of a square root. Such irrationals occur in the metric theory of the regular solids. An example of this occurrence is found in the fact that the side of a regular pentagon is equal to $\sqrt{10-2\sqrt{5}}\, r/2$, where r is the radius of the circumscribed circle. We find here no attempt to determine rational approximants to these irrationals; Euclid's main concern was to determine algebraic relations between the occurring irrationals.

Of great importance is the introduction of the postulate of parallels in the beginning of Euclid's work. It is known from remarks of Aristotle that the theory of parallels worried mathematicians. The introduction of a theorem about parallels as a postulate was an audacious device. It made possible the rigorous construction of this geometry, but caused much trouble to mathematicians through the ages until modern times.

Beside the *Elements*, Euclid wrote other works which are for the greatest part only preserved in fragments. Of importance for this review is his book on *Porisms*, some fragments of which we find with Pappus' work. (Pappus lived more than five hundred years after Euclid.) In this work, Euclid probably approached problems concerning functions, especially linear functions and their geometric equivalents as embodied in straight lines, circles, and pencils of straight lines passing through a common point.

APPLIED MATHEMATICS AND THE PRESENT CRISIS

R. G. D. RICHARDSON, Brown University

1. Introduction. It may be useful at this time to make certain inquiries regarding the present status of applied mathematics in America and the tendencies which should be fostered for the future. This great nation, now in the throes of war, finds startling deficiencies in its material, intellectual, and spiritual resources. There is scarcity of tin and chromium; of basic scientific knowledge in fields like aeronautics; of comprehension of what part America should play in establishing a world order. We are caught in a predicament which gives us serious food for thought in contemplating the long future. Is it not our duty as mathematicians to give some aspects of these matters most searching consideration?

Why has research in the applications not kept pace with that in pure mathematics? Is the present a strategic moment for a significant advance in that field? Is there an obligation on American Science to assume a greater share of the world's progress in this particular sector?

Other problems come to mind. In terms of educational policy after the war is over, what will result as a residue of the present participation by universities in war programs? Should mathematicians consider some more comprehensive sort of organization of their interests? The physicists have their Institute which is proving markedly effective during the war. The mathematicians are well organized for publication and research but not on the promotional side; in the present crisis, Washington found that it had to turn to physicists and others for some of the help which we would have furnished had we been more completely organized.

There is at the moment a marked trend toward increasing the group of professional mathematicians whose work lies in government agencies and industrial fields rather than in teaching. When the proper time comes, can we define the word "mathematician" in such a connection so that it has a meaning in the scientific world?

2. The deficiency in Applied Mathematics. Of those mathematicians whose names are starred in *American Men of Science*, the number who are now working in the field of applications is almost negligible. The percentage has decreased with each of the six issues and the new list to appear soon will not change this picture. Only one man (E. W. Brown) interested primarily in applied mathematics has been elected to the presidency of the American Mathematical Society since 1900, though before that time nearly all the presidents were in that field. An examination of the list of doctorates conferred in the past few years shows a very small percentage (certainly not more than 8%) in the applications. The great majority of our leading institutions make no provision for carrying students to the doctorate in applied mathematics. A scrutiny of American scientific journals will show only a small (though happily increasing) number of investiga-

tions into the fundamental reaches of this sector. To be sure, mathematical physics and statistics are well represented in America; investigations in electrodynamics and thermodynamics have achieved considerable success; but in the fields of mechanics (including such branches as fluid dynamics, elasticity, and plasticity), acoustics, and optics, the names are few and mostly of those of foreign birth.

3. A surprising situation. This deficiency is surprising in a nation that considers itself practical, above all. American engineering in many of its aspects has achieved a high level of performance. In physics and in astronomy America has built instruments and achieved results which are the marvel of the world. For these reasons, it would be hard to bring it home to the layman that America has had to rely largely, and still is relying largely, on foreign countries to build up the bases on which many of the advances in our mechanical engineering rest. But that such is the fact can easily be verified by any scientist.

It is surprising, too, because American research before 1890 was weighted toward the applications. Gibbs with his epoch-making theories which form the bases for much of modern physical chemistry; Newcomb in mathematics and astronomy—in his day the best known of American scientists internationally; G. W. Hill, in mathematical astronomy, who brought renown to this country: it would seem that these products of American scholarship might well have influenced American mathematics to proceed in quite another direction from that which it actually took.

In 1900, physics and engineering in America were essentially descriptive and there were opportunities to influence the theoretical development of these subjects. These opportunities were unfortunately neglected by the mathematicians; the physicists and engineers themselves proceeded to develop the mathematics underlying their respective disciplines and with such marked success that a large proportion of the applied mathematicians of the present day are the result of their training. Moreover, physicists have cultivated almost exclusively such sectors as wave mechanics, statistical mechanics, cosmic astrophysics; and the resulting researches have been printed in physics publications and only rarely in mathematical journals. Since many of these investigations are essentially mathematical in character and have little reference to experimentation, this development strikes one not only as unfortunate but as bizarre.

4. Causes of the deficiency. In the period since 1900, America has made unprecedented strides in mathematical research as a whole. Nevertheless, while this country is now easily the world leader in many branches of pure mathematics and stands high in some divisions of applied mathematics, it lags far behind in some other divisions. In Applied Mechanics there are a few excellent men, but they are widely scattered throughout the industries and the universities. For a quarter of a century the deficiency in this direction has been realized by the leaders of mathematical thought in the country and attempts have been made by two or three universities to build up a continuing school where young

men interested in certain fields of applied mathematics can get the very best broad training. The Massachusetts Institute of Technology, the University of Wisconsin, and a few other institutions have over a long period cultivated the applications; and more recently California Institute of Technology has entered this field. Industrial concerns, in particular the Bell Telephone Laboratories, have assembled groups of men skilled in applied mathematics who have made noteworthy contributions. But for one reason or another (probably in part because of the tradition built up by the mathematicians who went early to Germany for training, in part because of lack of consistent coöperation between universities, and in part because of lack of funds), these various attempts to build up strong departments of applied mathematics in the universities have not achieved complete success and nothing outstanding as a center of instruction in this field has persisted. An able presentation of these facts was made by Thornton C. Fry of the Bell Telephone Laboratories in his 1940 report on mathematics in industry, prepared under the auspices of the *Committee on Survey of Research in Industry*.* This report has had wide circulation and has evoked strong expressions of approval.

As the result of the influence of European mathematicians, including Sylvester, Klein, Kronecker, and Hilbert, the prospective leaders of American mathematics—such as Moore, White, Osgood, Bôcher, Van Vleck, and Pierpont—when they came back to America after having studied abroad during the period 1890-1910, brought with them an enthusiasm for rigor which was to galvanize instruction and to develop this country into a leader in pure mathematics. But no one with enough influence was able to transport the ideas seething in the mind of Felix Klein, who realized that if Germany was to be strong, the country would have to foster institutes of applied mathematics like that in Applied Mechanics at Göttingen established under his influence.

There are many reasons why young men have gone largely into pure rather than applied mathematics. In the first place, there has been an unhappy cleavage between the groups of leaders in the two fields, to the disadvantage of both. In the second place, it is probably fair to state that it is more difficult to do significant mathematical research in applied mathematics than it is in some of the fields of pure mathematics. Furthermore, our system of mathematics in the high schools and colleges does not prepare the minds of prospective mathematicians for interest in applications in the field of mechanics, as does, for example, the British system. In the American high schools, mathematics has been divorced from physics to the detriment of both. Attitudes of mind formed in the adolescent period are not easy to change. College teaching has fostered this divorce. The exigencies of the war are changing this attitude, but it remains to be seen if the impetus given to the fusion of these disciplines will persist.

In Germany, Britain, and France there have been at least half a dozen institutions where applied mathematics was sedulously cultivated. For example,

* Research—A National Resource—II, Section VI, Part 4, pp. 268–288—A House Document, 77th Congress. Printed also by the Bell Telephone Laboratories.

there are great research institutes for aeronautics in Germany where even before the war many hundreds of persons (mathematicians, engineers, etc.) were concentrated. To illustrate the trend, it might be pointed out that since the beginning of the present war a new Institute has been established in Göttingen to study a particular branch of aeronautics known as Problems of Unsteady Flow and it has issued numerous important memoirs. There are other great German centers also to which engineering problems that involve mathematics can be sent and where both the theoretical and the experimental phases can be explored. These centers are adequately supported by the government and the industries. We have nothing in America of precisely the same kind. It is true that there are universities and technical schools where the highest grade of engineering instruction is carried on and where research is actively prosecuted. It is true also that there have been Research Institutes to which practical engineering problems on a somewhat lower level are assigned by industrial organizations. Furthermore, in a host of private industrial laboratories there has been high-grade research carried on. And to the extent that personnel is available, these activities are being expanded for the war.

But there has been in America almost no institution to which a young man could go for the broadest training in the advanced reaches of mathematics applied to engineering, and in which he could catch the spirit of research and learn the necessary techniques. Probably there should eventually be several institutions of this sort in this vast country; but at present there are not enough mathematicians expert in this applied field to staff any considerable number of such schools. Besides, there are decided advantages in concentrating intensive instruction in a very few institutions until strong centers are established; this seems especially true in the present emergency.

5. Effect of the emergency. The war has greatly intensified the need for remedying America's inadequacies in industrial mathematics. It has made the most striking demands upon a host of industries related to war activities and has simultaneously isolated us from communication with other centers of leadership in this field. Even the British institutions where applied mathematics is cultivated are virtually shut off from us by the enormous demands made upon them by their own country.

Thus the normal needs for orderly industrial development are greatly augmented by the present national emergency. Adequate exploitation of aerodynamics and other fields bearing directly upon defense activities awaits the basic work of mathematicians. A program must have the double purpose of serving the nation's immediate war needs and of pointing the way to a means for attacking some of the more difficult problems of organization of engineering research which lie ahead. America cannot afford to lag behind either now or in the period of reconstruction when competition is bound to be of the very keenest. We should be ready to assume our rightful place in this science as in others; South America, Asia and, to a lesser extent, Europe must look to us for leadership.

6. The growing realization of need. There are many persons influential in American science who have been for some time convinced that greatly increased attention should be given to the applications; that only as theory and practice stimulate and supplement one another can either achieve permanent strength worthy of our nation. The large influx of foreigners during the past two decades has brought to our shores some outstanding figures in the applications to Mechanics; we may cite Friedrichs, Den Hartog, Kármán, Mises, Prager, Reissner, and Timoshenko. These and others form a nucleus for instruction and research which we trust betokens a far-reaching development.

The country long since has passed the stage when it is necessary to demonstrate the vital importance of fundamental scientific research in agriculture and industry. It has been a question, however, as to whether the nation would support the rapid expansion of research in a sector such as Mechanics until it could visualize clearly the possibility of adding not only to theoretical knowledge, but also to practical application. The war is increasingly proving that such contributions can be made; mathematics is serving the spectacular developments taking place in aeronautics and other war endeavors. Problems in ballistics are enlisting groups of mathematicians at Aberdeen and elsewhere; radio research makes its calls to the Radiation Laboratory and to other centers; the Taylor Model Basin has mathematical problems in ship construction; Langley Field has enlisted a number of men to work on urgent questions in aeronautics; this list could be greatly expanded. The call for men of ability and adaptability in all of these fields greatly exceeds the supply.

A careful study of the need for Applied Mathematics was made in the summer of 1941 by a committee* consisting of eminent scholars and administrators: Marston Morse, of the Institute for Advanced Study, President of the American Mathematical Society, Chairman; M. J. Kelly, Research Director of the Bell Telephone Laboratories; G. B. Pegram, Dean of the Graduate School, Columbia University; Theodore von Kármán, Director of the Aeronautics Laboratory, California Institute of Technology; Warren Weaver, Director for Natural Sciences, Rockefeller Foundation. The findings of this committee reinforced the report of Fry and made important constructive suggestions for remedying the situation. It was pointed out that the critical situation called for something more than the ordinary evolution of educational methods in this sector. Since this report was written, America has entered the war and the recommendations of the committee would doubtless now be sharpened and intensified.

7. The objectives of a new development. The United States has for years been training an enormous number of undergraduate engineers and these men prove a highly useful factor in the development of our resources. On the other hand, compared with a country like Germany, there are relatively few engineers

* See an article in the American Journal of Physics, Vol. 11, No. 2, 67–73, April 1943, where more details of that investigation are given. See also an article by Rufus Oldenberger entitled Pure and Applied Mathematics in The Journal of Engineering Education, January, 1943, pp. 432–437.

with a training as extensive as that of men in other professions. It is fair to compare the training of engineers with that of physicians, since both professions demand a theoretical and a practical knowledge, which are obtained by a combination of university instruction and practical experience under guidance. The emergency reveals how inadequate are our reserves of personnel with three or four years of graduate training in engineering. If America is to compete on the higher levels of research in engineering, it seems clear that more of our ablest young men should be getting additional training of a graduate school character. Extensive grounding in classical physics and applied mathematics must be combined with a first-hand knowledge of some important practical problems. A prospective mathematical engineer would still have to acquire practical experience, just as a physician must serve as an interne; but he would be able to progress rapidly to problems beyond the mere routine.

Men teaching mathematics in engineering schools should have an understanding of, and sympathy with, the outlook of their colleagues in the engineering faculties. Many of them should have had their work for the doctorate primarily in applied mathematics. Moreover, every candidate for the degree of Ph.D. in mathematics should have a modicum of training in applied mathematics just as he should be prepared in analysis, in geometry, and in algebra. In a country reputed to have a genius for material development, such suggestions seem very moderate. Participation in courses of high grade in the applied field would aid in giving the proper attitude to those not concerned in delving deeply themselves into this area.

The student body for such a venture must consist of men who have a flair for the practical in science as well as for the theoretical. Talented men must be enlisted at the end of their college careers as well as at later stages of their scientific development. They should exhibit unusual aptitude for the physical sciences as well as all-round competence. One potent reason why more of the abler neophytes graduating from engineering schools have not proceeded to further study is the competition of the industries; in order to enlist such men it is necessary to have adequate funds for fellowships.

8. A new journal. At various times during the last fifteen years plans for a new periodical in the field of applied mathematics have been drawn up; but, for one reason or another, no decisive step was taken until recently. Much of the literature has been in foreign languages and this proves a bar in the case of many engineers; it has seemed clear that a larger proportion of current publication in applied mathematics must be in the English language. There are many articles published in physics and engineering journals that have significant mathematical content; this is particularly true in the *Journal of Applied Mechanics*. Nevertheless, there has been a need for an additional periodical devoted to research involving advanced mathematical treatment of engineering problems. Working with representatives of journals in allied fields, a group of leaders has inaugurated the *Quarterly of Applied Mathematics*. Its welcome by *Mechanical Engi-*

neering is expressed in a recent editorial.* The new Quarterly has enlisted a staff of editors and collaborators of such high quality as to inspire confidence in its future. The first number has evoked much commendation.

9. Activities of American mathematicians relating to the war. It will be recalled that the American Mathematical Society and the Mathematical Association of America appointed a joint Committee on War Preparedness in September, 1939, only a few days after war was declared in Europe. Much effort has been given to planning the utilization of our national mathematical resources. Advocates of the Kilgore Bill now pending in Washington claim that science is poorly organized for assistance to the war effort and that altogether too small a fraction of scientists are enlisted. On the other hand, W. L. Lawrence, writing in the New York *Times* for January 3, 1943, states that 87% of mathematicians engaged in research have turned their attention to war work; the truth of the matter probably lies between these two extremes. As individuals, mathematicians have in general responded promptly to the call for assistance in the emergency. More than one hundred have left their teaching positions to give full time to research in war problems; an equal number are in uniform, engaged in research or giving instruction to the armed forces; probably as many more are engaged part-time in active investigation for the government or industry; and the great bulk of the remainder are teaching men in college programs in connection with the armed services. All this emphasis on the practical side of mathematics cannot fail to leave an impression on future instruction. The profession needs wise leadership in determining what permanent influence this diversion of effort will leave behind. It would be folly to let the pendulum swing back to the opposite pole and abandon the present gains made in scientific prestige by our participation in practical applications. We have resources ample enough to cultivate many diverse interests, including various branches of applied, as well as of pure, mathematics.

10. Professional mathematicians not in the academic field. Today the term "engineer" or "chemist" denotes a certain proficiency at the professional level; tomorrow the designation "physicist" will be similarly introduced and standardized and the day after that, the appellation "mathematician." While there will never be a call comparable to that for physicists or chemists, the need for such professional mathematicians, already considerable, will grow as the complexity of industrial research increases. What can we as mathematicians do to direct into the proper channel men who have talent for the applications of mathematics to the development of engineering? How can standards of competence be set and maintained? What mark of certified approval can be adopted by a group of institutions preparing such men?

11. New developments at Brown University. As a modest contribution to the amelioration of the situation, Brown University inaugurated in June, 1941 a

* May, 1943, p. 312.

program of Advanced Instruction and Research in Mechanics, which is now in its seventh term. This effort is supported by the ESMWT of the U. S. Office of Education and the Carnegie Corporation; there is also a liberal grant for fellowships from the Rockefeller Foundation.

For at least two or three years previous to 1941, there had been at Brown University a conviction that America could not avoid war, that physics and engineering and the underlying mathematics were bound to play a leading part in its prosecution, and that steps should be taken to strengthen one of the weakest links in the American scientific chain. The immediate stimulus of this enterprise was, however, the Fry report cited above.

It is instructive to consider the original objectives of this effort. To be sure, these at the moment are to a considerable extent obscured by the fact that the onset of war has made it essential that the program for the bulk of the participants should be abbreviated and speeded up. Among the basic ideas originally laid down were the improved recruitment and training of applied mathematicians and a stimulation of the interest in applied mathematics among mathematicians and engineers. The primary object was the enlisting of very high grade men and giving them thorough and deep training for careers as teachers of applied mathematics, that training to include practical experience in experimental sciences. This would involve three or four years of graduate study of which perhaps one year would be in practical work. An original subsidiary aim, thrust more prominently into the foreground by the war, was the preparation of a small number of persons for employment as mathematicians in industry and government agencies. From this group a smaller number can be expected to extend the frontiers of knowledge by new developments arising out of the practical problems on which they work. This latter phase of the plan is proving to have real significance in the progress of American science.

During the first session of the program, the University appointed an Evaluating Committee to consider the problem from a national standpoint. The report of this committee, with Marston Morse as chairman, is referred to above. Its main recommendations have been followed.

The main group of courses* is centered around Mechanics, which term includes advanced work in fluid dynamics, elasticity, plasticity, aerodynamics, theory of vibrations, theory of structures, and so forth. Other fields of applied mathematics are being included from time to time.

The staff of half a dozen or more is recruited from the ablest men available. The average number of students in attendance has been in excess of fifty, the enrollment in the summer session being greater than that during the academic year. One quarter were already in possession of the doctorate. The clientele ranges from students who have just completed their undergraduate training in mathematics, physics, or engineering to mature men who wish to change their fields of research from pure to applied mathematics. Of the 200 who have en-

* A pamphlet listing the courses will be sent on a written request addressed to the Graduate School, Brown University.

rolled, 40 have gone into research in government agencies concerned with aeronautics, ship construction, gun construction, radar, etc.; 20 are engaged in research in industries connected with the war; 10 have special assignments in connection with the armed forces; 25 are continuing as engineers in industry; the remainder are instructors and graduate students in universities.

More than 25 research papers have been completed and others are under way; some deal with immediate practical problems which have arisen in the prosecution of the war, while others are of a more fundamental character. As a significant feature of the program, a limited number of research problems are being investigated for government agencies and war industries.

Besides the instruction in lectures and seminars, there have been numerous single talks and short series of lectures by visiting experts, as well as conferences on topics of current research interest (Non-Linear Mechanics, Ballistics, etc.). As a by-product of the instruction, the School has mimeographed ten series of lectures which, without any advertising, have been widely distributed to industries and universities. Some of the lectures have been put into book form and others will be eventually. Such treatises will prove to be of real significance to future progress in their respective fields. The School is coöperating in preparing a resumé of some of the excellent recent work of the Russians in the field of Mechanics; this will probably be published soon by the Taylor Model Basin and thus made available to engineers.

The founding of a new journal has been referred to above. Brown University has made a liberal subvention; there are already eight hundred subscribers. After an initial period, it is hoped that the journal will become practically self-supporting.

The experiment at Brown University is a lively one; it has already evoked much interest, and it permits a hope that it will prove to be a contribution to the advancement of American science.

GEODESICS AND PLANE ARCS ON AN OBLATE SPHEROID

L. E. WARD, University of Iowa

1. Introduction. Current interest in navigation has suggested the following question. Let P_1 and P_2 be two points on an oblate spheroid, s_{12} the length of the geodesic connecting them, and σ_{12} the length of the elliptic arc P_1P_2 which is the intersection with the spheroid of the plane determined by P_1, P_2, and the center of the spheroid. If it is assumed that P_1 and P_2 are not on the same meridian or on the equator, by how much will σ_{12} exceed s_{12}?

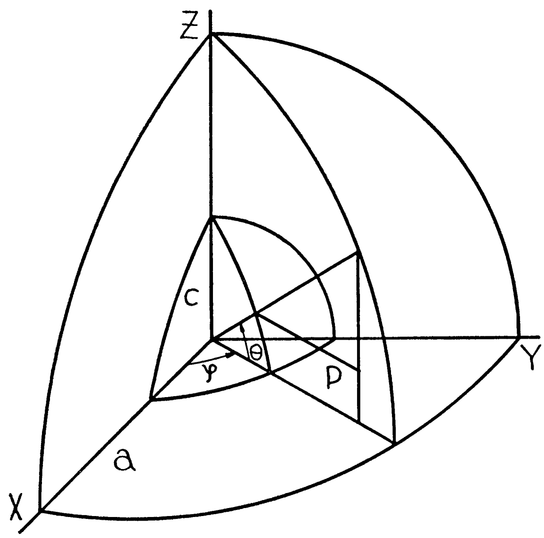
This question will be discussed by expressing the two lengths in question in terms of the coordinates of P_1 and P_2, and the semi-major axis and eccentricity of the spheroid. With the idea of applying the results to the earth, the eccentric-

ity will be regarded as the principal infinitesimal, and the first few terms of the expansions of s_{12} and σ_{12} as series in positive integral powers of the eccentricity will be found. It will be shown that $\sigma_{12}-s_{12}$ is an infinitesimal of order four referred to the eccentricity, a result believed to be new. Finally, a numerical example will be considered.

2. The geodesic arc. The angles θ and ϕ indicated in the figure will be used as parameters. In terms of these the parametric equations of the spheroid are

$$x = a \cos\theta \cos\phi, \qquad y = a \cos\theta \sin\phi, \qquad z = c \sin\theta,$$

where a and c are the semi-major and semi-minor axes respectively. It is to be noted that the latitude λ of a point on the spheroid is not equal to θ in general, but is connected with θ through the relation $a \tan\lambda = c \tan\theta$.



The length of any arc on the spheroid is given by the integral

$$\int \sqrt{(a^2 \sin^2\theta + c^2 \cos^2\theta)d\theta^2 + a^2 \cos^2\theta d\phi^2}.$$

It is found convenient to use θ as the variable of integration, so that an arc length can be expressed as the integral $\int L d\theta$, where

$$L = \sqrt{a^2 \sin^2\theta + c^2 \cos^2\theta + a^2\phi'^2 \cos^2\theta}, \qquad \phi' = d\phi/d\theta.$$

Along geodesics Lagrange's equation

$$\frac{d}{d\theta}\frac{\partial L}{\partial \phi'} - \frac{\partial L}{\partial \phi} = 0$$

must be satisfied. Since $\partial L/\partial\phi=0$, a first integral of Lagrange's equation is $\partial L/\partial\phi'=\text{const.}$, namely,

$$\phi' \cos^2 \theta = hL, \tag{1}$$

where h is the constant of integration. This equation can be written

$$d\phi = \frac{ah}{\cos\theta}\sqrt{\frac{1-e^2\cos^2\theta}{\cos^2\theta - a^2h^2}}\,d\theta, \tag{1$'$}$$

where $e^2=(a^2-c^2)/a^2$ is the square of the eccentricity of the spheroid.

We may suppose that Cartesian axes have been selected so that P_1 has $\phi=0$ and $\theta=\theta_1\geqq 0$ for its coordinates. Further, we may suppose P_2 located so that ϕ increases monotonically along the geodesic from P_1 to P_2. We will suppose also that P_2 is so located that along this geodesic θ increases, at least at first; this so that in (1$'$)h shall be positive.

Let us measure the arc s of the geodesic so that $s=0$ at P_1 and s increases with ϕ. This makes $ds/d\phi>0$. Since $L=ds/d\theta$, (1) can be written $ah=a\cos^2\theta d\phi/ds$; and since $ds\geqq a\cos\theta d\phi$, it is clear that $0\leqq ah<1$. Hence there exists an acute angle α such that $\cos\alpha=ah$. In terms of α we may write (1$'$) in the form

$$d\phi = \frac{\cos\alpha}{\cos\theta}\sqrt{\frac{1-e^2\cos^2\theta}{\cos^2\theta-\cos^2\alpha}}\,d\theta. \tag{2}$$

If along any portion of the geodesic θ decreases when ϕ increases, the right hand member of (2) must be replaced by its negative when (2) is applied to such portion.

Equation (2) shows that ϕ and θ continue to increase together until $\theta=\alpha$, at which location $d\theta=0$. This means that at $\theta=\alpha$, the geodesic is tangent to a parallel of latitude. For a certain distance beyond $\theta=\alpha$ the angle ϕ continues to increase while θ decreases. Since in the right hand member of (1$'$) we may replace θ by $-\theta$ without altering the coefficient of $d\theta$, we infer that the geodesic is symmetric about the plane of the equator. Accordingly the geodesic undulates between the parallels of latitude $\theta=\alpha$ and $\theta=-\alpha$, being successively tangent to one and then to the other.

The relation between ϕ and θ along the geodesic is

$$\phi = \int_{\theta_1}^{\theta}\frac{\cos\alpha\sqrt{1-e^2\cos^2\psi}}{\cos\psi\sqrt{\cos^2\psi-\cos^2\alpha}}\,d\psi, \tag{3}$$

and this holds for $\theta_1\leqq\theta\leqq\alpha$. On the next phase of the curve we must write

$$\phi = \int_{\theta_1}^{\alpha}\frac{\cos\alpha\sqrt{1-e^2\cos^2\psi}}{\cos\psi\sqrt{\cos^2\psi-\cos^2\alpha}}\,d\psi + \int_{\theta}^{\alpha}\frac{\cos\alpha\sqrt{1-e^2\cos^2\psi}}{\cos\psi\sqrt{\cos^2\psi-\cos^2\alpha}}\,d\psi. \tag{4}$$

The change in ϕ when θ increases from θ_1 to α, $\phi_{1\alpha}$, is obtained from (3) by taking

$\theta=\alpha$. If in the resulting integral the change of variable $\sin\psi=\sin\alpha\cos\psi'$ is made, the formula is obtained

$$\phi_{1\alpha} = \frac{\sqrt{1-e^2\cos^2\alpha}}{\cos\alpha}\int_0^{\beta_1}\frac{\sqrt{1-k^2\sin^2\psi}}{1+\tan^2\alpha\sin^2\psi}\,d\psi,$$

in which

$$\beta_1 = \operatorname{arc\,cos}\frac{\sin\theta_1}{\sin\alpha} \quad\text{and}\quad k^2 = \frac{e^2\sin^2\alpha}{1-e^2\cos^2\alpha}\cdot$$

Let $s_{1\alpha}$ designate the length of the geodesic arc from P_1 to the first point where this arc is tangent to the circle $\theta=\alpha$. It is easily found that

$$\begin{aligned} s_{1\alpha} &= a\sqrt{1-e^2\cos^2\alpha}\int_0^{\beta_1}\sqrt{1-k^2\sin^2\psi}\,d\psi \\ &= a\sqrt{1-e^2\cos^2\alpha}\left[\beta_1-\tfrac{1}{4}e^2\sin^2\alpha(\beta_1-\sin\beta_1\cos\beta_1)+\cdots\right] \\ &= a\left[\beta_1-\tfrac{1}{4}e^2(\beta_1+\beta_1\cos^2\alpha-\sin^2\alpha\sin\beta_1\cos\beta_1)+\cdots\right], \end{aligned} \tag{5}$$

where in each case the terms not written are of the fourth and higher orders in e.*

For convenience we regard α as given and choose $P_2(\phi_2, \theta_2)$ on the geodesic whose equation is (3). If P_2 is on the arc $s_{1\alpha}$, we have $s_{12}=s_{1\alpha}-s_{2\alpha}$, while if P_2 is on the extension of this arc beyond $\theta=\alpha$, but not too far along, $s_{12}=s_{1\alpha}+s_{2\alpha}$. We shall deal further with the latter situation, so that, from (4),

$$\phi_2 = \left[\int_{\theta_1}^{\alpha}+\int_{\theta_2}^{\alpha}\right]\frac{\cos\alpha\sqrt{1-e^2\cos^2\psi}}{\cos\psi\sqrt{\cos^2\psi-\cos^2\alpha}}\,d\psi.$$

If this integrand be expanded in powers of e and integrated, the series for ϕ_2 is obtained,

$$\phi_2 = \gamma_1+\gamma_2-\tfrac{1}{2}e^2(\beta_1+\beta_2)\cos\alpha+\cdots, \tag{6}$$

where

$$\gamma_i = \operatorname{arc\,cos}(\tan\theta_i/\tan\alpha), \qquad i = 1, 2.$$

By analogy with (5) a series for $s_{2\alpha}$ can be written, and the formula for s_{12} is obtained by addition.

$$\begin{aligned} s_{12} = a\bigl[\beta_1+\beta_2-\tfrac{1}{4}e^2\{&(\beta_1+\beta_2)(1+\cos^2\alpha) \\ &-\sin^2\alpha(\sin\beta_1\cos\beta_1+\sin\beta_2\cos\beta_2)\}+\cdots\bigr]. \end{aligned} \tag{7}$$

3. The elliptic arc. The equation of the plane of the elliptic arc can be written as $Ax+By+Cz=0$ or

$$Aa\cos\phi+Ba\sin\phi+Cc\tan\theta=0. \tag{8}$$

* In the infinite series to be used herein terms of order greater than 2 in e will be systematically omitted.

In order that this plane shall contain the points P_1 and P_2, the coefficients A, B, C may be taken to be $A=c\sin\phi_2\tan\theta_1$, $B=c(\tan\theta_2-\tan\theta_1\cos\phi_2)$, $C=-a\sin\phi_2$.

From (8) we find

$$\begin{aligned}\sin\phi &= -\left[A\sqrt{a^2(A^2+B^2)-c^2C^2\tan^2\theta}+cBC\tan\theta\right]/a(A^2+B^2),\\ \cos\phi &= \left[B\sqrt{a^2(A^2+B^2)-c^2C^2\tan^2\theta}-cAC\tan\theta\right]/a(A^2+B^2),\\ d\phi &= cC\sec^2\theta d\theta/a(A\sin\phi-B\cos\phi),\end{aligned}$$

whence

$$d\phi^2=\frac{c^2C^2\sec^4\theta}{a^2(A^2+B^2)-c^2C^2\tan^2\theta}d\theta^2.$$

Thus the square of the element of the elliptic arc is

$$ds^2=\left[a^2\sin^2\theta+c^2\cos^2\theta+\frac{a^2c^2C^2\sec^2\theta}{a^2(A^2+B^2)-c^2C^2\tan^2\theta}\right]d\theta^2. \tag{9}$$

It is convenient to introduce the angle θ', which is defined to be the maximum of θ along the elliptic arc P_1P_2. This angle is given by $cC\tan\theta'=-a(A^2+B^2)^{1/2}$. It is readily found that

$$\sin\theta'=\left[\frac{\tan^2\theta_1-2\tan\theta_1\tan\theta_2\cos\phi_2+\tan^2\theta_2}{\tan^2\theta_1-2\tan\theta_1\tan\theta_2\cos\phi_2+\tan^2\theta_2+\sin^2\phi_2}\right]^{1/2}. \tag{10}$$

In preparation for the expansion of $\sin\theta'$ in powers of e, we find, using (6)

$$\begin{aligned}\sin\phi_2 &= \sin(\gamma_1+\gamma_2)-\tfrac{1}{2}e^2(\beta_1+\beta_2)\cos\alpha\cos(\gamma_1+\gamma_2)+\cdots,\\ \cos\phi_2 &= \cos(\gamma_1+\gamma_2)+\tfrac{1}{2}e^2(\beta_1+\beta_2)\cos\alpha\sin(\gamma_1+\gamma_2)+\cdots.\end{aligned}$$

These are substituted into (10). If the relation

$$\tan^2\theta_1-2\tan\theta_1\tan\theta_2\cos(\gamma_1+\gamma_2)+\tan^2\theta_2=\tan^2\alpha\sin^2(\gamma_1+\gamma_2),$$

which is easily verified, is made use of, and the resulting expression expanded into a series in powers of e, the useful formula

$$\sin\theta'=\sin\alpha-\frac{e^2(\beta_1+\beta_2)\cos^3\alpha\sqrt{(\sin^2\alpha-\sin^2\theta_1)(\sin^2\alpha-\sin^2\theta_2)}}{2\sin\alpha\cos\theta_1\cos\theta_2\sin(\gamma_1+\gamma_2)} \tag{11}$$

is obtained. We note in passing that (11) shows that $\theta'<\alpha$.

In terms of θ' equation (9) becomes

$$ds^2=a^2\frac{1-e^2(\sin^2\theta'-\sin^2\theta)}{\sin^2\theta'-\sin^2\theta}\cos^2\theta d\theta^2.$$

Accordingly the length of the elliptic arc P_1P_2 is

$$\sigma_{12}=a\left[\int_{\theta_1}^{\theta'}+\int_{\theta_2}^{\theta'}\right]\sqrt{\frac{1-e^2(\sin^2\theta'-\sin^2\theta)}{\sin^2\theta'-\sin^2\theta}}\cos\theta d\theta.$$

If $\kappa = e \sin \theta'$ and ψ_1 and ψ_2 are defined by

$$\sin \theta' \cos \psi_1 = \sin \theta_1, \qquad \sin \theta' \cos \psi_2 = \sin \theta_2, \tag{12}$$

and if the variable of integration is changed in accordance with the equation $\sin \theta = \sin \theta' \cos \psi$, we find

$$\sigma_{12} = a\left[\int_0^{\psi_1} + \int_0^{\psi_2}\right]\sqrt{1 - \kappa^2 \sin^2 \psi}\, d\psi, \tag{13}$$

which is in the standard form of an elliptic integral for an elliptic arc.

Expanding the integrand in (13) and integrating term by term yields

$$\sigma_{12} = a[\psi_1 + \psi_2 - \tfrac{1}{4}e^2 \sin^2 \theta' \{\psi_1 + \psi_2 - (\sin \psi_1 \cos \psi_1 + \sin \psi_2 \cos \psi_2)\} + \cdots]. \tag{14}$$

In order to compare this result with (7) the series in powers of e for ψ_1 and ψ_2 are needed.

From (12) and (11) we find

$$\begin{aligned}\cos \psi_i = \frac{\sin \theta_i}{\sin \alpha} &+ \frac{e^2(\beta_1 + \beta_2) \cot^3 \alpha \tan \theta_i}{2 \cos \theta_{3-i} \sin (\gamma_1 + \gamma_2)} \\ &\cdot \sqrt{(\sin^2 \alpha - \sin^2 \theta_i)(\sin^2 \alpha - \sin^2 \theta_{3-i})} + \cdots, \quad i = 1, 2.\end{aligned} \tag{15}$$

Hence

$$\psi_i = \beta_i - \frac{e^2(\beta_1 + \beta_2) \cos^3 \alpha \tan \theta_i}{2 \sin^2 \alpha \cos \theta_{3-i} \sin (\gamma_1 + \gamma_2)}\sqrt{\sin^2 \alpha - \sin^2 \theta_{3-i}} + \cdots, \quad i = 1, 2, \tag{16}$$

and

$$\sin \psi_i = \sin \beta_i + \cdots, \qquad i = 1, 2, \tag{17}$$

no additional terms being needed in (17).

From (16) we obtain

$$\psi_1 + \psi_2 = (\beta_1 + \beta_2)(1 - \tfrac{1}{2}e^2 \cos^2 \alpha + \cdots), \tag{18}$$

and from (15) and (17)

$$\sin \psi_1 \cos \psi_1 + \sin \psi_2 \cos \psi_2 = \sin \beta_1 \cos \beta_1 + \sin \beta_2 \cos \beta_2 + \cdots, \tag{19}$$

no further terms being needed in this expansion.

Accordingly, combining (18) and (19) with (14),

$$\begin{aligned}\sigma_{12} = a[\beta_1 + \beta_2 - \tfrac{1}{4}e^2\{&(\beta_1 + \beta_2)(1 + \cos^2 \alpha) \\ &- \sin^2 \alpha(\sin \beta_1 \cos \beta_1 + \sin \beta_2 \cos \beta_2)\} + \cdots].\end{aligned}$$

This agrees with (7) so far as the written terms go, and thus completes the proof of our main result.

4. A numerical problem. In the numerical problem here described the earth is taken to be a spheroid with $a=3962.80$ miles, $c=3949.56$ miles, and $e^2=.006674$.* We select $\theta_1=30°$, $\alpha=75°$, and, for simplicity, $\theta_2=30°$.

These values give $\cos\beta_1=(6^{1/2}-2^{1/2})/2$ and $\cos\gamma_1=(2-3^{1/2})/3^{1/2}$, while $k^2=.006230$. With these we compute

$$\int_0^{\beta_1}\sqrt{1-k^2\sin^2\psi}\,d\psi = \beta_1 - \tfrac{1}{2}k^2\int_0^{\beta_1}\sin^2\psi d\psi - \tfrac{1}{8}k^4\int_0^{\beta_1}\sin^4\psi d\psi - \cdots$$
$$= \beta_1 - .0009100.\dagger$$

Further we find $(1-e^2\cos^2\alpha)^{1/2}=1-.0002235740$. Hence

$$s_{12} = 2a(1-.0002235740)(\beta_1-.0009100) = 2a(\beta_1-.0011393). \tag{20}$$

This is about 8128 miles.

It is now necessary to compute ϕ_2 from (4). We find

$$\phi_2 = 2\gamma_1 - e^2\beta_1\cos\alpha - \tfrac{1}{4}e^4\cos\alpha\int_0^{\beta_1}(1-\sin^2\alpha\cos^2\psi)d\psi - \cdots$$
$$= 2\gamma_1 - .00177457.$$

Because of the fact that $\theta_2=\theta_1$, equation (10) reduces to $\sin\theta'=(1+\cot^2\theta_1\cos^2\phi_2/2)^{-1/2}$. In our example this becomes $\sin\theta'=(1+3\cos^2\phi_2/2)^{-1/2}$. But

$$\cos\phi_2/2 = \cos(\gamma_1 - .00088728) = \cos\gamma_1 + .00087654,$$
$$\cos^2\phi_2/2 = (7-4\sqrt{3})/3 + .00027197.$$

Thus $\sin\theta'=(8-4\sqrt{3}+.00081591)^{-1/2}=\sin\alpha-.00036745$. This shows that θ' is only about 5 minutes less than α, that is, the plane path has a maximum deviation of 5 minutes from the geodesic path. Consequently the geodesic path reaches a point about 6 miles farther north than the plane path does.

We can now calculate ψ_1 from (12). We find $\cos\psi_1=\cos\beta_1+.00019699$ and $\psi_1=\beta_1-.00023025$. From (13)

$$\sigma_{12} = 2a\left[\psi_1 - \tfrac{1}{2}\kappa^2\int_0^{\psi_1}\sin^2\psi d\psi - \tfrac{1}{8}\kappa^4\int_0^{\psi_1}\sin^4\psi d\psi - \cdots\right]$$

But $k^2=e^2\sin^2\theta'=.0062225$, $\int_0^{\psi_1}\sin^2\psi d\psi=.291740$, and $\int_0^{\psi_1}\sin^4\psi d\psi=.137753$. Hence $\sigma_{12}=2a(\beta_1-.0011386)$.

Consequently $\sigma_{12}-s_{12}=2a(.0000007)$ miles $=29$ feet.

* These values are taken from Vega, Logarithmen, 1914.

† All decimal fractions are correct to the number of significant figures given.

QUADRATIC FORMS WITH LINEAR CONSTRAINTS

H. B. MANN, Columbia University

Let

$$\begin{Vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1} & \cdots & a_{nn} \end{Vmatrix}$$

be the matrix of a quadratic form. Let

$$h_i = \begin{vmatrix} a_{11} & \cdots & a_{1i} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{i1} & \cdots & a_{ii} \end{vmatrix}. \tag{1}$$

The form $f=\sum_{i=1}^{i=n}\sum_{j=1}^{j=n}a_{ij}x_ix_j$ is positive definite if and only if

$$h_i > 0 \qquad \text{for } i = 1 \cdots n \tag{2}$$

It is negative definite if and only if

$$(-1)^i h_i > 0 \qquad \text{for } i = 1 \cdots n \tag{2'}$$

A proof of this theorem can be found for instance in Kowalewski *Determinantenrechnung.*

Hancock in *Theory of Maxima and Minima* proves: If $f=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}x_ix_j$ and the linear constraints of rank p, $\sum_{j=1}^{n}b_{\alpha j}x_j=0$ for $\alpha=1, \cdots, p<n$ are given then f is positive for all values satisfying the constraints if and only if in the equation

$$\begin{vmatrix} a_{11}-\lambda & \cdots & a_{1n} & b_{11} & \cdots & b_{p1} \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ a_{n1} & \cdots & a_{nn}-\lambda & b_{1n} & \cdots & b_{pn} \\ b_{11} & \cdots & b_{1n} & 0 & \cdots & 0 \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ b_{p1} & \cdots & b_{pn} & 0 & \cdots & 0 \end{vmatrix} = B_0(-\lambda)^{n-p}+B_1(-\lambda)^{n-p-1}+ \cdots +B_{n-p} \tag{3}$$

the coefficients $B_0, B_1, \cdots, B_{n-p}$ all have the same sign. It is negative definite if the signs of the coefficients $B_0, B_1, \cdots, B_{n-p}$ alternate.

However a much simpler set of necessary and sufficient conditions corresponding to (2) and (2′) can be derived which, so far as the author knows, has not been incorporated in any textbook.

We shall prove the following theorem: Let $f=\sum_{i=1}^{n}\sum_{k=1}^{n}a_{ik}x_ix_k$ be subject to the linear constraints $\sum_{j=1}^{n}b_{\alpha j}x_j=0$, $\alpha=1, \cdots, p<n$. Let

$$\begin{vmatrix} b_{11} & \cdots & b_{p1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ b_{1p} & \cdots & b_{pp} \end{vmatrix} \neq 0 \quad \text{and} \quad h_i = \begin{vmatrix} a_{11} & \cdots & a_{1i} & b_{11} & \cdots & b_{p1} \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ a_{i1} & \cdots & a_{ii} & b_{1i} & \cdots & b_{pi} \\ b_{11} & \cdots & b_{ii} & 0 & \cdots & 0 \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ b_{p1} & \cdots & b_{pi} & 0 & \cdots & 0 \end{vmatrix}.$$

The form f is positive for all values satisfying the constraints if and only if $(-1)^p h_i > 0$ for $i = p+1 \cdots n$. It is negative for all such values if and only if $(-1)^{p+i} h_i > 0$ for $i = p+1 \cdots n$.

To prove this theorem we prove two lemmas.

LEMMA 1. *If f and the constraints are subjected to a transformation $x_i = \sum_{k=1}^n \gamma_{ik} y_k$, $i = 1, \cdots, n$ and $f = \sum_{i=1}^n \sum_{k=1}^n a'_{ik} y_i y_k$ and the constraints are given by $\sum_{j=1}^n b'_{\alpha j} y_j$, $\alpha = 1, \cdots, p$ then*

$$h'_n = \begin{vmatrix} a'_{11} & \cdots & a'_{1n} & b'_{11} & \cdots & b'_{p1} \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ a'_{n1} & \cdots & a'_{nn} & b'_{1n} & \cdots & b'_{pn} \\ b'_{11} & \cdots & b'_{1n} & 0 & \cdots & 0 \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ b'_{p1} & \cdots & b'_{pn} & 0 & \cdots & 0 \end{vmatrix} = \begin{vmatrix} \gamma_{11} & \cdots & \gamma_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \gamma_{n1} & \cdots & \gamma_{nn} \end{vmatrix}^2 h_n.$$

Proof. We consider the quadratic form

$$F = \sum_{i=1}^n \sum_{k=1}^n a_{ik} x_i x_k + \sum_{\alpha=1}^p \sum_{j=1}^n b_{\alpha j} x_{n+\alpha} x_j.$$

Its discriminant is h_n. We subject F to the transformation

$$x_i = \sum_{k=1}^n \gamma_{ik} y_k \qquad \text{for } i = 1, \cdots, n$$

$$x_{n+\alpha} = y_{n+\alpha} \qquad \text{for } \alpha = 1, \cdots, p;$$

then

$$F = \sum_{i=1}^n \sum_{k=1}^n a'_{ik} y_i y_k + \sum_{\alpha=1}^p \sum_{j=1}^n b'_{\alpha j} y_{n+\alpha} y_j$$

which proves our lemma.

LEMMA 2. *If f is subjected to a transformation of the form*

$$x_i = \sum_{j=1}^n \gamma_{ij} y_j, \quad \textit{for } i \leqq \bar{m}, \qquad x_i = \sum_{j=i}^n \gamma_{ij} y_j, \quad \textit{for } i > \bar{m},$$

so that $\gamma_{ij}=0$ *for* $i>\bar{m}$, $j<i$; *then*

$$h_m' = \begin{vmatrix} \gamma_{11} & \cdots & \gamma_{1m} \\ \vdots & & \vdots \\ \gamma_{m1} & \cdots & \gamma_{mm} \end{vmatrix}^2 h_m \qquad \textit{for } m \geqq \bar{m}.$$

Proof. The terms of f containing x_i for $i>m$ do not contribute anything to the form $f_1' = \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}' y_i y_j$. On the other hand

$$x_i = \sum_{j=1}^{n} \gamma_{ij} y_j = \sum_{j=1}^{m} \gamma_{ij} y_j + \sum_{j=m+1}^{n} \gamma_{ij} y_j \qquad \text{for } i = 1, \cdots, m.$$

The second part of the right hand side of this equation does not contribute anything to f_1'. Hence f_1' is obtained from

$$f_1 = \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} x_i x_j$$

by the transformation

$$x_i = \sum_{j=1}^{m} \gamma_{ij} y_j \qquad (i = 1, \cdots, m).$$

But then by Lemma 1, we have

$$h_m' = \begin{vmatrix} \gamma_{11} & \cdots & \gamma_{1m} \\ \vdots & & \vdots \\ \gamma_{m1} & \cdots & \gamma_{mm} \end{vmatrix}^2 h_m.$$

We can now prove our theorem. We put

$$\begin{aligned} y_1 &= b_{11}x_1 + \cdots + b_{1n}x_n \\ &\cdots\cdots\cdots\cdots \\ y_p &= b_{p1}x_1 + \cdots + b_{pn}x_n \\ y_i &= x_i \qquad \text{for } i > p. \end{aligned}$$

The determinant of this substitution is given by

$$\begin{vmatrix} b_{11} & \cdots & b_{1p} & b_{1p+1} & \cdots & b_{1n} \\ \vdots & & \vdots & & & \\ b_{p1} & \cdots & b_{pp} & b_{pp+1} & \cdots & b_{pn} \\ 0 & \cdots & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ \vdots & & & & & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{vmatrix} = \begin{vmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & & \vdots \\ b_{p1} & \cdots & b_{pp} \end{vmatrix} = c.$$

Since this determinant is different from 0 by hypothesis we can express the y's in terms of the x's but we have

$$x_i = y_i \qquad \text{for } i > p;$$

hence the conditions of Lemma 2 are satisfied for $\bar{m}=p$. We have then $f=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}' y_i y_j$ with the restrictions $y_\alpha=0$ for $\alpha=1, \cdots, p$ and by Lemma 2,

$$h_m' = \begin{vmatrix} a_{11}' & a_{1p}' & a_{1m}' & 1 & 0 & \cdots & 0 \\ \vdots & & & 0 & \ddots & & \vdots \\ \vdots & & & \vdots & & \ddots & 0 \\ a_{p1}' & \cdots & a_{pm}' & 0 & \cdots & 0 & 1 \\ \vdots & & & 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots & & & \vdots \\ a_{m1}' & \cdots\, a_{mp}' \,\cdots & a_{mm}' & 0 & \cdots & \cdots & 0 \\ 1 & 0\cdots 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \vdots & & & & \\ \vdots & & \vdots & & & & \\ 0 & \cdots 0 \; 1 & 0 & \cdots & \cdots & \cdots & 0 \end{vmatrix} = c^2 h_m \qquad \text{for } m \geqq p.$$

Applying the conditions (2) and (2′) we see that the sign of f for all values satisfying the constraints is determined by the signs of the determinants

$$h_{p+1}'' = \left| a_{p+1\,p+1}' \right|, \quad h_{p+2}'' = \begin{vmatrix} a_{p+1\,p+1}' & a_{p+1\,p+2}' \\ a_{p+2\,p+1}' & a_{p+2\,p+2}' \end{vmatrix}, \cdots, \; h_n'' = \begin{vmatrix} a_{p+1\,p+1}' & \cdots & a_{p+1\,n}' \\ \vdots & & \vdots \\ a_{n\,p+1}' & \cdots & a_{nn}' \end{vmatrix}.$$

But

$$h_m' = (-1)^p h_m'' \qquad \text{for } m > p.$$

For if h_m' is expanded by the columns $m+1 \cdots n$ and the rows $m+1 \cdots n$ we obtain

$$h_m' = (-1)^{(m+2)p+(m-p+2)p} \begin{vmatrix} a_{p+1\,p+1}' & \cdots & a_{p+1\,m}' \\ \vdots & & \vdots \\ a_{m\,p+1}' & \cdots & a_{mm}' \end{vmatrix} = (-1)^p h_m'',$$

and this proves our theorem.

It may be observed that the necessity of the conditions of our theorem but not their sufficiency follows also from the theorem in Hancock's book mentioned at the beginning of this paper.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The department of Discussions and Notes is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

QUOTIENTS IN GALOIS FIELDS

J. B. SHAW, University of Illinois

In the Galois Field $GF[p^n]$ let x be a primitive mark and set $p^n-1=m$, then $x^m=1$. The members of the field are the various powers of x. These are also reduced by the moduli to residues which are polynomials in x of degree lower than n. Usually these residues are the items of study. However it is interesting to notice also the unreduced quotients of the various powers of x by the modulus $z=f(x)$, where $f(x)$ is a polynomial properly chosen of degree n.

Let $x^a=r_a+zq_a$, $x^m=1+zq$. We consider q to be written with detached coefficients reduced modulo p, and filled in with zeros at the beginning to make up m terms. Then in detached coefficients q_a will be the first a terms of q, and if we multiply q by the residue r_a we will merely shift the first a terms cyclically to follow the others. For,

$$r_aq = x^aq - zqq_a = x^aq + q_a - x^mq_a.$$

The first term annexes a zeros to q, the next substitutes for these the first a terms of q, and the last term erases the same terms at the beginning.

Example 1. When $n=1$ we have a familiar theorem in integers written on the base x. Thus,

$$10^6 = 1 + 7(142857),$$

and if we multiply 142857 by the residues of the powers of 10, we merely shift these digits cyclically. Similarly,

$$10^{16} = 1 + 17(0588235294117647).$$

The multiples of q by $1, \cdots, 16$ move these figures cyclically.

Example 2. In $GF[3^3]$, $z=x^3+0-x+1$ (3). Dividing x^{26} by z we find the coefficients of q

$$0\ 0\ 1\ 0\ 1\ 2\ 1\ 1\ 2\ 0\ 1\ 1\ 1\ 0\ 0\ 2\ 0\ 2\ 1\ 2\ 2\ 1\ 0\ 2\ 2\ 2.$$

If we multiply this by $r_6=1+1+1$, we have the same set with the first 6 shifted to follow the others.

THE GAME OF NIM

L. S. RECHT, Scott Field, Illinois

The note on the game of Nim in the January, 1942 issue of this MONTHLY is capable of an interesting generalization. It was established in Bouton's article* that the necessary and sufficient condition that a set of n numbers x_i $(1 \leqq i \leqq n)$ represents a "losing combination" is:

Represent each x_i in the binary system,

$$x_i = \sum_{m=0}^{\infty} b_{im} 2^m, \qquad 0 \leqq b_{im} \leqq 1.$$

Then for every fixed m, $\sum_{i=1}^{n} b_{im}$ is even.
This condition is equivalent to:

Represent each x_i in the system 2^q,

$$x_i = \sum_{j=0}^{\infty} a_{ij} 2^{qj}, \qquad 0 \leqq a_{ij} \leqq 2^q - 1.$$

Then for every fixed j, the set of numbers a_{ij} is itself a "losing combination" according to the above condition based on the binary system.

For $q = 1,2$, this representation is the binary and quaternary already discussed. Thus, the numbers 0; 1,1; 2,2; 3,3; 1,2,3 lose their appearance of arbitrariness.

Proof. Represent each x_i in the system 2^q,

$$x_i = \sum_{j=0}^{\infty} a_{ij} 2^{qj}, \qquad 0 \leqq a_{ij} \leqq 2^q - 1.$$

Represent each a_{ij} in the binary system,

$$a_{ij} = \sum_{k=0}^{\infty} b_{ijk} 2^k = \sum_{k=0}^{q-1} b_{ijk} 2^k.$$

Then

$$x_i = \sum_{j=0}^{\infty} \sum_{k=0}^{q-1} b_{ijk} 2^{qj+k} = \sum_{m=0}^{\infty} b_{im} 2^m, \qquad m = qj + k.$$

This is the binary representation of x_i. By Bouton's above-mentioned condition, the numbers x_i constitute a "losing combination" if and only if all the sums $\sum_{i=1}^{n} b_{im}$ are even. Since the b_{im} are identical with the b_{ijk}, all the suns $\sum_{i=1}^{n} b_{im}$ are even if and only if all the sums $\sum_{i=1}^{n} b_{ijk}$ are even. By Bouton's condition again all the sums $\sum_{i=1}^{n} b_{ijk}$ are even if and only if each of the sets of numbers a_{ij}, where j is fixed, constitutes a "losing combination." From which the conclusion follows: The numbers x_i constitute a "losing combination" if and only if each of the sets a_{ij}, where j is fixed, constitutes a "losing combination."

* C. L. Bouton, Annals of Math., ser. II, vol. 3, 1901, p. 35.

THE SHORTEST WAY FOUND QUICKLY

W. R. Ransom and J. V. Breakwell, Tufts College

A slide rule which will calculate distance and direction for great-circle flights very quickly is described below.

The data are latitude and longitude *from*, and latitude and longitude *to*. Their *differences* are found in the usual manner, the length of the great circle is denoted by Z, and the angle between the track and the meridian by θ. The slide rule is based upon the formulas:

$$\text{hav}(Z) = \text{hav}(\text{diff. lat.}) + \cos(\text{lat. from}) \cos(\text{lat. to}) \text{hav}(\text{diff. long.})$$

$$\sin(\theta) = \sin(\text{diff. long.}) \cos(\text{lat. to}) \csc(Z).$$

The whole thing may be made from a sheet of heavy bond paper $8\frac{1}{2}$ by 11 inches. For convenience it will be thought of as divided into eight equal strips whose boundaries will be referred to as follows (see Fig. 1):

$0 = T$ (top edge), 1, 2, $3 = S$ (the slide), $4 = M$ (the middle),
$5 = C$ (the cut), $6 = F$ (a fold), 7, and $8 = B$ (the bottom).

The sheet is first folded (the long way) so that the top meets the bottom with the middle outside. Then the bottom is folded up to the middle so that the fold F comes outside. Next the top is folded to meet F with the slide S inside. Last, the fold F is brought over to meet the middle so that the crease C can be cut along to separate the sheet into *frame*, 0 to 5, and *slide*, 5 to 8.

The scales are all placed on what has been referred to as the *outside* and may be described as follows:

Along 1, above it, a double log cos scale from 85° to 0° and then (in red) from 0° to 85°. Twice log cos 85° $= -2.1194$ and this scale is made 21.194 *cm.* long.

Along 2, above it, a single log sin scale from 5° to 90° and back to 175°, numbered also (in red) as a log cos scale from 85° to 0°. This scale is to be made 10.597 *inches* long to correspond with the scale on 6.

Along 4, under it, a scale of nat hav from 0° to 180°, numbered also (in red) as a scale of nautical miles (100 mi. = 1° 40′). This scale should be exactly 10 inches long to correspond with the scale on 7.

Along 6, under it, a scale of log sin to match that along 2.

Along 7, under it but the other side up, a scale of nat hav to match that along 4, and mark its 0° end with an arrow.

Along 8, above it and the same side up as the scale along 7, a scale of log hav from 180° to 5°. Log hav 5° $= -2.72064$, and this scale is therefore made 27.2064 *cm.* long to correspond with the scale on 1.

Complete directions for use are placed directly upon the appropriate scales as follows (see Fig. 2):

Above the scale over 1, at the left, "Set Latitude-From over Difference-Longitude," and at the right, "Under Latitude-To read Auxiliary Angle."

Above the scale along 7, at the left, "Set Arrow over Auxiliary Angle," and at the right, "Under Difference-Latitude read Length of Great Circle Z."

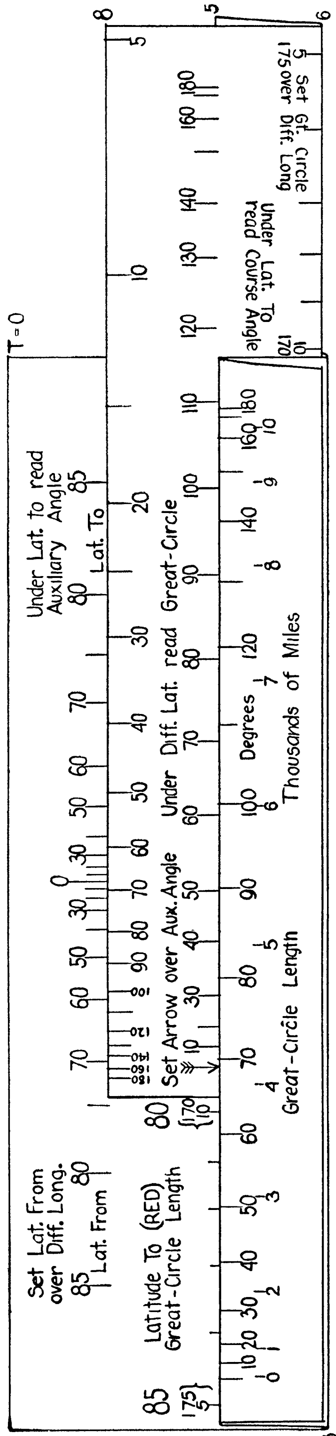


FIG. 2

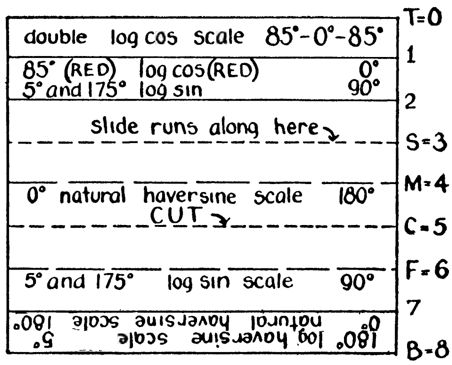


FIG. 1

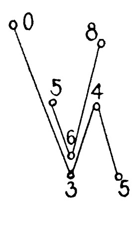


FIG. 3a

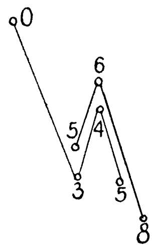


FIG. 3b

Under the scale along 6, at the left, "Set Great-Circle-Length Z over Difference-Longitude," and at the right, "Under Latitude-To (red) read Course-Angle θ." The following note may be added: "The *course* is θ if NE, $180°-\theta$ if SE, $180°+\theta$ if SW, and $360°-\theta$ if NW."

Along the inside of the fold $S=3$ write "This is the Runway Fold." Along the upper side of $F=6$ write "To find Length of Great Circle, let this fold slide along Runway Fold." Along the under side of $C=5$ write the other side up "To find the Course, let this edge slide along the Runway Fold."

To use the device, the fold $F=6$ of the slide is set into the fold to run along $S=3$, and the directions in sight are followed and the length of the great circle is found (see Fig. 3a). Then the slide is turned the other edge up and its edge $C=5$ is set into the fold to run along $S=3$, and the scales along 2 and 6 are used to find the course (see Fig. 3b).

If made upon heavy paper, such as is used for charts, about three times as long and carefully ruled, such a device might furnish as much accuracy as an airplane can use.

The same arrangement of slides and scales can also be labelled so as to furnish a quick solution for altitude and azimuth when hour-angle, declination, and latitude are known.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Spherical Trigonometry and Tables. By W. A. Granville, P. F. Smith, and J. S. Mikesh. New York, Ginn and Company, 1942. 18+60+9+43 pages. \$1.25.

The numbering of the pages in this book indicates that it is part of a text on Plane and Spherical Trigonometry by the same authors. There are twenty-nine articles preceded by an introduction containing formulas from Plane Trigonometry and applications to sailing. Articles one through twenty one are essentially the same as are found in the 1909 edition of Granville's Plane and Spherical Trigonometry. Some of the problems have been changed. But it is difficult to improve on that earlier edition. The remaining articles treat of applications to the celestial sphere and have been revised to parallel apperantly current United States Navy ideas on this subject. Additional problems have been added, and, with a set of four place tables, the book provides a thorough understanding of the subject.

B. R. BEISEL

Analytical Geometry and Calculus. By H. B. Phillips. Cambridge, Mass., Lew A. Cummings Company, 1942. 10+457 pages.

In many undergraduate curricula in mathematics the haste is so great to get to the calculus that less and less attention is being paid to formal courses in the basic material of algebra, trigonometry, and analytic geometry. This is particularly true in courses of study designed primarily for the scientific student where an early working knowledge of calculus is essential. Many such programs start with analytic geometry on the assumption that the beginning student already has some knowledge of algebra and trigonometry and swiftly pass on to the calculus. Only a few freshman courses spend anything like a year on the combined subject matter of algebra, trigonometry, and analytic geometry and some even begin with calculus itself spending time, when and as needed, on the necessary preliminary material. Various combination texts have been devised for such programs of study.

Of recent years more and more of the regular textbooks in calculus have included some chapters on these fundamental topics. The book under review is of this type, designed, as the author states in the preface, " . . . to provide a course in analytic geometry and calculus for students of science and engineering." But this text is even more inclusive: it contains other topics, a study of which will be of extreme benefit to the scientific student. There are chapters devoted to Limits and Continuity (Chap. I), Determinants (Chap. V), Vectors (Chap. X), Complex Numbers (Chap. XIV), and Functions of a Complex Variable (Chap. XV). Other chapters cover the usual material found in a standard treatment of the calculus. There is no discussion of differential equations.

Another unusual feature of this book is the large number of problems, ranging up to well over a hundred following each chapter. Answers to all problems are provided at the end of the book. An outstanding feature of any book by Phillips is the clarity of the language. There is no prolixity. Student and teacher alike will be pleased with the careful choice of words. They will also be pleased to find no mention of Duhamel's Theorem.

Some readers may wish to take mild issue with the order of the subject matter. While this reviewer approves of the introduction of some elementary ideas of integration (Chap. III) immediately after a little differentiation (Chap. II), yet he prefers an extensive treatment of areas, volumes, centers of gravity, moments, etc. only after a discussion of multiple integration (the last chapter, Chap. XVIII). Throughout the book special attention is paid to "sign." For example, the integral of cot u is carefully written $\int \cot u\, du = \ln|\sin u| + c$. On page 343, second display line from the bottom, read $\phi(t)$ instead of $\phi(1)$; this seems to be a case of broken type. The printing is well done but a few of the figures could be improved. One wishes that a short table of integrals had been included.

C. O. OAKLEY

An Outline of Probability and its Uses. By M. C. Homes. Minneapolis, Burgess Publishing Co., 1936. 8+119 pages. $1.50.

The author presents material on the following subjects: a priori and a posteriori probability, compound and alternative probability, permutations, combinations, probability pertaining to repeated trials, hypergeometric law, frequency distribution curves, measures of central tendency, measures of dispersion, skewness, Tchebycheff's Theorem, laws of large and small numbers, Gaussian error function, probability paper, expectation, Bayes Theorem, sampling, probable errors of statistical constants, Stirling's formula, Lexian Theory, Pearson curves, goodness of fit, correlation, table of factorials and error function table.

The author has attempted to do too much in this small book—hence the presentation of each topic is too brief, several of the proofs are inadequate and there are too few applications. The statements in several instances are incorrect and the problems are not thought provoking enough for a text on probability. In reading the first paragraph in the book one gets the impression that it is written primarily for students preparing to study physics, yet there are very few problems pertaining to physics.

Probability paper is well explained, the diagrams pertaining to the number of successes in k trials are instructive and the Beta diagram is a help in interpreting ideas concerning Pearson frequency curves.

An experienced teacher can use this book with supplementary material.

W. D. Baten

Plane Trigonometry with Tables. By P. L. Evans. New York, Ginn and Company, 1941. 8+84 pages. $1.25.

This text is one of the series, "Mathematics for Technical Training" and it may be considered to fulfill the purpose for which it is intended. The work is divided into nine chapters plus a supplementary chapter on the Slide Rule. This and the chapter on logarithms should appeal to the technical student.

In the preface we read, "little has been omitted from the usual course." This is the true case, but in some instances the result is not much more than a handbook style of writing. For example, the introduction covers angle, degree, radian, mil, area of a sector, quadrant, rectangular and polar coordinates. Another chapter of five pages treats of the functions of one angle, inverse trigonometric functions and trigonometric equations. The drawings in the chapter on graphic representation and line representation of trigonometric functions are excellent. A complete set of tables accompanies the text.

The book is up-to-date and it should be of great value to vast numbers of young people who are studying eagerly while performing important tasks in to-day's world.

B. R. Beisel

The mechanical make-up of the book is clear, but somewhat crowded. The figures are very well done. They are unnumbered and never cited. The illustrative examples, which form a fairly large and very important portion of the text are in smaller type, requiring more of an effort to read than the main body of the text.

No typographical errors were discovered.

VIRGIL SNYDER

An Introduction to Managerial Business Statistics. By H. P. Hartkemeier. New York, Thomas Y. Crowell Company, 1943. 14+207 pages. $1.75.

This is a paper bound work book, punched for insertion in a ring binder and designed for use in a course of business statistics which requires no mathematical background and involves no mathematics except the arithmetic necessary to properly fill out the prepared "completion sheets." The author characterizes the book in the preface when he says that the book, " . . . gives directions for the immediate practical application of the elementary techniques which every business man needs, . . . "

L. A. DYE

Blueprint Reading at Work. By W. W. Rogers and P. L. Welton. New York, Silver Burdett Company, 1942. 8+136 pages. $1.28.

This book, designed to provide the basic information necessary for the interpretation of a blueprint, is written in an interesting and easy-to-understand manner. A careful study of its material by secondary school pupils or apprentices in the metal working industries should develop easily the ability to read blueprints.

The course treats the following topics: Three-view drawings, two-view drawings, one-view drawings, sectional drawings, screw threads and screw gears, cams, welding and welding practices, sheet metal layout and practices. Each topic is broken down into unit lessons of which there are thirty in all. Each lesson is composed of four parts: (1) a presentation of the problem, (2) a blueprint on the problem, (3) a set of questions on the blueprint, (4) a sketching problem or an objective test.

The mathematics needed is very simple. It consists of the addition or subtraction of common fractions whose denominators are factors of 64, the addition or subtraction of three-place decimal fractions, the addition or subtraction of angles measured in the sexagesimal system, and substitution in simple formulas.

The book is presented in a wire binding and is printed in blue ink which gives it a realistic appearance. The authors have done a thorough piece of work and have turned out a text which merits wide usage and which will certainly find an important place in war-time courses.

M. C. HARTLEY

The Fundamental Principles of Mathematical Statistics. By H. H. Wolfenden. Published for the Actuarial Society of America by the Macmillan Co. of Canada, Toronto, 1942. 15+379 pages. $5.00.

This book was written to meet the needs of the actuaries. Its author states that "an attempt has been made to assemble and coordinate those portions of mathematical statistics which are really needed, both by actuarial beginners in their studies and by qualified actuaries in the solution of the problems which arise in practice." This is supplemented by a rather extensive historical analysis of the development of the theory of probability, statistics, and graduation, with complete references to original papers.

The form of presentation is novel. The main body of the book is only 148 pages and is followed by three appendices: A. History, B. Mathematics and Interpretations, C. Applications, altogether about 200 pages. A twenty-page bibliography rounds out the volume. According to the preface, "The body of the treatment has been designed as a condensed presentation only, from which the principal ideas may be acquired in an orderly and easy manner, the subsidiary questions being dealt with by reference to separate portions of the book." All derivations are placed in appendix B and the main part of the book contains only a description of the problems and the results.

The subject covered includes the binomial, the Poisson, and the normal distributions, theory of small samples including the methods of Student and Fisher, the multinominal distribution and a survey of the standard frequency curves extended to include the curves used in actuarial practice. This is followed by a short chapter on curve fitting and one on the tests of the goodness of fit. The following five pages on *recent* (italics ours) researches are hardly more than a syllabus of the literature. The main body of the book is concluded by an outline of a course in graduation. This section is a valuable review of the literature for an actuary who is already familiar with his subject. Summation and interpolation formulas are suppressed in the text and the subject can be comprehended only by consulting the references.

Appendix A supplies historical notes and references to the main part of the book with special emphasis on the development of the normal law of errors. This section might be valuable to those interested in the development of the subject from the beginning, as careful references are given to the early literature.

Appendix B supplies the derivations omitted in the text. Appendix C provides additional comments and illustrations, using actuarial data as material. The bibliography is divided into two lists, containing material of historical and of present value respectively.

It seems unfortunate that a book written by an author thoroughly familiar with his subject should contain a number of lax and misleading statements. On page 8, for instance, we read that Bernoulli's theorem may "evidently" be interpreted as meaning $\lim_{n\to\infty} s/n = p$, where p is the a priori probability of success and s is the number of successes in n trials. The text contains terms which are only carelessly defined or not defined at all.

While great emphasis is placed on the controversies and discussions of the past, modern statistics is virtually excluded from the text. The analysis of variance is given 18 lines and fiducial limits are disposed of in slightly more than a page.

The book is valuable as a historical summary and as a source of references, but it is not suitable for study. A beginner would soon be confused by the maze of cross-references and the lack of logical order in presentation. He would find no exercises on which to test his knowledge. An advanced student would be discouraged by the lax use of mathematics and by the lack of consistent, straightforward development of theory.

We can conclude with the hope that, perhaps with the aid of the bibliographic work done in this book, a well integrated textbook on statistics will be written which will satisfy the mathematician as well as the educator, and will convey to the student the results of modern research in a clear, concise form.

B. A. LENGYEL

Mathematics in Human Affairs. By F. W. Kokomoor. New York, Prentice-Hall, Inc., 1942. 754 pages. $5.35.

The author's preface sets forth the purpose of this volume, as well as its main features, in the following manner:

"For a long time it has been the belief of the author that a book both *about* and *of* mathematics could be written informally and nontechnically enough to enable a person of average ability and with almost no previous preparation to master it. This book is the outcome of that belief.

"The text starts with the simplest concepts and ends in the calculus. The general plan of each chapter is to present a discussion of the topic at hand, then give a few of its historical connections, descriptive material about it for orientation purposes, numerous illustrations of how it enters into the work of the world, and a development of methods of solution. Then follows a study guide containing suggestions, questions, and problems; and a set of fifty multiple-choice exercises, each having five possible answers, one of which is correct."

In the light of these statements, it is pertinent to range this book with a rapidly growing body of texts which aim to make mathematics more appealing to the general reader. It cannot be doubted that such attempts are really caused by a serious educational situation. This is evidenced, for example, by the fact that more than twenty of our states have made the study of mathematics optional in the high school. As a result, many of our young people, including those entering college, are almost illiterate today in that field. How to deal with this lack of mathematical training, either at the college level or in the practical pursuits of everyday life, has long been a pressing problem. The war emergency did not create this educational crisis, but merely made it more spectacular. As a result there seems to be a growing conviction, at least for the time being, that the virtual elimination of mathematics from many school programs cannot be defended

in an age of industry, science, and technology, not to mention mechanized warfare.

Naturally, the remedies that have been proposed for the correction of this state of affairs differ greatly in educational soundness and in scholarly quality.

The volume under discussion presents to the reader a series of broadly informative sketches which outline key topics chosen from the fields of arithmetic, algebra, geometry, trigonometry, and analytics. These are followed by a brief introduction to the domains of statistics, of differentiation and integration, and by some glimpses of "the way ahead." Applied problems are interspersed throughout the text. The style is essentially that of an informal chat. Dramatic settings are provided by continual reference to historical backgrounds and allusions.

An appraisal of this extensive treatise may well begin by applauding the author's evident intent to develop a real interest in the study of mathematics as a significant human enterprise, as well as some competence in dealing with basic mathematical concepts and techniques. Such a program, at best, represents a task of formidable dimensions. Even the most noted previous efforts along this line, such as those of Paul La Cour, Tannery, Auerbach, Colerus, Hogben, have left much to be desired. And the present volume, in the opinion of the reviewer, shares certain defects of its predecessors. Thus, while some of the topics are treated with commendable skill and completeness, a good deal of the material is so sketchy that neither real insight nor genuine power can be expected to result. The optimistic feeling of the author that "a person of average ability and with almost no previous preparation" will be able to master his book is not warranted. On the contrary, much of his discussion cannot be read profitably without a foundation equivalent to three years of high school mathematics. So vast is the territory covered that neither coherence nor systematic growth is achieved. The reader is confronted with a kaleidoscope of fascinating snapshots, but essential concepts, principles, and skills are not lifted into prominence. As a result, a clear perspective of the framework of mathematics as a system of ideas does not emerge. Not infrequently, the text is too discursive. Finally, while the multiple-choice exercises, at the end of each chapter, exhibit much variety and demand considerable versatility, many pages could readily have been saved by the use of more economical and equally significant types of testing.

A number of obvious slips or typographic errors were noted. Thus, in the computation on page 340, the letter "b" is omitted. Again, on page 662, "dx" is omitted in the last paragraph. On pages 639 and 640, in sections (3) and (4) under Test I, the word "and" is omitted.

In spite of the shortcomings pointed out above, which are more or less unavoidable in a cursory examination of such a large field, the text will serve a useful purpose if it induces in its readers a real desire to continue their mathematical studies.

WILLIAM BETZ

Differential Equations. By H. W. Reddick. New York, John Wiley and Sons, 1943. 9+245 pages. $2.50.

This is a text intended for use in both liberal arts colleges and in engineering schools. The applications emphasize the use of the theory in situations of especial interest to physicists or engineers.

The discussion is restricted to ordinary differential equations which are of the first degree in the derivative of highest order, or which can be reduced to that form immediately. There is no mention, for instance, of Clairaut's equation, and the treatment of singular solutions is limited to an incidental remark in connection with a geometrical problem solved in the text (Article 25, Example 3).

The author makes no attempt to give a complete mathematical discussion; he entirely avoids such topics as general existence theorems. As a text in the formal technique of setting up and solving certain types of ordinary differential equations, intended primarily for undergraduates interested more in the varied applications of the subject than in abstract mathematical theory, it is a good book. Within its scope it gives reasonable attention to the demands of mathematical rigor.

There is a large collection of problems, for which answers are given. The examples solved in the text are numerous and are well chosen to illustrate effectively the topics under consideration. In the early chapters the author gives full explanations of these examples. The amount of detail given diminishes as the student progresses through the book and gains in proficiency. This is, in general, an excellent plan. Some instructors may feel, however, that the author is a little too cautious in this respect in the early part of the book, and that some of the explanations in this section might well be made more concise.

The first two chapters, *Preliminary Ideas* and *The Formation of Differential Equations*, are devoted to such topics as general and particular solutions, and various methods of finding differential equations having given general solutions.

Chapter 3, *Differential Equations of the First Order*, includes a discussion of separable equations, integrating factors, equations with homogeneous coefficients, linear equations, Bernoulli's equation, equations with linear coefficients, and the equivalence of solutions. The applications discussed include a simple problem in dynamics, chemical reactions and chemical solutions, atmospheric and oceanic pressure, steady-state heat flow, electric circuits, and the tension in a rope wound on a cylinder. There are also some of the usual geometric applications, including the problem of the tractrix (Article 25, Example 2). In the last paragraph of the discussion of this problem, the author states that the equation

$$\pm x + C = \sqrt{k^2 - y^2} - k \cosh^{-1} \frac{k}{y}$$

is unchanged when y is replaced by $-y$. Here k has been defined as the length of the tangent, and so is positive. Hence if negative values of y are to be admitted

it would be preferable to write the transcendental term as $-k \cosh^{-1}|k/y|$; this change can be justified from the derivation of the equation.

Chapter 4 is devoted to *Linear Equations with Constant Coefficients.* Operational methods are used to solve the homogeneous equation. The method of undetermined constants and variation of parameters are each used in the treatment of the non-homogeneous equation. The applications discussed in this chapter include simple harmonic motion, damped and forced vibrations and further work with electric circuits. There is also a particularly interesting discussion of the deflection of beams, including a derivation of the differential equation.

Chapter 5, *Some Special Higher Order Equations*, contains a treatment of certain higher order equations which are solvable by special devices. These methods are applied to the study of rectilinear motion under the inverse square law and to the problem of the suspended cable.

Chapter 6 is devoted to *Simultaneous Equations.* Systems of two first order equations are discussed and the geometric interpretation is emphasized. Systems of two linear equations of any order are treated by an operational method. The motion of a projectile with resistance proportional to velocity is studied. Cyclic systems of differential equations are given especial emphasis, and this theory is applied to a specialized form of Einstein's equations.

Chapter 7, *Series Solutions*, develops the usual method of solving certain homogeneous linear equations of the second order by means of power series, or the series of Frobenius. Bessel's equation is used to obtain the series for $J_n(x)$ for the case in which n is a non-negative integer.

In reading the book the reviewer noted the following errata:

Page 95, first line of Problem 13 (*a*): For Ex. 3, Art. 5, read Ex. 3, Art. 25. (N.B. The solution of this problem includes not only the curves obtained in the illustrative example cited, but also their reflections in the x-axis.)

Page 95, last line of Problem 16. For If read Is.

Page 128, first formula of Example 5. For d^2x/dx^2 read d^2x/dt^2.

Page 129, last line of Example 5. For

$$v_{\max} = \tfrac{1}{3}(112 - 28)112^{-4/3} = 26(112)^{-4/3} = 0.0482 \text{ ft/sec.}$$

read

$$v_{\max} = \tfrac{1}{3}(112 - 28)112^{-4/3} = 28(112)^{-4/3} = 0.0519 \text{ ft/sec.}$$

Page 177, last line. For

$$t = \frac{1}{k}\sqrt{\frac{a}{2}}\left(\sqrt{x(a-x)} + a\cos^{-1}\frac{x}{a}\right)$$

read

$$t = \frac{1}{k}\sqrt{\frac{a}{2}}\left(\sqrt{x(a-x)} + a\cos^{-1}\sqrt{\frac{x}{a}}\right),$$

and for

$$t = \frac{1}{k}\sqrt{\frac{a}{2}}\left(\sqrt{x(x-a)} + a\cosh^{-1}\frac{x}{a}\right)$$

read

$$t = \frac{1}{k}\sqrt{\frac{a}{2}}\left(\sqrt{x(x-a)} + a\cosh^{-1}\sqrt{\frac{x}{a}}\right).$$

Page 202, fifth line after formula (7). For

$$\omega = -\tfrac{1}{2} + i\sqrt{3/2}, \qquad \omega^2 = -\tfrac{1}{2} - i\sqrt{3/2}$$

read

$$\omega = -\tfrac{1}{2} + i\sqrt{3}/2, \qquad \omega^2 = -\tfrac{1}{2} - i\sqrt{3}/2.$$

Page 208, seventh line: For

$$wp\frac{dp}{dx} - p^2 = k^2w^4$$

read

$$wp\frac{dp}{dw} - p^2 = k^2w^4.$$

Page 215, last line. For so ution read solution.

F. W. PERKINS

A Primer of Formal Logic. By J. C. Cooley. New York, Macmillan Co., 1942. 11+378 pages. $3.00.

Dr. Cooley's book is a valuable text for introducing students to the techniques of modern symbolic logic. It is clear and yet concise. It includes full treatments of the sentential calculus and the lower functional calculus. Beyond this, it contains a discussion of the nature of deductive systems and a chapter on the definition of various classes of numbers and on the setting up of number systems. In the model system for logic used as an illustration, there are 15 postulates with the result that proofs, though complete and formal, are brief. There is an unusually clear account of the need for some such device as a theory of types for a higher functional calculus, and a brief, simplified description of Russell's theory. Included also are accounts of the class definition of number, and of Peano's postulates for the natural number system.

The book has the advantage of being replete with everyday examples of the abstract forms being treated. Extensive and valuable exercises for each chapter add still more concrete illustrative material. The author leaves unmentioned the many philosophical problems connected with the developments he covers, and concentrates on the treatment of logic as an abstract symbolic calculus.

C. A. BAYLIS

Basic College Mathematics. A General Introduction. By C. W. Munshower and J. F. Wardwell. New York, Henry Holt and Co., 1942. 11+606 pages.

This first year college mathematics text presents material for a unified course covering algebra, trigonometry, analytical geometry, and the elements of the calculus. As the authors state in their preface, this book is intended to serve either for a terminal course or for an introduction to a course in the calculus. The book achieves a unification of the subject matter through the concept of function. There is nothing new in this treatment, but it is the opinion of the reviewer that the authors have done an unusually good piece of work both with respect to organization and presentation of the traditional topics of freshman mathematics.

First, there is a rather long introductory chapter which contains a review of certain elementary ideas and an introduction to several new concepts, such as: rate of change, mathematical induction, infinite sequences and infinite series. Chapters II to VII, inclusive, develop all the basic theory of simple algebraic functions usually found in analytic geometry, college algebra and the calculus (the analysis of implicit functions of the second degree is postponed until a much later chapter). The concepts of limit, derivative, differential, integral as anti-derivative, and the differential equation are all introduced in chapter III in connection with simple power functions. Integration as summation is also presented in an early chapter. After a study of transcendental functions, the conics, and the fundamentals of geometry of functions of three variables, the authors conclude with a chapter on determinants and one on permutations, combinations, and probability. Following the answers to odd-numbered problems is a collection of tables and a group of reference formulas.

The reviewer believes that this is both logically and pedagogically a sound text. It is truly unified and not a mere collection of topics as are many so called "unified" texts. The illustrative material is clear and interesting. The book contains a wide selection of problems from business, economics, military science, navigation, psychology and sociology. Pedagogically, the chief criticism of the book is the difficulty which the introductory chapter presents to a student new to the subject matter. It seems to the reviewer that some of the material contained in the long chapter might have been saved for a later chapter. Certainly it would have been wiser to have deferred mathematical induction and the proof of the binomial theorem.

ROBERTA F. JOHNSON

Plane Trigonometry, Solid Geometry, and Spherical Trigonometry. (With tables.) By W. W. Hart and W. L. Hart. Boston, D. C. Heath and Company, 1942. 7+280+129 pages. $2.60.

This book, which is as well written as one familiar with the Hart textbooks would expect, is designed to meet a very evident present day need. It stresses the

practical applications of trigonometry and solid geometry without sacrificing any material necessary in the background of a student wishing to continue with college mathematics. A knowledge of high school algebra and plane geometry is presupposed.

Somewhat more than half of the book, which is divided into three parts, is devoted to a complete course in plane trigonometry with a brief chapter on polar coordinates. The second part, approximately sixty pages, contains theorems and exercises on planes, lines, and spheres in space and an appendix listing the important theorems of plane geometry. The third part is devoted to spherical trigonometry with some applications to navigation.

It seems to the reviewer that this book, the whole, or in part, could well be used as a preliminary course for students who wish to study navigation. To cover the book completely would necessitate a five-hour semester course. However, the material is so arranged that parts could easily be omitted should a shorter course be desired.

BEATRICE L. HAGEN

Table of the Coefficients of Everett's Central-Difference Interpolation Formula. (Tracts for Computers, No. V, Department of Statistics, University of London, University College.) By A. J. Thompson. Second edition. Cambridge (England), University Press, 1943. 8+32 pages. 5 shillings.

Everett's formula uses the differences of even order in the horizontal lines through two consecutive tabular entries in a diagonal difference table. When carried out through differences of order $2k$, it represents a polynomial of degree $2k+1$ coinciding with the tabulated function values for $2k+2$ values of the argument symmetrically distributed with respect to the interval in which interpolation is to be performed. This polynomial naturally is uniquely determined by its interpolating property, except as to mode of representation. The Everett form, written in terms of two variables whose sum is unity, has notable advantages in the way of symmetry of structure and convenience of application. It can be obtained from the Gauss formula which fits for the values $0, 1, -1, 2, -2, 3, \ldots$ of the argument successively, by replacing each difference of odd order in the Gauss formula by its expression in terms of the even differences of the next lower order.

The main table of the Tract under review gives the coefficients by which the second, fourth, sixth, and eighth differences are multiplied in Everett's formula, to ten decimal places, with second differences, for values of the argument from 0 to 1 at intervals of .001. Supplementary tables give coefficients of higher order and a larger number of decimal places, in many cases terminating decimals at full length, for a smaller number of values of the argument, also values for use near the beginning or end of a table, where the "central" differences required for the ordinary use of the formula are not available. "In this new edi-

tion," according to the Foreword of the Editor, E. S. Pearson, "the arrangement of the fundamental Table I has been altered so that all the coefficients for any three-decimal interpolation appear on the same line, at one opening. A column giving the coefficients of the eighth order has been added. The remaining tables are new and will, it is believed, be found useful in a variety of more specialized problems." An Introduction gives suggestions for the systematic treatment of problems in which something more than direct substitution in the formula is required.

DUNHAM JACKSON

Essentials of Plane and Spherical Trigonometry. By A. H. Sprague. New York, Prentice-Hall, Inc., 1942. 9+169 pages. $1.35.

In keeping with the times this book is brief, practical, and clear; yet it is thorough enough to cover completely the essentials of plane and spherical trigonometry.

The reviewer is impressed with the minimum of abstract theory and with the problems presented and the development of the formulas. The essential areas have been covered in such a way as to gain a mastery of the subject matter in a moderately short time. Its very compact size and completeness appeal to me greatly and the problems and illustrations are clear and to the point. The author has been able to combine brevity and completeness in the right proportion.

No previous knowledge of logarithms is assumed and the first chapter is devoted to the study of exponents and logarithms. The trigonometric functions of an acute angle are defined and sufficient applications are given to familiarize the student with the subject before the functions of the general angle are introduced.

The author stresses the law of sines and the law of cosines and has included an excellent table of squares and square roots that serve as adequate tools for the solution of the oblique triangle. In deference to usage, however, he has added a chapter on supplemental topics that contains the usual law of tangents, the "r" formulas, and a discussion of circular measure. An excellent feature is that the fundamental formulas and identities are placed in a single chapter, and abundant problem material on solving identities is included.

The usual material, treated as simply as possible, is included in the division on spherical trigonometry. There is an excellent derivation of the cosine-haversine law with problems illustrating its use. Four place tables of haversines and logarithms of haversines are included. The section on terrestrial and celestial spheres together with the problems included should be of interest to the student of aviation and navigation.

Four place tables of logarithms and of the trigonometric functions are included.

W. B. MOYE

NEW BOOKS RECEIVED

Plane Trigonometry with Tables. By D. H. Ballou and F. H. Steen. Boston, Ginn and Co., 1943. 6+84 pages. $2.00.

College Physics. Revised Edition. By H. A. Perkins. New York. Prentice-Hall, Inc., 1943. 10+593 pages. $4.00.

Basic Mathematics for Aviation. By F. Ayres, Jr. Boston, Houghton Mifflin Co., 1943. 225 pages. $2.35.

Concise Spherical Trigonometry with Applications and Review of Solid Geometry and Plane Trigonometry. By J. R. Hammond. Boston, Houghton Mifflin Co., 1943. 13+256 pages. $2.20.

Elementary Statistical Methods. By Helen M. Walker. New York, Henry Holt and Co., 1943. 15+368 pages. $2.75.

The Slide Rule and Logarithms. By E. J. Hills. Boston, Ginn and Co., 1943. 14+108 pages. $0.75.

An Introduction to Managerial Business Statistics. By H. P. Hartkemeier. New York, Thomas Y. Crowell Co., 1943. 14+207 pages. $1.75.

Vector and Tensor Analysis. By H. V. Craig. New York and London, McGraw-Hill Book Co., Inc., 1943. 14+434 pages. $3.50.

Miscellaneous Physical Tables. Planck's Radiation Functions and Electronic Functions. New York, Work Projects Administration, 1941. 7+61 pages. $1.50.

Applied Mathematics for Technical Students. By M. S. Corrington. New York and London, Harper and Bros., 1943. 12+226 pages. $2.20.

Mathematical Statistics. By S. S. Wilks. Princeton University Press, 1943. 12+284 pages. $3.75.

Elementary Mathematics for the Machine Trades. By J. J. Weir. New York and London, McGraw-Hill Book Co., 1943. 8+193 pages. $1.60.

Basic Physics for Pilots and Flight Crews. By E. J. Knapp. New York, Prentice-Hall, Inc., 1943. 6+118 pages. $1.25.

Essentials of Plane and Spherical Trigonometry. By C. Bell and T. Y. Thomas. New York, Henry Holt and Co., Inc., 1943. 6+152 pages. $1.80.

Analytical Geometry of Three Dimensions. By W. H. McCrea. Edinburgh and London, Oliver & Boyd, Ltd.; New York, Interscience Publishers, Inc., 1942. 7+144 pages. $1.75.

Infinite Series. By J. H. Hyslop. Edinburgh and London, Oliver and Boyd, Ltd.; New York, Interscience Publishers, Inc., 1942. 11+120 pages. $1.75.

An Introduction to Plane Geometry with Many Examples. By H. F. Baker. Cambridge, The University Press; New York, The Macmillan Co., 1943. 8+382 pages. $4.00.

Manual de Estadistica Vital. By F. E. Linden. Montevideo, 1942. 13+76 pages.

Selected Topics in Higher Mathematics for Teachers. Association of Teachers of Mathematics, 1943. 107 pages. $0.50.

The Mathematics of Physics and Chemistry. By H. Margenau and G. M. Murphy. New York, D. Van Nostrand Co., Inc., 1943. 12+581 pages. $6.50.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 581. *Proposed by D. K. Kazarinoff, University of Michigan*

A farmer died and left his two sons a herd of cattle which they sold. The number of dollars received per head was the same as the number of heads. With the proceeds of the sale the sons bought sheep at \$10 each and one lamb for less than \$10. The sheep and the lamb were divided between the two brothers so that each received the same number of animals. How much should the son who received only sheep pay to the son who received the lamb in order that the division should be equitable?

E 582. *Proposed by V. Thébault, San Sebastián, Spain*

Find a square of ten digits such that the two numbers formed by the first five and last five digits are consecutive.

E 583. *Proposed by N. A. Court, University of Oklahoma*

Given four spheres (A), (B), (C), (D), with centers A, B, C, D, let a plane parallel to ABC cut DA, DB, DC in points U, V, W. Show that the radical axis of the three spheres having U, V, W for centers and coaxal with the respective pairs of spheres (D) and (A), (D) and (B), (D) and (C), coincides with the radical axis of the spheres (A), (B), (C).

E 584. *Proposed by C. O. Oakley, Haverford College*

(a) Find the smallest integer N such that, if the units digit is transposed from right to left, a number M is obtained where $M=5N$.

(b) Find the largest integer which satisfies these conditions (without complete repetition of digits).

E 585. *Proposed by A. H. Stone, Purdue University*

Let a circle with center O meet the sides BC, CA, AB of a triangle in the pairs of points X and X', Y and Y', Z and Z'. Let M be the Miquel point of XYZ (*i.e.*, the point of concurrence of circles AYZ, BZX, CXY), and M' be that of $X'Y'Z'$. Prove that $OM=OM'$.

If, further, the lines AX, BY, CZ concur, say in P, and consequently the lines AX', BY', CZ' concur, say in P', prove that PP' and MM' are parallel.

SOLUTIONS

An Integral-valued Function

E 546 [1942, 683]. *Proposed by W. E. Buker, Pittsburgh Public Schools*

Show that $\frac{1}{5}x^5+\frac{1}{3}x^3+\frac{7}{15}x$ is an integer for every integral value of x.

I. *Solution by L. R. Ford, Illinois Institute of Technology.* We can prove the following general theorem. *If a polynomial $f(x)$ of degree m has integral values for $m+1$ successive integral values of x, then it is integral for every integral value of x.*

Let $f(x)$ be integral for the integral values a, $a+1, \cdots, a+m$. We form a difference table based on these values. All the entries in this table are formed by repeated subtractions and so are integers. Using Newton's interpolation formula, we have

$$f(a+n) = f(a) + n\Delta f(a) + \binom{n}{2}\Delta^2 f(a) + \cdots + \binom{n}{m}\Delta^m f(a).$$

If n is an integer, the binomial coefficients $\binom{n}{r}$ are integers. The second member is thus a sum of products of integers, and so is an integer. This establishes the theorem.

If

$$f(x) = \tfrac{1}{5}x^5 + \tfrac{1}{3}x^3 + \tfrac{7}{15}x,$$

then

$$f(0) = 0, \quad f(\pm 1) = \pm 1, \quad f(\pm 2) = \pm 10, \quad f(\pm 3) = \pm 59,$$

and the theorem applies.

II. *Solution by H. M. Gehman, University of Buffalo.* Since

$$\begin{aligned} 3x^5 + 5x^3 + 7x = 3(x-2)(x-1)x(x+1)(x+2) \\ + 4\cdot 5(x-1)x(x+1) + 15x, \end{aligned}$$

we need only the theorem that the product of n consecutive integers is divisible by n, to see that the number

$$(3x^5 + 5x^3 + 7x)/15 = \tfrac{1}{5}x^5 + \tfrac{1}{3}x^3 + \tfrac{7}{15}x$$

is an integer.

III. *Solution by C. F. Strobel, North Carolina State College of Agriculture and Engineering.* The statement is true for $x=0$. We will assume it true for $x=n$ (a non-negative integer), and prove that an integer is obtained for $x=n+1$. Expanding

$$f(n+1) = \tfrac{1}{5}(n+1)^5 + \tfrac{1}{3}(n+1)^3 + \tfrac{7}{15}(n+1)$$

and collecting terms, we get

$$f(n+1) = (\tfrac{1}{5}n^5 + \tfrac{1}{3}n^3 + \tfrac{7}{15}n) + (n^4 + 2n^3 + 3n^2 + 2n + 1).$$

The first bracket contains an integer by assumption. The second contains a polynomial with integral coefficients. Thus $f(n+1)$ is an integer.

For negative values of x we need only observe that $f(-n) = -f(n)$.

IV. *Solution by Alan Wayne, Flushing, New York.* Write the function in the form $\frac{1}{5}(x^5-x)+\frac{1}{3}(x^3-x)+x$. Then by Fermat's Theorem (which says that x^p-x is divisible by p if p is prime), the function is an integer for integral x.

Also solved by R. K. Allen, D. H. Browne, William Douglas, D. M. Dribin, J. D. Elder, C. W. Emmons, Howard Eves, S. E. Field, W. L. Fields, W. C. Foreman, R. E. Greenwood, B. Hamming, Free Jamison, Irving Kaplansky, Meyer Karlin, Elmer Latshaw, William McGavock, Walter Penney, S. G. Roth, W. C. Rufus, E. D. Schell, J. E. Sherwood, H. E. Spencer, E. P. Starke, E. E. Strock, W. R. Utz, G. A. Williams, and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4089. *Proposed by J. H. Butchart, Grinnell College*

The tangent to Kiepert's hyperbola at a vertex of the triangle is the harmonic conjugate of the symmedian with respect to the altitude and median from that vertex. It meets the corresponding side of the medial triangle at a point on the tangent to the hyperbola at the centroid.

4090. *Proposed by N. A. Court, University of Oklahoma*

The polar plane, with respect to the "quasipolar" sphere of a tetrahedron (T), of a point on the circumsphere of (T) trisects the segment joining the Monge point of (T) to the diametric opposite of the given point on the circumsphere.

Note. The "quasipolar" sphere (Q) of a tetrahedron (T) has for its center the Monge point M of (T) and for the square of its radius one third of the power of M for the circumsphere (O) of (T). See Bull. Am. Math. So. vol. 48, 1942, p. 583.

4091. *Proposed by Morgan Ward, California Institute of Technology*

Given the three series

$$z-\frac{z^5}{2\cdot 4\cdot 5}+\frac{z^9}{2\cdot 4\cdot 6\cdot 8\cdot 9}-\frac{z^{13}}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 13}+\cdots,$$

$$\frac{z^3}{2\cdot 3}-\frac{z^7}{2\cdot 4\cdot 6\cdot 7}+\frac{z^{11}}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 11}-\frac{z^{15}}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14\cdot 15}+\cdots,$$

$$\frac{z^2}{1\cdot 2}-\frac{z^6}{1\cdot 3\cdot 5\cdot 6}+\frac{z^{10}}{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 10}-\frac{z^{14}}{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 14}+\cdots,$$

prove that the sum of the squares of the first two series is double the third series.

4092. *Proposed by V. Thébault, San Sebastián, Spain*

For the triangle ABC let (A_1, B_1, C_1), (A_2, B_2, C_2), $\cdots$, (A_n, B_n, C_n) be the centers of squares constructed exteriorly (or interiorly) on the sides (BC, CA, AB), (B_1C_1, C_1A_1, A_1B_1), $\cdots$, (B_nC_n, C_nA_n, A_nB_n) of the corresponding triangles. (1) Show that the center ω_1 of the circle orthogonal to the circles with centers A, B, C and radii B_1C_1, C_1A_1, A_1B_1 coincides with the center of the nine point circle of ABC. (2) Find the locus of centers $\omega_2, \omega_3, \cdots, \omega_n$ of the circles orthogonal to the circles with centers A, B, C and with radii (B_2C_2, C_2A_2, A_2B_2), (B_3C_3, C_3A_3, A_3B_3), $\cdots$, (B_nC_n, C_nA_n, A_nB_n).

Correction. In the first line of 4079 [1943, 264] replace "complexes" by "simplexes."

SOLUTIONS

Similar Triangles on Sides of a Quadrilateral

4034 [1942, 263]. *Proposed by V. Thébault, San Sebastián, Spain*

On the sides AB, BC, CD, DA of a convex quadrangle $ABCD$ equilateral triangles with vertices A', B', C', D' are constructed exteriorly (or interiorly). Show that the diagonals $A'C'$ and $B'D'$ of quadrangle $A'B'C'D'$ are perpendicular (or equal) according as the diagonals AC and BD of $ABCD$ are equal (or perpendicular), and conversely.

Solution by J. R. Musselman. In this MONTHLY vol. 40 (1933) p. 159, I proved that if we construct externally on the sides of a positive n-gon of type M directly similar triangles, their n vertices would form a positive n-gon of type M. The converse is also true. For $n=4$, the n-gon is a pseudo-square, *i.e.*, a quadrilateral whose diagonals are both equal and perpendicular, and the theorem would read: If on the sides of a positive ordered pseudo-square we construct directly similar triangles, their four vertices are a positive pseudo-square. The converse is also true.

Let us denote the coordinates of the points A, B, C, D, by the complex

numbers a, b, c, d respectively, and let the triangles all be similar to some triangle $X_1X_2X_3$ whose vertices have the coordinates x_1, x_2, x_3. It follows that

$$\begin{vmatrix} a' & b & a \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} b' & c & b \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} c' & d & c \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} d' & a & d \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

whence

$$\begin{vmatrix} c' - a' & d - b & c - a \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} d' - c' & a - c & d - b \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

or

$$(x_2 - x_3)(c' - a') = (x_3 - x_1)(b - d) + (x_1 - x_2)(a - c),$$

and

$$(x_2 - x_3)(d' - c') = (x_3 - x_1)(c - a) + (x_1 - x_2)(b - d).$$

If the diagonals $A'C'$ and $B'D'$ are to be equal, then

$$\begin{aligned}(1)\quad &[(x_3-x_1)(b-d)+(x_1-x_2)(a-c)][(\bar{x}_3-\bar{x}_1)(\bar{b}-\bar{d})+(\bar{x}_1-\bar{x}_2)(\bar{a}-\bar{c})] = \\ &\qquad [(x_3-x_1)(c-a)+(x_1-x_2)(b-d)][(\bar{x}_3-\bar{x}_1)(\bar{c}-\bar{a})+(\bar{x}_1-\bar{x}_2)(\bar{b}-\bar{d})].\end{aligned}$$

This will be satisfied for arbitrary $X_1X_2X_3$ if

$$(1a)\qquad (b - d)(\bar{b} - \bar{d}) = (c - a)(\bar{c} - \bar{a}),$$

and

$$(1b)\qquad (b - d)(\bar{a} - \bar{c}) = (c - a)(\bar{b} - \bar{d}),$$

which state that AC and BD are equal and perpendicular.

If the diagonals $A'C'$ and $B'D'$ are to be perpendicular, then

$$\begin{aligned}(2)\quad &[(x_3-x_1)(b-d)+(x_1-x_2)(a-c)][(\bar{x}_3-\bar{x}_1)(\bar{c}-\bar{a})+(\bar{x}_1-\bar{x}_2)(\bar{b}-\bar{d})] = \\ &\qquad [(x_3-x_1)(a-c)+(x_1-x_2)(d-b)][(\bar{x}_3-\bar{x}_1)(\bar{b}-\bar{d})+(\bar{x}_1-\bar{x}_2)(\bar{a}-\bar{c})].\end{aligned}$$

This will be satisfied for arbitrary $X_1X_2X_3$ if

$$(2a)\qquad (b - d)(\bar{a} - \bar{c}) = (c - a)(\bar{b} - \bar{d})$$

and

$$(2b)\qquad (b - d)(\bar{b} - \bar{d}) = (c - a)(\bar{c} - \bar{a}).$$

Therefore if we construct directly similar triangles on the sides of the quadrilateral $ABCD$ the condition that the diagonals $A'C'$ and $B'D'$ be either equal or perpendicular implies that AC and BD be both equal and perpendicular and

hence, from my earlier theorem, $A'C'$ and $B'D'$ cannot be equal without being perpendicular, or cannot be perpendicular without being equal.

It is possible to choose particular triangles $X_1X_2X_3$ so that the condition of equality of $A'C'$ and $B'D'$ does not imply perpendicularity. For condition (1a) or (2a) will not exist if

$$(x_1 - x_2)(\bar{x}_1 - \bar{x}_2) = (x_3 - x_1)(\bar{x}_3 - \bar{x}_1)$$

which implies $\overline{X_2X_1} = \overline{X_1X_2}$. Consequently we have the theorem:

If we construct exteriorly (or interiorly) similar isosceles triangles on the sides of a quadrilateral ABCD, the diagonals $A'C'$ and $B'D'$ will be equal if AC and BD are perpendicular; the diagonals $A'C'$ and $B'D'$ will be perpendicular if AC and BD are equal.

Secondly conditions (1b) and (2b) will not hold if

$$(x_3 - x_1)(\bar{x}_1 - \bar{x}_2) = (x_2 - x_1)(\bar{x}_3 - \bar{x}_1),$$

which implies that $X_1X_2X_3$ is a right triangle with a right angle at X_1. Consequently we have:

If we construct exteriorly (or interiorly) directly similar right triangles on the sides of ABCD as hypotenuses, the diagonals $A'C'$ and $B'D'$ will be equal if AC and BD are equal; the diagonals $A'C'$ and $B'D'$ will be perpendicular if AC and BD are perpendicular.

The converses of these two theorems may be proven.

Thirdly, conditions (1a) and (1b) or (2a) and (2b) will be automatically satisfied for the particular triangle $X_1X_2X_3$ if it be an isosceles right triangle with right angle of X_1. This exceptional case gives the theorem:

If we construct exteriorly (or interiorly) directly similar isosceles right triangles on the sides of any quadrilateral ABCD as hypotenuses, the vertices of the right angles forming a quadrilateral $A'C'B'D'$, then the diagonals $A'C'$ and $B'D'$ are always equal and perpendicular independently of any relation between the diagonals AC and BC.

Solved also by Howard Eves.

Editorial Note. The proposer stated that the theorem of the problem is an application of more general theorems which will appear in the *Annales de la Société Scientifique de Bruxelles.* Eves used rectangular coordinates in his proof of the theorem of the problem. We could also use vectors or complex coordinates, and it requires just a trifle more labor to generalize the problem for directly similar triangles in place of equilateral. It may appear simpler to use complex coordinates and to state the results in vector form as follows. Let a, b, c, d be the complex coordinates of A, B, C, D in positive order of rotation, forming a quadrilateral, convex or not. Directly similar triangles are constructed exteriorly (or interiorly) on its sides, for example the triangle ABA' with angle $BAA' = \theta$,

$-\pi \leqq \theta < \pi$, and with the ratio of lengths of sides $AA'/AB = p > 0$. Then $a' = (1-p\lambda)a + p\lambda b$, $\lambda = e^{\theta i}$; and we find that

$$(B'D')^2 - (A'C')^2 = (1 - 2p\cos\theta)(BD^2 - AC^2) + 4p(p - \cos\theta)BD\cdot AC,$$
$$B'D'\cdot A'C' = p(\cos\theta - p)(BD^2 - AC^2) + (1 - 2p\cos\theta)BD\cdot AC,$$

where the means the scalar product of the indicated vectors. The determinant of the coefficients of the right-hand members of the equations is zero if and only if each coefficient is zero, and then $p = \cos\theta = \sqrt{2}/2$. This exceptional case gives the theorem at the end of the above solution. A discussion of the remaining cases is easy.

A Family of Surfaces

4028 [1942, 202]. *Proposed by P. D. Thomas, Norman, Oklahoma*

Find the equation of a family of surfaces, each surface satisfying $(OP)^2 = (OQ)^2$, where O is the origin, P is any point on the surface, and Q is the point in which the normal to the surface at P meets the xy-plane.

Find the envelope of the family.

Solution by the Proposer. The points P and Q have coordinates (x, y, z) and $(x+pz, y+qz, 0)$ respectively, where $p = \partial z/\partial x$, $q = \partial z/\partial y$. Imposing the given condition gives

$$x^2 + y^2 + z^2 = x^2 + 2pxz + p^2z^2 + y^2 + 2yqz + q^2z^2,$$

which reduces to

$$F = z(1 - p^2 - q^2) - 2(px + qy) = 0. \tag{1}$$

To solve the partial differential equation (1) employ the Charpit equations,

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q} = \frac{-dp}{F_x + pF_z} = \frac{-dq}{F_y + qF_z}, \quad \text{where } F_x = \frac{\partial F}{\partial x}, \quad \textit{etc.}$$

Consider the third of these equations, supplying the values of the partials from (1). This equation is

$$dz/(-2p^2z - 2px - 2q^2z - 2qy) + dp/ - p(1 + p^2 + q^2) = 0. \tag{4}$$

The denominator of the first member of (4) may be written $-z(1+p^2+q^2)$ in the light of equation (1). Hence (4) becomes $dz/-z(1+p^2+q^2)+dp/-p(1+p^2+q^2) = 0$, or simply $dz/z+dp/p=0$, and integrating we find that $p=a/z$. This value of p placed in equation (1) and the resulting equation solved for q gives

$$q = \frac{-y \pm (y^2 - 2ax + z^2 - a^2)^{1/2}}{z},$$

and using $dz = pdx + qdy$, we get

$$\frac{ydy - adx + zdz}{\pm (y^2 - 2ax + z^2 - a^2)^{1/2}} = dy;$$

and integrating, $\pm(y^2-2ax+z^2-a^2)^{1/2}=y+b$. Rationalizing this last result gives

$$z^2 = 2ax + 2by + a^2 + b^2,$$

a two-parameter family of parabolic cylinders. The elimination of p and q among $F=0$, $F_p=0$, $F_q=0$, gives for envelope the isotropic cone $x^2+y^2+z^2=0$.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Professor A. A. Albert of the University of Chicago has been elected to membership in the National Academy of Sciences.

Professor Emeritus Arthur Pelletier of the École Polytechnique in Montreal has received the honorary degree of Doctor of Mathematics from the University of Montreal.

Dr. O. B. Ader and Dr. F. C. Gentry have been appointed visiting assistant professors at the University of New Mexico.

Professor R. C. Archibald of Brown University has become professor emeritus.

Assistant Professor R. G. Archibald of Queens College has been promoted to an associate professorship.

Assistant Professor H. R. Cooley of New York University has been promoted to an associate professorship.

Professor D. R. Curtiss of Northwestern University has become professor emeritus.

Assistant Professor Nelson Dunford of Yale University has been promoted to an associate professorship.

Dr. Paul Erdös has been appointed visiting lecturer at Purdue University.

Associate Professor J. S. Frame of Allegheny College has been appointed professor of mathematics and head of the department of mathematics at Michigan State College.

Dr. Abe Gelbart of North Carolina State College, Dr. P. R. Halmos of the University of Illinois, and Dr. Hans Samelson of the University of Wyoming have been appointed assistant professors at Syracuse University.

Assistant Professor J. W. Green of the University of Rochester has been appointed mathematician at the Aberdeen Proving Grounds.

W. R. Hardman and Dr. Ivan Niven of Purdue University have been promoted to assistant professorships.

Professor W. L. Hart, who was on leave of absence from the University of Minnesota, has returned to that institution.

Associate Professor W. L. Hutchings of Rollins College has been appointed to an associate professorship at Whitman College, Walla Walla, Washington.

C. J. Kirchen of State Teachers College, Mankato, Minnesota, has been appointed product control statistician at the U. S. Rubber Company Ordnance Plant, Eau Claire, Wisconsin.

W. A. Lafferty has been appointed teacher of mathematics at John Burroughs School, Clayton, Missouri.

Miss Mary Ann Lee has been appointed teaching assistant at Cornell University.

Dr. B. A. Lengyel of the College of the City of New York has been appointed to an assistant professorship at the University of Rochester.

Associate Professor C. I. Lubin of the University of Cincinnati has accepted a position with the Kellex Corporation of New York City.

Assistant Professor W. T. Martin of Massachusetts Institute of Technology has been appointed head of the department of mathematics at Syracuse University.

Associate Professor A. F. Moursund of the University of Oregon has been promoted to a professorship.

Professor Ruth M. Peters has been granted a leave of absence from Lake Erie College to serve as a research mathematician of the Applied Mathematics Group at Columbia University.

Dr. S. E. Rauch of Santa Barbara Teachers College and Dr. A. E. Poole of Montana School of Mines have been appointed assistant professors of mathematics and physics at the University of Oregon.

Dr. P. C. Rosenbloom of Brown University has been awarded a National Research Fellowship on a participating basis at Indiana University for the year 1943–44.

Assistant Professor F. C. Smith of the College of St. Francis in Joliet, Illinois, is now in the actuarial department of the Lincoln National Life Insurance Company in Fort Wayne, Indiana.

Assistant Professor F. H. Steen of Allegheny College has been promoted to an associate professorship and appointed head of the department.

Professor J. L. Synge of the University of Toronto has been appointed chairman of the department of mathematics at Ohio State University to replace Professor H. W. Kuhn who has retired. Also he has been elected a fellow of the Royal Society.

Dr. A. D. Wallace of the University of Pennsylvania has been promoted to an assistant professorship.

Assistant Professor G. C. Vedova of the University of Maryland has been appointed visiting associate professor at Haverford and is teaching in the pre-meteorological unit of the army air corps quartered there.

Dr. Max Wyman of the National Research Council of Canada has been appointed lecturer in mathematics at the University of Alberta.

Dr. Leo Zippin of Queens College has been promoted to an assistant professorship.

The following appointments to instructorships are announced:

Cornell College (Naval Flight Preparatory School): Frances Freese

New Mexico State College (Army Specialized Training Program): N. W. Wells (instructor in physics)

New York University: Dr. P. B. Norman

Purdue University: Paul Irick, Dr. W. P. Reid

Reed College: Dr. R. W. Shephard

U. S. Navy Pre-flight School (Chapel Hill, N. C.): C. W. Trigg

University of Buffalo: Dr. Paul Civin, Ruth A. Brendel

University of Minnesota: Ernest Johnston

University of Oregon (Army Programs): Mrs. Esther F. Alkire, G. R. Costello, Mrs. Lola R. Hamilton, Bessie V. Kamarad, Mrs. Jeannette Lund, Marie Ridings, J. J. Rowland, W. G. Scobert.

Yale University: Dr. S. P. Avann, Dr. W. H. Durfee, Dr. C. E. Rickart.

Professor Guido Fubini, formerly professor of mathematics at the University of Turin and since 1939 a member of the Institute for Advanced Study, died on June 6, 1943.

Professor Emeritus G. A. Harter of the University of Delaware died July 22, 1943.

Dr. H. E. Hawkes, professor of mathematics at Columbia University and dean of the college, died on May 4, 1943. He was a charter member of the Mathematical Association.

Professor R. L. Menuet of Tulane University died on May 9, 1943.

Dr. W. H. Metzler, formerly professor of mathematics and dean of the Graduate School of Syracuse University and later dean of the State Teachers College at Albany, N. Y., died on April 18, 1943.

Professor Emeritus W. F. Osgood of Harvard University died July 22, 1943. He was a charter member of the Mathematical Association.

H. M. Showman, lecturer and registrar of the University of California at Los Angeles, died June 24, 1943. He was a charter member of the Mathematical Association.

Professor Emeritus Clara E. Smith of Wellesley College died on May 12, 1943. She was a charter member of the Mathematical Association.

Professor Emeritus H. S. White of Vassar College died May 20, 1943.

Dr. J. E. Williams, professor of mathematics and dean of the faculty of Virginia Polytechnic Institute, died on April 19, 1943. He was a charter member of the Mathematical Association.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

THE UNITED STATES ARMED FORCES INSTITUTE

In recognition of the educational needs of men in the armed forces, the War Department established the Army Institute at Madison, Wisconsin, in March, 1942. Personnel of the United States Navy, the Coast Guard, and the Marine Corps were also eligible to enroll for courses. In February, 1943, the name was changed to the United States Armed Forces Institute. Since that time the Institute has provided an opportunity for personnel of the armed forces to continue their education so as to contribute to their military efficiency, and to benefit them upon their return to civil life.

The Institute offers the prospective student a choice of two plans of registration; he may enroll in any one of the sixty-four courses in the eight fields of study offered directly under the auspices of the Institute, or he may enroll for an extension course offered by any one of the seventy-nine colleges and universities cooperating in the program. A uniform fee of two dollars is charged for any Institute course; in the case of university extension courses, the government pays one half the total cost of tuition and texts, not to exceed twenty dollars for any one course.

In mathematics, the Institute has organized courses in arithmetic, algebra, geometry, trigonometry, analytic geometry, and calculus. A total of forty-four courses are available in pure or applied science. Other subjects treated are English, the social studies, and business.

It is urged that school officials assist students about to enter military service by planning future programs of education. To quote Colonel F. T. Spaulding, Chief of the Education Branch, "The objective might well be to plan with each student prior to induction a duration program of courses, approved courses which will be accepted for graduation or degree requirements."

Additional information upon the program may be obtained by writing to the Commandant, United States Armed Forces Institute, Madison, Wisconsin. All Army and Navy establishments have a supply of catalogs and application forms.

THE MICHIGAN DIVISION FOR EMERGENCY TRAINING

The University of Michigan recently created within the institution a Division for Emergency Training. Its function is to accommodate special programs and extraordinary groups of trainees whose presence upon the Michigan campus is a result of the war. In particular, the Division aids the several schools and col-

leges in arranging special military programs, avoiding, as far as possible, interference with regular offerings. Complete records are kept of the work which the special students are doing so that they may complete their educational plans with a minimum of delay after the war.

The Division has also been interested in the development of emergency curricula for civilians. One of the first programs set up was the Basic Curriculum Preparatory to War Service. Addressed to recent high school graduates and selected high school seniors, this curriculum offers an integrated course in mathematics and physics, an integrated course in history, English, and American institutions, and a course in physical education. The program extends over a period of twenty-four weeks, thus enabling most high school graduates to complete the work before entering the armed forces. Students spend more than thirty hours each week in class study under the supervision of their instructors; also they live together with an experienced counselor. The intensive course in physics and mathematics covers the equivalent of a year's work in each of these fields.

PRE-INDUCTION COURSES IN MATHEMATICS

Soon after the start of the war, the United States Office of Education began receiving urgent and repeated requests from individuals and organizations throughout the country to give the secondary schools detailed suggestions for the teaching of mathematics for pre-induction purposes. In December, 1942, the Office in cooperation with the President of the National Council of Teachers of Mathematics appointed a committee to make a survey of the mathematical needs of the armed forces, and upon this basis to make a report concerning what the schools might do for the emergency. The committee consisted of Virgil S. Mallory, Professor of Mathematics, New Jersey State Teachers College at Montclair; William D. Reeve, Professor of Mathematics, Teachers College, Columbia University; Giles M. Ruch, Chief, Research and Statistical Service, U. S. Office of Education; Raleigh Schorling, Professor of Education, University of Michigan; and Rolland R. Smith, Specialist in Mathematics for the Public Schools of Springfield, Massachusetts, and President of the National Council of Teachers of Mathematics. Dr. Smith served as chairman of the Committee.

It was the thought of the Committee that detailed analyses of a large number of post-induction training and field manuals would provide a first approximation to the mathematical needs of the military establishments, or at least afford a basis for evaluating the general character of the curricular modifications in secondary school mathematics dictated by the emergency. To this end, the Committee requested and received the fullest cooperation of the Army, the Navy, and the Civil Aeronautics Administration.

The final report of the Committee entitled "Pre-Induction Courses in Mathematics" was published in *Education for Victory*, the Office of Education journal, for April first, and also appeared in the March number of *The Mathematics Teacher*. Reprints of the report may be had at ten cents each postpaid by writing to The Mathematics Teacher, 525 W. 120th St., New York, N. Y.

FROM THE WAR POLICY COMMITTEE

Important developments in pure and applied mathematics are taking place during the war period. Unusual conditions prevail in the mathematics departments of our colleges and universities. Many new connections are being established by mathematicians with public affairs, with industry, and with business. During normal times all these matters would receive adequate attention. New mathematical results would find their way into the scientific periodicals; likewise, the problems raised by the extraordinary demands made upon the colleges of the country would be discussed at meetings and in journals devoted to such matters.

Most of this reporting and discussion is impossible at the present time. Publicity is out of the question for most of the important work that is being done. Very few people have the time, if they had the inclination, to write or speak about their experiences and their problems. But it should need no argument to convince mathematicians of the necessity of keeping complete records of the questions which they have considered and of the solutions that are being obtained.

If a person, or a group of persons, could be found willing to undertake the task of gathering the information which has a bearing on these matters, and if there were much hope that such information could be secured at the present time, it would be our obvious obligation to undertake this task. But neither of these hopes has much chance of realization.

Thus, it becomes of urgent importance that every mathematician, whose work is in any way affected by the war, keep a complete and careful record of his personal activities and experiences. If this is done, the future work of the historian of mathematics may be greatly facilitated. Unless it is done, much that is valuable and interesting will fade out; the extremely important role played by mathematicians during the national emergency will be forgotten, and significant achievements will be lost.

For these reasons the War Policy Committee of the American Mathematical Society and of the Mathematical Association of America urges the members of these organizations to keep a careful and detailed record of their scientific and educational war work.

ARNOLD DRESDEN,
Secretary Ad Interim

THE ARMY SPECIALIZED TRAINING PROGRAM

Since late March of the present year, the Army Specialized Training Program (ASTP) has been in operation upon college and university campuses in this country. As of July 22, 190 collegiate institutions had been invited to participate in the project. Reflecting a rapid rate of progress in the development of the ASTP, authorization has been issued to Commanding Generals of Service Commands to negotiate contracts to expand a number of existing units as well as to establish new units at other institutions. Virtually all of the new units

established during the summer were to be activated in time for the start of the term on August 9.

The processing of soldiers into the Program has been taking place at a rate well ahead of schedule. More than 60,000 soldiers were located in Army Specialized Training Units in late July, and upward of 20,000 others were to enter the Program in the term starting in August. In addition, approximately 17,000 soldiers, (July 22), were registered in Specialized Training and Reassignment (STAR) Units.

At STAR Units, selected soldiers are housed, classified, instructed, and finally are assigned to specific courses of study in the ASTP at a level and in a field for which they seem to be best qualified. Thirty-one STAR Units have been designated. Soldiers remain at these Units for a period of from five to thirty days, and are then sent to Army Specialized Training Units. In the latter Units, trainees receive prescribed instruction in specific curricula for one or more twelve-week terms. The ASTP gives the young soldier an opportunity for medical instruction, dental training, veterinary medical training, basic phase instruction, training in advanced engineering, foreign area and language study, training in advanced personnel psychology, or preprofessional training. The majority of participating institutions give instruction in the basic phase. In many cases, a college has units in more than one field of study.

In June, the War Department announced the creation of the Army Specialized Training Reserve Program under which qualified high school graduates between 17 and 18 years of age will be granted military scholarships providing for basic phase instruction in the Army Specialized Training Program at selected colleges and universities. The Reserve Program will be limited to those volunteers who received qualifying scores on the preinduction test administered last April 2, and to those who qualify in similar tests to be given in the future. The next test is scheduled for early November.

A maximum quota of 25,000 of these ASTP Reservists has been established by the War Department. This is in addition to the quota of 150,000 set for the number of soldiers participating in the Army Specialized Training Program at any one time. Unlike ASTP trainees, the Reservists will not be on active duty, nor will they receive basic military training before entering the ASTP Program. Instead, they will be Enlisted Reservists on inactive duty and will wear civilian attire.

The Army Specialized Training Reserve Program is aimed to provide a direct flow of qualified young men toward Army Specialized Training prior to their entering active military duty. There will thus be established a constant reservoir of men with aptitudes and capacity for college-level training to meet the needs of the various Services for high-grade technicians, specialists, and candidates for officer training.

At the end of the term in which the trainee reaches his eighteenth birthday he will be placed on active military duty and will be sent to an Army Replacement Training Center for the prescribed basic military training. On completion

of that training, he will be sent, if qualified, to a STAR Unit where he will be screened for continuation in the ASTP in a particular field of study and upon the highest level for which he is found qualified. He will then be assigned to an Army Specialized Training Unit.

The ASTP Reservists' work load will be similar to that of the regular ASTP trainee. The ASTR physical training program will be modified to take account of the younger age of the Reservists. The total work week will include approximately 57 hours of supervised activity, made up of 24 hours of classroom and laboratory work, 24 hours of required study, and the balance devoted to military and physical instruction. Where a student is sent to an institution at which there is an ROTC unit, he will be given ROTC instruction.

The military scholarship will provide for payment of tuition, messing, housing, and such medical service as is customary at the institution. Instruction in the Reserve Program began this summer. The first group of eligibles were chosen from those who qualified in the preinduction test administered last April 2, and who had not attained their eighteenth birthday prior to August 15.

In July, the War Department announced a modification of the original plan of training ROTC students called to active duty. It is now possible to permit this group to be returned to school for additional academic training designed to increase their value as future officers.

Second year advanced ROTC students who have graduated are now being placed in officer candidate schools as rapidly as vacancies become available. Second year advanced ROTC students who have not graduated are being permitted to return to school on an inactive status, at their request, to continue their academic work. This group will be permitted to remain in college to complete the semester or quarter in progress on December 31, 1943, unless sooner graduated. Second year advanced ROTC students awaiting assignment to Officer Candidate Schools and those who do not elect to return to college will be held in replacement training centers, where they will be utilized as assistant instructors.

First year advanced ROTC students reported during the summer at replacement training centers for basic military training. Upon completion of this training they are being returned to college under the supervision of the Army Specialized Training Division, pending the availability of vacancies in officer candidate schools. During this period, which may be long enough to provide two or more quarters, they will be given academic instruction designed to make them more useful officers in their branch of Service.

A course in mathematics is taught during each of the three terms of the basic phase instruction of the ASTP; these courses are numbered AST-406, AST-407, and AST-408. For those who continue to the advanced phase in engineering, a fourth term of mathematics, AST-401, is taught. The topical outlines supplied for each of these courses appear below; they were drawn up by an advisory committee of mathematicians.

Mathematics AST-406

(Time Allowance: 6 hours per week)

Addition, subtraction, multiplication, and division of polynomials. Factoring of following types: (a) perfect squares; (b) difference of two squares; (c) x^2+ax+b; (d) factoring by grouping terms; (e) sum and difference of cubes. Operation with fractions. Evaluation of formulas: Include elementary mensuration of plane and solid figures. Linear equations in one and two unknowns. Exponents and radicals. Logarithms: Accuracy of computation is to be stressed; five-place tables recommended. Quadratic equations in one unknown. Also include solution of linear-quadratic pairs of simultaneous equations. Complex numbers. Ratio, proportion, and variation; use illustrative material from plane and solid geometry. Binomial theorem. Trigonometric functions. Recommended introduction of functions of a general angle immediately. Fundamental identities. Computation on right triangles. Components of a vector. Radian measure. (Also introduce the mil.) Graph of the sine and cosine function. Trigonometric identities including addition formulas for sine, cosine, and tangent. Double and half angle formulas. Solution of oblique triangles. Insist on accuracy of computation.

Suggested Texts: Brink, *Intermediate Algebra*; Brink, *Plane Trigonometry*; Crathorne and Lytle, *Plane Trigonometry*; Curtiss and Moulton, *Essentials of Trigonometry*; Peterson, *Intermediate Algebra for College Students*; Rietz and Crathorne, *College Algebra*, 4th Edition; Rosenbach and Whitman, *College Algebra*; Rosenbach and Whitman, *Plane Trigonometry.*

Mathematics AST-407

(Time Allowance: 5 hours per week)

Rectangular coordinates. Distance and slope formulas. The straight line. Curve and equation: The derivation of the equation of the curves from given conditions. A thorough discussion of the curve from its equation including such topics as excluded values, symmetry, vertical and horizontal asymptotes, etc., as a preliminary to the drawing of the graph. The circle. Translation and rotation of coordinate axes. The conics, (not more than six or eight lessons). Polar coordinates. A careful drill in plotting of graphs. Parametric equations. Transcendental equations. Include addition and multiplication of ordinates. Stress curves like $y=e^{ax}\cos bx$, etc. Solid analytic geometry: (a) Rectangular, cylindrical, and spherical coordinates; (b) Planes and straight lines; (c) Second degree surfaces. Sketch by means of sections parallel to axes.

Suggested Texts: Love, *Analytic Geometry*; Smith, Gale, and Neeley, *New Analytic Geometry*; Wilson and Tracey, *Analytic Geometry.*

Mathematics AST-408

(Time Allowance: 5 hours per week)

Definition of derivative. Finding the derivative by the increment process. Differentiation of algebraic functions. Application of the derivative to: (a) Equation of tangent and normal; (b) Rate; (c) Maxima and minima. Differentiation of transcendental functions with further applications of the derivative. Differentials with applications to approximation, etc. Integration of standard forms, integration by trigonometric substitution, integration by parts. The definite integral with applications, including areas and volumes.

Suggested Texts: Granville, Smith, and Longley, *Differential and Integral Calculus*; Love, *Differential and Integral Calculus*; Miller, *Calculus*; Sherwood and Taylor, *Calculus.*

Mathematics AST-401

(Time Allowance: 5 hours per week)

Formal integration completed with further applications, including pressure and work. Radius of curvature. Curvilinear motion. Infinite series: (a) Convergence and divergence; (b) Maclaurin and Taylor series; (c) Applications to computation. Partial differentiation with applications. Double and triple integration. Centers of gravity. Moments of inertia. Differential equations: (a) Linear of first order, (b) Second order with constant coefficients; particular integral to be found by method of undetermined coefficients. In differential equations, as throughout the whole work in calculus, emphasis is to be placed on applications to physical problems.

Suggested Texts: Granville, Smith, and Longley, *Differential and Integral Calculus*; Love, *Differential and Integral Calculus*; Miller, *Calculus*; Sherwood and Taylor, *Calculus.*

WOMEN IN AEROLOGY AND ELECTRONICS

The Navy is making an urgent search for women to perform important services in the Aerology and Electronics programs. In addition to the regular requirements for a candidate for Officers Training, the following qualifications are necessary.

For Aerology, college mathematics through integral calculus and one year of physics. For Radar, one year of college mathematics and one year of college physics.

After the regular Officers Training School, persons enlisted in these programs will continue with special work at various colleges and universities throughout the country.

The contribution to the war effort to be made through these programs is important. For further details consult your nearest Office of Naval Officer Procurement.

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE APRIL MEETING OF THE OHIO SECTION

The twenty-eighth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, on April 1, 1943. Contrary to the custom, only an afternoon session was held, the usual dinner and evening session being omitted. In the absence of the Secretary, Professor V. B. Caris served in his place at the meeting.

Thirty-six persons were in attendance, including the following twenty-seven members of the Association: W. E. Anderson, F. R. Bamforth, Grace M. Bareis, I. A. Barnett, Henry Blumberg, H. F. Bright, O. E. Brown, C. T. Bumer, V. B. Caris, Paul Cramer, Sister Mary C. Garvin, J. F. Heyda, Margaret E. Jones, H. K. Justice, A. C. Ladner, Edith J. McKissock, C. G. Maple, R. H. Marquis, Florentina Mathias, C. C. Morris, H. S. Pollard, Tibor Radó, S. E. Rasor, K. C. Schraut, I. L. Stright, M. B. Tolar, F. B. Wiley.

The following officers were elected for the coming year: Chairman, Tibor Radó, Ohio State University; Secretary-Treasurer, Rufus Crane, Ohio Wesleyan University; Member of the Executive Committee, C. R. Wylie, Jr., Ohio State University; Member of the Program Committee, F. B. Wiley, Denison University. It is expected that the next meeting will be held at the Ohio State University on Thursday, April 6, 1944.

The following program was presented:

1. *The pre-meteorological training program—an experiment and a challenge*, by Professor C. T. Bumer, Kenyon College.

Professor Bumer discussed the mathematical aspects of the pre-meteorological training program in the light of his contact with that work at Kenyon College. He remarked that, after slow development in America, as compared with the Scandinavian countries and Germany, the study of meteorology in this country began in earnest about twelve years ago. He pointed out that thousands of practical meteorologists must now be trained quickly to meet the needs of the armed forces, while at the same time thorough training in mathematics and physics must be provided so that the men will be able to understand the basic problems of meteorology. It was also stated that a thorough knowledge of differential equations is essential to the meteorologist, and that the interrelation of mathematics and physics is a characteristic feature of the training program.

2. *Determinants via mathematical induction*, by Professor F. B. Wiley, Denison University.

The content of this paper furnishes a concise introductory treatment of determinants, and also affords an opportunity for the student to obtain more than the usual amount of experience with mathematical induction. The elementary

properties of determinants were established by mathematical induction as consequences of the following definition:

$$| a_{ij} |_{i,j=1,\cdots,n} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} A_{ij}; \qquad | a_{11} | = a_{11}.$$

In particular, the theorems on the change of rows into columns, on the multiplication of determinants, on the solution of n simultaneous linear equations in n unknowns, and on the eliminant of a set of n homogeneous linear equations in n unknowns, were established by this same type of reasoning.

3. *On the completely numerical solution of differential equations*, by Professor O. E. Brown, Case School of Applied Science.

Professor Brown remarked that Milne's method for numerical approximation to the particular solution of the differential equation $y'=f(x, y)$ determined by the initial condition that the integral curve must pass through the point (x_0, y_0) requires, for starting the solution, the values of the ordinates y_0, y_1, y_2, and y_3 at the equally spaced points along the x-axis whose abscissas are x_0, x_1, x_2, and x_3. Two formulas are required, one for extending the table of known points, and one for checking and correcting the new entries. In this paper there were presented similar formulas for advancing and checking which require only one point, only two, and only three, by use of which the method may be applied from the beginning.

4. *Effects of the war upon mathematics in Ohio*, by Professor H. K. Justice, University of Cincinnati.

This paper was the report of a committee directed by Professor Justice. The speaker explained that the committee had sent questionnaires to the mathematics departments of all the colleges and universities of the state, soliciting information on new courses, enrollment, staff changes, and various activities related to the war effort. Then, with the assistance of Professors Anderson and Blumberg, he presided over an open discussion in which members exchanged views and reported war-time developments in mathematics. The committee then reviewed pertinent questionnaires from institutions not represented at the meetings.

5. *Whither American Mathematics?* by Professor Henry Blumberg, Ohio State University.

The content of Professor Blumberg's paper was an outgrowth of the study reported in the preceding paper. The speaker discussed the present challenge to American mathematicians, particularly in relation to liaison with the applications. He considered some of the difficulties in the way of greater attainments, and indicated along what lines these difficulties might be met, making a number of specific recommendations. He finally discussed the character of the general problem facing American mathematics, of which the liaison problem is only a part.

RUFUS CRANE, *Secretary*

THE ANNUAL MEETING OF THE MISSOURI SECTION

The annual meeting of the Missouri Section of the Mathematical Association of America was held in conjunction with the meeting of the Missouri State Teachers Association at Kansas City, Missouri, on Friday, December 4, 1942. The meeting consisted of a morning session which was held in the Continental Hotel.

The Secretary of the Section finds himself unable to provide a list of the members of the Association who attended the meeting. The following officers were elected for the coming year: President, Professor R. R. Middlemiss, Washington University; Secretary, W. E. Ferguson, University of Missouri.

The following papers were presented:

1. *Inscribing triangles in simple closed plane curves*, by Dr. J. V. Wehausen, University of Missouri.

The speaker shows that if C is a simple closed plane curve, if P is any point interior to the curve, and if θ is any angle, then there are two points A and B on C such that $PA=PB$, and angle APB is equal to θ. He proved also that if there is a triangle with sides of lengths a, b, and c, then there are three points P, Q, and R on the curve such that the distances PQ, QR, and RP are proportional to a, b, and c. The proofs of these two theorems were based upon the Jordan separation theorem. The proof of the first is similar to a proof given by H. E. Vaughan for the case in which $\theta=180°$. Vaughan's proof was printed in this MONTHLY, vol. 46, 1939, p. 657.

2. *A two dimensional representation of vectors and scalars*, by Dr. F. P. Beer, University of Kansas City.

Dr. Beer's paper will appear in a later issue of this MONTHLY.

3. *Mathematics in the C. P. T. program*, by W. E. Ferguson, University of Missouri.

In this paper it was explained that the elementary portion of the C. P. T. program calls for thirty-six hours of instruction in mathematics. In regard to preparation, it was stated that the men form a very heterogeneous group, some being high school graduates of fifteen years ago, while others hold advanced degrees in various fields. The mathematical training of some of the men consists of only two units in high school, while others have had as much as fifteen hours of college mathematics.

The course includes a review of fundamental operations in arithmetic and algebra, ratio and proportion, linear equations, formulas, time zones, graphs, scales, angle measurement, vectors, and the use of the Dalton Mark VII computer. Excluding the technical topics and applications, the course includes nothing beyond second year high school algebra. Various teaching difficulties were mentioned, and it was stated that students of average intelligence with three units of mathematics in high school succeed very well in the work.

4. *Why not a thorough revision of freshman mathematics?* by Father W. C. Doyle, Rockhurst College, introduced by Dr. Shanks.

Father Doyle described the course in freshman mathematics given at Rockhurst College. Distinctive features of the course described were an early introduction to analytic geometry, the inclusion of a large number of computational problems involving decimals (particularly in plotting curves), and decreased emphasis upon radicals and certain other phases of formal algebra.

5. *Pointless geometry*, by Dr. M. E. Shanks, University of Missouri.

This speaker outlined several postulate systems for geometry and topology which start with elements other than the point as the primitive concept. In particular, the postulate systems of Huntington and Wald were singled out as representative of what might be called geometries of "lumps." Topological spaces derived from lattices were also mentioned.

M. E. SHANKS, *Secretary*

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-sixth Summer Meeting, New Brunswick, N. J., September 11–13, 1943.

Twenty-seventh Annual Meeting, Chicago, Ill., November 27–28, 1943.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., April, 1944
ILLINOIS
INDIANA, Indianapolis, Oct. 29–30, 1943
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, Jan. 29, 1944
OHIO, Columbus, April 6, 1944
OKLAHOMA
PHILADELPHIA, Philadelphia, Nov. 27, 1943
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 11, 1944
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN

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AZIMUTH LINE OF POSITION

W. W. FLEXNER, Cornell University

1. Introduction. If X and S are points of the earth's surface, the *azimuth* at S of X is the angle which the great circle SX makes with the north direction of the meridian of S. If X is in known position, an observer who finds that the azimuth of X at his position is θ may conclude that he is somewhere on the *isoazimuth* (X, θ), *i.e.*, on the locus of earth points from which the azimuth of X is θ. It will be shown below how to use θ and the position of X to find on a large scale map (which need not include X) a straight line closely approximating the part of (X, θ) on the map. Such a line will be called an *azimuth line of position* because the observer's position must lie close to that line [**10**].

The large scale map used may be any projection on which terrestrial distances and angles can be laid off with reasonable accuracy. Hence the technique to be described may be used with Mercator networks, either printed ones or ones specially prepared as occasion arises; with Lambert maps such as those of the United States Civil Aeronautics Administration or the French War Office; with American Polyconics such as those of the United States Coast and Geodetic Survey showing the coastal waters of the United States, or those on which the United States Army Military Maps are based; or with British Ordnance Polyconics.

If the Greenwich Civil Time is known, the latitude and longitude of the earth point directly under a heavenly body, be it sun, moon or star, may be determined from the nautical almanac [**1**], so that point will serve as an X. If both the altitude above the horizon and the azimuth of the heavenly body at a point S were accurately known, the position of S could be found by solving a spherical triangle. (The solution is in general ambiguous but there will, except when the difference of longitude of X and S is nearly 90°, be no doubt as to which solution gives S.) In practice the measurement of altitude by means of a sextant can be made accurately and hence a reliable line of position, the Sumner line, can be found from this measurement [**1**]. The azimuth on the other hand, because of mechanical difficulties in sighting and because its correctness depends on knowing the direction of true north accurately, cannot be found reliably enough to make a solution of the astronomical triangle worth while. But crossing the azimuth line of position to be described here with the Sumner line will give a graphical solution of the astronomical triangle which separates the part of the result which is accurate (the Sumner line) from the part depending on less accurate measurement (the azimuth). Direct computation by spherical trigonometry gives no such indication of the direction in which error in the fix is to be expected.

In the daytime when only the sun is visible, the use of an azimuth line of position thus makes it possible to get on a large scale map a fix, approximate in the sense described, without waiting the two or three hours necessary to cross one Sumner line, carried forward by dead reckoning, with another. (See example

1, section 5.) In general the azimuth line makes with a Sumner line simultaneously obtained a large enough angle. (See end of section 3.)

If X is a radio transmitter in known position its azimuth at S may be measured and an azimuth line of position plotted. Until the accuracy of methods for obtaining a radio azimuth is improved it is hardly worth while to plot the azimuth line on a large scale map since the standard small scale map approximations are adequate. The method now used of approximating an isoazimuth by a straight line (rhumb line) on a small scale Mercator [**2**] is however very rough at large distances and so could not long survive improvement in the technique for determining the azimuth. The approximation of an izoazimuth by a straight line (approximate great circle) on a Lambert Conformal map at present in use [**2**] would be still less satisfactory at long distances.

Like most calculations involving a spherical surface the ones needed here to find an azimuth line of position are not trivial. The work below is, however, so arranged that tables depending on three variables similar to the "Tables of Computed Altitude and Azimuth" (United States Hydrographic Office Publication no. 214) could be prepared were it ever thought worth while. Use of such tables would make the computations trivial.

A reader interested only in the application of the results of this paper may now skip to sections 4 and 5.

2. Map projections on which isoazimuths have special character. The considerations of this paper are all based on a spherical earth. The formula [**3**]

$$\cot A \sin C = \cot a \sin b - \cos C \cos b \tag{2.1}$$

holds for any spherical triangle ABC where a is the side opposite angle A, *etc.* if ϕ is the angular distance of a point S from the north pole, and ϕ_x is the angular distance of a fixed point X from the north pole, and λ is the difference of longitude of S and X, (2.1) shows that S is on (X, θ) if and only if

$$\cot \theta \sin \lambda = \cot \phi_x \sin \phi - \cos \phi \cos \lambda, \tag{2.2}$$

so that (2.2) is the equation of (X, θ) in spherical coordinates. From (2.2) it readily follows that the isoazimuths form a three parameter family on the earth, that all of them except the equator pass through both the north and south poles, that (X, θ) makes at X the angle θ with the southerly direction of the meridian of X if the angle is measured towards S.

In 1833 J. J. Littrow discovered a map projection whose theory was later applied to the study of isoazimuths by Captain Weir and H. Maurer [**4**]. If (x, y) are rectangular cartesian coordinates, the equations of the map are:

$$x = \sin \lambda \csc \phi, \qquad y = \cos \lambda \cot \phi. \tag{2.3}$$

Substituting (2.3) in (2.2) yields an equation linear in x and y. Hence isoazimuths (X, θ) for all θ and for points X on the meridian $\lambda = 0$ are represented by straight lines, and every straight line of the map represents such an isoazimuth. The map is conformal except at the points $\lambda = \pm 90°$, $\phi = 90°$ and is a

two sheeted Riemann surface with algebraic singularities at these points. If the complex z-plane is the Littrow map and the w-plane is a polar stereographic map, then $z=w+1/w$. The Littrow map represents the meridians and parallels by confocal hyperbolas and ellipses, the meridian $\lambda=0$ being a straight line. The construction here made of the azimuth line of position depends on the theory of this map.

For the sake of completeness four other special maps will now be described though no use is made of them in the subsequent sections. Maurer [5] has constructed a conformal map in which isoazimuths of stations on $\lambda=0$ are represented by circles. If two points, X and Y, on the equator are given differing by less than 90° in longitude, Maurer [6] has also shown how to construct a map such that (X, θ) and (Y, θ) are straight lines for every θ. This map is conformal at X and at Y but not generally.

Dilloway [7] has described the construction of a map for any two stations X and Y such that on it (X, θ) and (Y, θ) are all represented by straight lines and that the map permits the angles θ to be laid off directly at X and Y. At and near the stations and the line joining them Dilloway's map will represent different points of the earth, sometimes near together, by the same map point. In other words the Dilloway map has a complicated singular region right at its center.

If (ρ, ψ) are polar coordinates in the plane, I have found that a map given by the equations

$$\rho = 2 \cot \phi, \qquad \psi = \lambda \tag{2.4}$$

carries the isoazimuths $(X, 0)$ and $(X, 90°)$ for all X into the straight lines of the map. The map image of $(X, 90°)$ is orthogonal to the map meridian of X. Moreover on this map (X, θ) is asymptotic as $\rho \to \infty$ to those lines through X which make the map angles $\pm\theta$ with the southerly direction of the map meridian of X. Since the map is not conformal, this does not imply that the map image of (X, θ) is tangent at X to its asymptotes. The map is two sheeted, one sheet for the northern and one for the southern hemisphere (one asymptote is in each hemisphere) and represents the whole equator by the point $\rho=0$.

3. Azimuth line of position. Let ϕ be the angular distance of a point A from the north pole;

ϕ_x be the angular distance of X from the north pole;

λ be the difference of longitude of the points A and X;

θ_c be the azimuth of X at A;

θ_0 be the azimuth of X at an arbitrary point S.

In the sequel A will be an "assumed position" for which ϕ and λ are therefore known and for which θ_c and the distance XA can therefore be found by spherical trigonometry; θ_0 will be the "observed" azimuth at the unknown position S.

Let M be a Littrow map constructed for the meridian of X. Then on M the curve (X, θ_c) and (X, θ_0) are represented by straight lines q and t making the angle $\theta_0-\theta_c$ at the map image X' of X, q going through the map image A'

of A, and t through the image S' of S (figure 1). Let i be the inclination of the map meridian of A, *i.e.*, the angle that meridian makes at A with the meridian of X. Then $\tan i = (dx/dy)$ for constant λ and so using (2.3)

$$\tan i = \tan \lambda \cos \phi. \tag{3.1}$$

Hence the angle α that q makes with the meridian of A is given by

$$\alpha = \theta_c + i. \tag{3.2}$$

Let D' be the foot of the perpendicular from A' to the map meridian of X. Then on map M, using (2.3), $X'A' = A'D' \csc \theta_c = (\sin \lambda \csc \phi) \csc \theta_c$. But if P is

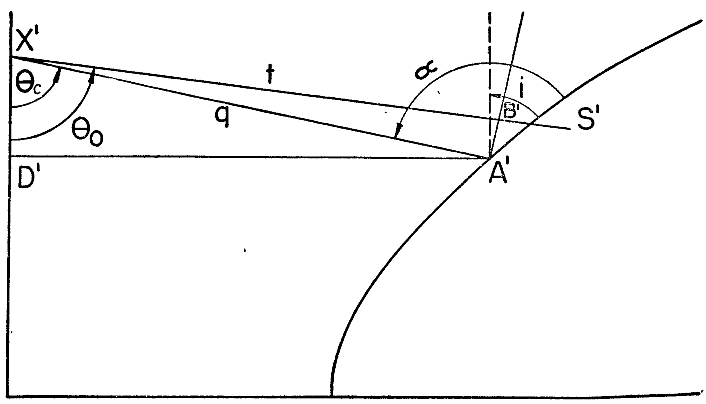

FIG. 1

the north pole and d is the great circle distance XA in angular units, the law of sines for the spherical triangle XPA gives $\sin \phi_x / \sin \theta_c = \sin d / \sin \lambda$ so

$$X'A' = \sin d \csc \phi \csc \phi_x. \tag{3.3}$$

On map M let the perpendicular at A' to q meet t in the point B'. Then $A'B' = X'A' \tan (\theta_0 - \theta_c)$ or, using (3.3)

$$A'B' = \sin d \csc \phi \csc \phi_x \tan (\theta_0 - \theta_c). \tag{3.4}$$

To approximate the true length r on the earth of the curve u corresponding to $A'B'$ it is necessary to divide (3.4) by the scale of the map M at A'. If ds and dS are arc length on map M and on the earth, then (2.3) yields

$$dS = \frac{10{,}800 \sin \phi}{\pi\sqrt{\cot^2 \phi + \cos^2 \lambda}}\, ds \text{ nautical miles.} \tag{3.5}$$

But if

$$\tan \psi = \cos \lambda \tan \phi, \quad \text{then} \quad \sqrt{\cot^2 \phi + \cos^2 \lambda} = \cot \phi \sec \psi. \tag{3.6}$$

So (3.4) and (3.5) yield

$$(3.7) \qquad r = \left[\frac{10{,}800}{\pi} \sin d \tan \phi \csc \phi_x \cos \psi\right] \tan (\theta_0 - \theta_c).$$

Notice that i, θ_c and the square bracket of (3.7) depend only on the positions of X and A and so could be tabulated in terms of λ, lat. A and lat. X in tables similar to H.O. 214.

Let K be a map on which in a small region terrestrial angles by direct measurement and (scaled) distances may be adequately approximated, and let A from now on be the map K image of A. Let n be a straight line on K making the angle α (towards X) with the map meridian at A. Then n is, by what has gone before, the tangent at A to the image on map K of (X, θ_c). Let p be a straight line on K perpendicular at A to n. For short distances p closely approximates the curve u of the earth. Let B be a point on p at (scaled) distance r from A. If $\theta_0 > \theta_c$ put B on the northerly side of A; if $\theta_0 < \theta_c$ on the southerly. (If p runs east-west put B east of A if and only if S is east of X as shown by θ_0.) Then B closely approximates the image on map K of B'. Let m be a line on B making the angle $\theta_0 - \theta_c$ with n in such a way that m and n, if produced, would meet on the X side of A. Then m closely approximates a tangent to (X, θ_0) and so is the required azimuth line of position.

In order that the line AB be short and the approximation good, it is necessary that A be not too far from S, so A should be near the dead reckoning position of S (see the examples, section 5). Since $\theta_0 - \theta_c$ is then small, the angle between an azimuth line m and a simultaneous Sumner line is about $90° - i$. Formula (3.1) shows under what conditions this is so small as to magnify the error unduly (see section 1).

4. Summary of directions for drawing azimuth line of position. Let K be a map on which terrestrial angles and distances may be laid off. Let L_x be the latitude and λ_x the longitude of a point X from which light or radio signals are received. As described in section 1, X may be either a terrestrial point under a bright heavenly object or a radio transmitter.

i. Determine the bearing θ_0 at your position of X, measuring θ_0 toward X from the north direction of your meridian.

ii. Choose near your dead reckoning position an assumed position A in lat. L_a, long. λ_a. Let λ be the difference of longitude of X and A.

iii. Let θ_c be the azimuth of X at A, again measured towards X, and let d be the great circle distance XA. These two quantities can be determined by solving the spherical triangle XAP where P is the north pole [8].

iv. On map K lay off a line n through A making the angle α (toward X) with the map meridian of A where

$$(4.1) \qquad \alpha = \theta_c + i$$

$$(4.2) \qquad \tan i = \pm \tan \lambda \sin L_a \begin{cases} + \text{ sign if } L_a \text{ is } N, \\ - \text{ sign if } L_a \text{ is } S. \end{cases}$$

Draw a line p through A perpendicular to n. Let ψ and r be such that:

$$\tan \psi = \cos \lambda \cot L_a \tag{4.3}$$

$$r = \frac{10{,}800}{\pi} \sec L_x \cot L_a \cos \psi \sin d \tan (\theta_0 - \theta_c) \text{ nautical miles.} \tag{4.4}$$

Locate a point B by laying off on p a length r from A in the northerly direction if $\theta_0 > \theta_c$, southerly if $\theta_0 < \theta_c$. If p runs east-west, put B east of A if and only if you are east of X. The direction of X will be shown by θ_0.

v. Through B draw a line m making the acute angle $\theta_0 - \theta_c$ with n in such a way that m produced and n produced would meet on the side of p at which X lies. Then m is your azimuth line of position.

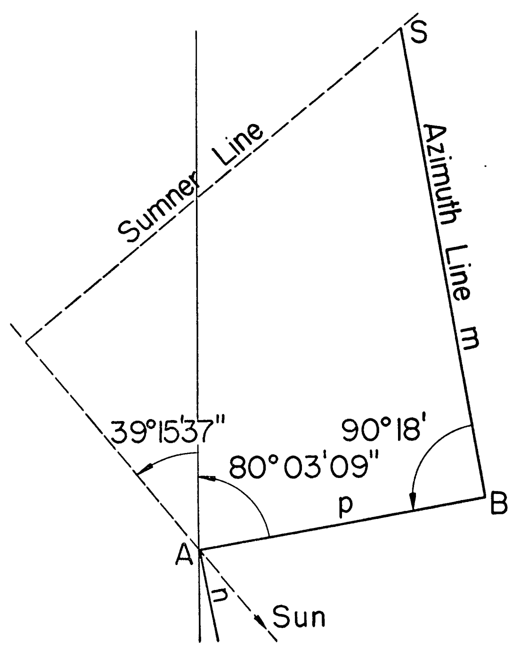

FIG. 2 (Scale: 1 inch to 14″ of Longitude)

5. Examples. The first example illustrates the obtaining of an approximate fix as the intersection of a Sumner line and an azimuth line of position based on simultaneous observations of the altitude and azimuth of the sun's center. In the second example a fix is obtained by crossing azimuth lines of position on two radio transmitters. All standard spherical trigonometry computations, including those for the Sumner line in example 1, are here omitted, only the results being given. The first column of the tabular computation refers to example 1, the second two columns refer to example 2, transmitters X and Y

respectively. The quantities to be plotted on map K are underlined. In figures 2 and 3 maps K are large scale Mercator networks, the longitude scales being as shown on each figure, the latitude scales then being obtained from the table of "Meridional Parts or Increased Latitudes" which is table 3 of the Bowditch "American Practical Navigator" (H.O. no. 9). Distances were plotted in the standard way by using the latitude scale.

The "observed" data were computed by spherical trigonometry from the true positions of the observers which are: in example 1, 42°28′N, 76°31′W; in example 2, 36°N, 60°W; and so, especially in example 2, they are much more accurately known than they would be in practice.

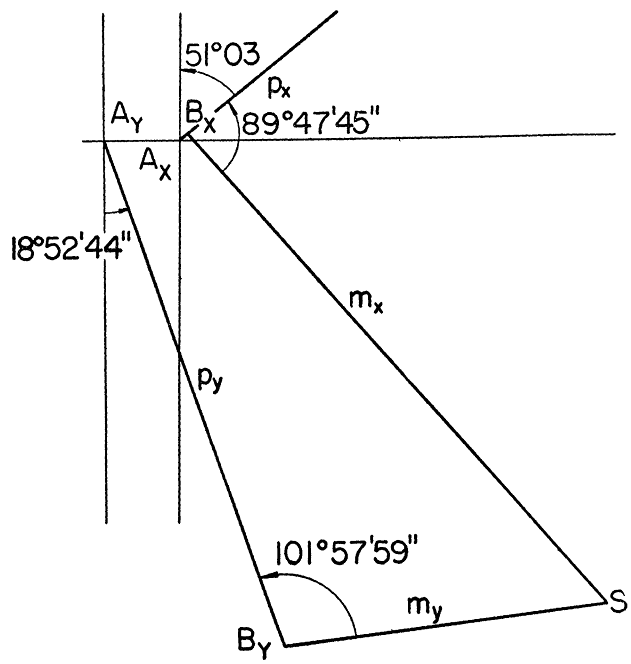

FIG. 3 (Scale: 1 inch to 1° of Longitude)

Example 1. On January 20, 1939 at Greenwich Civil Time $14^h38^m00^s$ an observer near Ithaca, N. Y. (42°26′27″N, 76°29′00″W), finds that the altitude of the sun's center is 17°15′40″ and the azimuth of the sun's center from his position is 140°44′55″ E of N. From these data determine the observer's position.

Example 2. A ship in dead reckoning position 38°12′N, 62°35′W finds that radio transmitter X on Mt. Desert Rock (43°58′08″N, 68°07′44″W) bears 35°27′46″W of N and that transmitter Y on Cape Hatteras (35°15′17″N, 75°31′16″W) bears 88°49′23″ W of N. From these data determine the ship's position.

In the first example one assumed position is used, in the second, two, A_x for transmitter X and A_y for transmitter Y. The coordinates of the assumed positions appear in the tabular computations.

		Example 1	Example 2 Transmitter X.	Transmitter Y.
	L_a	$42°N$	$38°N$	$38°N$
	λ_a	$76°45'30''W$	$62°07'44''W$	$62°31'16''W$
	L_x	$20°13'48''S$	$43°58'08''N$	$35°15'17''N$
	λ_x	$36°45'30''W$	$68°07'44''W$	$75°31'16''W$
	$\lambda=\|\lambda_a-\lambda_x\|$	40°	6°	13°
	d	72°22'29''	7°29'30''	10°46'20''
	θ_0	141°02'23''	35°27'46''	88°49'23''
	θ_c	140°44'23''	35°15'31''	100°47'22''
	$\theta_0-\theta_c$	18'00''	12'15''	−11°57'59''
(4. 2)	log tan λ	9.92381	9.02162	9.36336
	log sin L_a	9.82551	9.78934	9.78934
	log tan i	9.74932	8.81096	9.15270
	i	29°18'46''	3°42'	8°05'22''
(4. 3)	log cos λ	9.88425	9.99761	9.98872
	log cot L_a	0.04556	0.10719	0.10719
	log tan ψ	9.92981	0.10480	0.09591
	ψ	40°23'24''	51°50'48''	51°16'32''
(4. 4)	log cos ψ	9.88176	9.79082	9.79628
	log sec L_x	0.02765	0.14284	0.08800
	log sin d	9.97912	9.11522	9.27162
	log cot L_a	0.04556	0.10719	0.10719
	$\log \frac{10,800}{\pi}$	3.53627	3.53627	3.53627
	log tan $(\theta_0-\theta_c)$	7.71900	7.55187	9.32623
	log r	1.18936	0.24421	2.12559
	r	15.47	1.755	−133.53
(4. 1)	$\alpha=\theta_c+i$	170°03'09''	38°57'	108°52'44''

6. Analytic methods. In closing, two additional methods of accurately locating position by azimuths must be mentioned. The first is that in E. J. Willis' *Methods of Modern Navigation* [9] and obtains a fix without use of any map. To get analytically an azimuth line of position for the isoazimuth (X, θ_0), Willis writes the equations of: 1) the great circle through X cutting the parallel of dead reckoning latitude at the angle $90°-\theta_0$, and thus he finds the latitude and longitude of the point A in which that great circle cuts that parallel; 2) the great circle through X cutting the meridian of dead reckoning longitude at the angle θ_0, which gives the point B common to that meridian and the second great circle; 3) the great circle arc AB which, since A and B are on (X, θ_0), approxi-

mates (X, θ_0). The intersection of two great circle arcs obtained by step 3 from two stations can then be found analytically and provides a close approximation to the ship's position.

Another method which may be used with or without a map was suggested to me by Willis' analytic treatment of the Sumner line (*loc. cit.*). Differentiation of formula (2.2) of this paper yields:

$$(6.1)\qquad \begin{aligned} -\csc^2\theta \sin\lambda d\theta = (&\cot\phi_x \cos\phi + \cos\lambda \sin\phi)d\phi \\ &+ (\sin\lambda\cos\phi - \cot\theta\cos\lambda)d\lambda. \end{aligned}$$

If θ, ϕ and λ are computed for an assumed position and $d\theta = \theta_0 - \theta$, the equation in $d\phi$ and $d\lambda$ which results from (6.1) is that of a straight line whose $d\phi$ and $d\lambda$ intercepts are:

$$(6.2)\qquad d\phi = \frac{-\csc^2\theta\sin\lambda d\theta}{\cot\phi_x\cos\phi + \cos\lambda\sin\phi}$$

$$(6.3)\qquad d\lambda = \frac{\csc^2\theta\sin\lambda d\theta}{\cot\theta\cos\lambda - \sin\lambda\cos\phi}.$$

This straight line is in the tangent plane to the earth at the assumed position and the points $(d\phi, d\lambda)$ on the line represent to the first order of approximation those points $(\phi+d\phi, \lambda+d\lambda)$ of the earth on the isoazimuth (X, θ_0). This line may now be approximated by a line having the intercepts (6.2) and (6.3) on any map which represents distances near the assumed position with reasonable accuracy. It should be remarked, however, that plotting a line by its intercepts may require a much smaller scale than plotting by the method of sections 3–5 (see for instance example 2 in which one of the four intercepts is far off the map).

If the same assumed position is used for two isoazimuths, the two equations (6.1) may be solved simultaneously for $d\phi$ and $d\lambda$. Then $(\phi+d\phi, \lambda+d\lambda)$ is the "fix." Similarly if the same assumed position is used for an azimuth and a Sumner line, equation (6.1) and the corresponding equation in terms of differentials for the Sumner line, as given by Willis, may be solved simultaneously. These computations can be carried out without use of maps.

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B. H. Brown, "Conformal and equiareal world maps," this MONTHLY, 42, 1935, 227–231.
5. H. Maurer, Marine Luftflotten Rundschau, Berlin, 1933, See also EK pp. 66, 75.
6. H. Maurer, Annalen der Hydrographie, 1919, p. 16. See also EK p. 75 and fig. 16.
7. A. J. Dilloway, "Cartographic solution of great circle problems," J. Royal Aeronaut. Soc. 46, 1942, pp. 26–28.
8. If tables like those described in section 3 were available for whole degrees of L_a and λ and for half degrees of L_x, A should be chosen so that L_a and λ come out in whole degrees. The present H.O. 214 tables will serve to solve XAP using, in entering the tables, L_a for Lat., L_x for Dec. (wide interpolation may be necessary here since not all half degrees are tabulated) and λ for $H.A.$ Then if the names of L_a and L_x are the same, $\theta_c = Az$ (are opposite, $\theta_c = 180° - Az$) and $d = 90° -$alt.
9. D. Van Nostrand Co., 1928.
10. See also W. Immler, "Die Kartenbeschickung der Funkpeilung in der winkeltreuen Kegel-karte," Annalen der Hydrographie 68, 1940, 8, pp. 282–292; and P. de Vanssay de Blavous, "The position at sea by radiogoniometric bearings taken on board," Hydrographic Review 10, 1933, pp. 87–107.

WHAT IS A LATTICE?

GARRETT BIRKHOFF, Harvard University

1. Order relations. With respect to one or more relations, most mathematical systems are lattices. Thus the real numbers are a lattice with respect to the *order* relation $x \leqq y$ (x is less than or equal to y); the non-negative integers are a lattice with respect to the relation $x | y$ (x is a *divisor* of y); the subsets of any class form a lattice if one relates them by the *inclusion* relation $X \subset Y$ (X is a subset of Y).

The assertion that the above three systems are *lattices* really means that the relations specified have a number of common formal properties. Thus the order relation between real numbers evidently has the following properties,

P1. For all x, $x \leqq x$, (Reflexivity)
P2. If $x \leqq y$ and $y \leqq x$, then $x = y$, (Anti-symmetry)
P3. If $x \leqq y$ and $y \leqq z$, then $x \leqq z$. (Transitivity)

Moreover statements P1–P3 remain true if the relation $\leqq$ of inequality between real numbers is replaced by the relation $|$ of divisibility between non-negative integers or by the relation $\subset$ of inclusion between subsets.

To summarize, the three relations $\leqq$, $|$, $\subset$, as defined, share properties P1–P3 *and all logical consequences of these properties*, however indirect.

2. A matter of economy. It seems to me a waste of paper to develop all properties of the inequality relation using one notation and terminology, to discuss divisibility using a second set of symbols and technical terms, and to repeat this procedure in treating the algebra of classes. Instead, I would take full ad-

vantage of basic analogies, and create a general *lattice theory* of relations satisfying P1–P3. This theory could contain as special cases many of the properties of inequality, divisibility, and inclusion; thus it would have advantages both of *unity* and *economy*.*

A trivial instance of this economy arises in deriving the properties of the relation $x<y$ and its analogs. By $x<y$, we mean that $x\leqq y$ but $x\neq y$. It is an easy but tiresome exercise to prove from P1–P3 that, for example, $x<y$ and $y\leqq z$ imply $x<z$; that $x_1<x_2<x_3<x_1$ is impossible, and so on. If lattice theory cannot entirely free us from making these tiresome proofs, it can at least free us from the need of going through them three or more times in different language. If we define a *proper divisor* of y to be an integer x such that $x|y$ yet $x\neq y$, we can apply *gratis* to this relation every property of $x<y$ proved from P1–P3. The same remark applies to the concept of a *proper subset*.

3. The principle of duality. A remarkable, but more sophisticated economy effected by lattice theory consists in its general Duality Principle, which includes as special cases duality principles in logic, projective geometry, number theory, *etc.* By the *dual* of a relation satisfying P1–P3, we mean simply its converse in the usual sense. Thus the dual of $\leqq$ is $\geqq$, the dual of $x|y$ is "x is a multiple of y," and the dual of $X\subset Y$ is $X\supset Y$.

In its simplest form, the Duality Principle states that the dual (converse) of any relation satisfying P1–P3 itself satisfies P1–P3. From this it follows that all definitions and theorems are either dual in pairs or self-dual.

For example, by a *lower bound* of a set X of elements X_i with respect to a relation $\leqq$ satisfying P1–P3, we mean simply an element u such that $u\leqq x_i$ for all x_i in X. The dual concept is that of *upper bound*, or element v such that $x_i\leqq v$ for all x_i in X. Analogs for divisibility are the dual concepts of common divisor and common multiple; what are the analogs for the inclusion relation between sets?

Similarly, by the *greatest lower bound* or *meet* of x, y with respect to $\leqq$, we mean an element u (denoted $x\cap y$) which is (i) a lower bound of x and y, and (ii) satisfies $u\geqq v$ for any other lower bound v of x and y. Thus in the case of inequality, $x\cap y$ is simply the *lesser* of x and y; in the case of divisibility, it is the *g.c.d.* of x and y; with inclusion, $X\cap Y$ is the *intersection* of X and Y, or set of all points in both X and Y.

The reader should have little trouble in defining the dual concept of the *least upper bound* or *join* of x and y, which is denoted $x\cup y$. This evidently specializes to the *greater* of x and y, the *l.c.m.* of x and y, and the *union* or sum of X and Y in the three examples we have taken.

* Similar unifications, achieved by abstract group theory, field theory, and ideal theory, constitute the characteristic feature of modern algebra. The importance of economy in organizing scientific knowledge is described by the physicist and philosopher Ernst Mach in his "Science of Mechanics," Chicago, 1893, p. 6.

4. Lattices. We are now ready to define precisely what we mean when we speak of a lattice.

DEFINITION. *A lattice is a system L of elements* $x, y, z, \cdots$ *together with a relation which satisfies* P1–P3 *and also the postulate*

L. *Any two elements x and y have a meet* $x \cap y$ *and a join* $x \cup y$ *in L.*

THEOREM. *The following algebraic identities are true in any lattice*:

L1. *For all x*, $x \cap x = x \cup x = x$,

L2. $x \cap y = y \cap x$ *and* $x \cup y = y \cup x$,

L3. $x \cap (y \cap z) = (x \cap y) \cap z$ *and* $x \cup (y \cup z) = (x \cup y) \cup z$.

L4. $x \cap (x \cup y) = x \cup (x \cap y) = x$.

We omit the proof. Conversely, any system satisfying L1–L4 becomes a lattice if one defines $x \leqq y$ to mean $x \cup y = y$ (or equivalently, to mean $x = x \cap y$). It is largely because of this interpretation of lattices in terms of binary operations that lattice theory can be regarded as a branch of algebra.

In fact, there is an analogy between the operations $\cap$, $\cup$ in a lattice and the operations $\times$, $+$ of ordinary arithmetic. Like multiplication, the operation $\cap$ is commutative and associative; similarly for $\cup$ and addition (*cf.* L2–L3). Further, in the three examples cited, there is valid the analog

L6′. $x \cap (y \cup z) = (x \cup y) \cup (x \cap z)$

of the *distributive* law $x(y+z) = xy + xz$ of arithmetic.

However, the distributive law does not hold in all lattices. For example, take the points, lines, and planes of projective space, together with the void set O and space I. These are a lattice with respect to inclusion; $X \cap Y$ is the intersection of X and Y, while $X \cup Y$ is the linear sum of X and Y—that is, the set of all points on lines joining X and Y. It is easily checked that if x, y, z are three distinct points on a line L, then $y \cup z = L$ whence $x \cap (y \cup z) = x \cap L = x$, whereas $x \cap y = x \cap z = O$ and so $(x \cap y) \cup (x \cap z) = O$.

Indeed, it is curious and not easily proved fact, that if three elements in a lattice satisfy L6′, then they satisfy its dual,

L6″. $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$,

whose arithmetic analog $x + yz = (x+y)(x+z)$ is not true.

In the preceding example, the elements O and I are universal lower and upper bounds, respectively; that is, for all x, $O \leqq x \leqq I$. Such elements need not exist in all lattices (although they do in all finite lattices). Thus with real numbers we must adjoin the imaginary numbers $-\infty$ and $+\infty$ to get universal bounds; on the other hand, 1 is a common divisor and 0 a common multiple of all non-negative integers.

If universal bounds exist, it is easily shown that they are somewhat analogous to 0 and 1 in arithmetic. Thus we can show from P1–P3 and our definitions that, for all x,

$$O \cap x = O, \quad O \cup x = x, \quad \text{and} \quad I \cap x = x.$$

These are analogs of $0 \cdot x = 0$, $0 + x = x$, and $1 \cdot x = x$.

5. Applications in function theory and in logic. It would be possible to con-

tinue most indefinitely giving examples and properties of lattices. But I hope that the preceding examples will give some idea of the generality and simplicity of the notion of a lattice. In order to round out the picture, I shall conclude by mentioning two other lattices which play basic roles in function theory and logic.

The continuous real functions $f(x)$ of a real variable form a lattice, if one defines $f \leqq g$ to mean $f(x) \leqq g(x)$ for all x. Here $f \cap g$ is the function $h(x)$ which, for all x, is the lesser of $f(x)$ and $g(x)$; we define $f \cup g$ dually. Thus $f \cup -f$ is the *absolute* $|f(x)|$ of $f(x)$.

Finally, in mathematical logic, attributes (good, rich, female, *etc.*) form a lattice. Here $p \leqq q$ is interpreted to mean p is *implied* by q, $p \cap q$ to mean p *or* q, $p \cup q$ to mean p *and* q. In this lattice (often called a *Boolean algebra*), a fundamental role is also played by the operation p' (meaning *not* p). It satisfies

L7. $(p')' = p,\ p \cap p' = 0,\ p \cup p' = I,$

$$(p \cap q)' = p' \cup q', \quad \text{and} \quad (p \cup q)' = p' \cap q'.$$

It is noteworthy that the same laws are satisfied by sets, if one lets X' denote the *complement* of X (set of points not in X); further, they are satisfied in projective geometry if one lets x' denote the *polar* of x. In fact, it can be proved that the chief difference between projective geometry and Boolean algebra (or the algebra of logic) is that the distributive laws L6′–L6″ of Boolean algebra must be replaced in projective geometry by the weaker *modular* law.

L5. If $x \leqq z$, then $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$.

Since $x \leqq z$, clearly $x \cup z = z$, so that the conclusion of L5 assumes the *self-dual* form $x \cup (y \cap z) = (x \cup y) \cap z$; thus it is also equivalent to $(x \cup y) \cap z = (x \cap z) \cup (y \cap z)$.

The self-dual modular law L5 is important for another reason. It is satisfied by the normal subgroups of any group, the ideals of any ring, *etc.*, and may be made the basis of most of the known decomposition theorems of modern algebra. But this is a long story!

AN EARLY REFERENCE TO DIVISION BY ZERO

C. B. BOYER, Brooklyn College

1. The Hindu legend. In this MONTHLY for 1924 we read* that Brahmagupta, writing in 628 A.D., was the first person to speak of division by cipher, although he gave no quotient. We read further that Bhaskara in 1152 termed such a quotient infinite. This is a reiteration of statements which one can find in almost any history or reference work in mathematics.† It is but one aspect of the impression generally prevailing that "the arithmetic of zero is entirely the Hindu contribution to the development of mathematical science. With no

* H. G. Romig, "Early history of division by zero," this MONTHLY, 31, 1924, 387–389.

† See Encyclopédie des sciences mathématiques, I, 1[1], 1904, p. 33; Moritz Cantor, Vorlesungen über Geschichte der Mathematik, I, 2nd ed., Leipzig, 1894, p. 576; H. T. Colebrooke, Algebra, with Arithmetic and Mensuration, from the Sanscrit of Brahmagupta and Bhaskara, London, 1817, pp. 137–138, 339–340; B. Datta and A. N. Singh, History of Hindu Mathematics. A Source Book. Part I. Numeral Notation and Arithmetic, Lahore, 1935, pp. 238–243.

other early nations do we find any treatment of zero."* Such categorical assertions are serious misrepresentations of the true situation.

It is now well known that symbols for empty places in the positional representation of numbers—corresponding in this respect to our zero—were used by the Babylonians, the Greeks, and the Mayas. Such use antedates historical evidence for the Hindu zero by some thousand years. Nevertheless, such symbols did not appear alone, or independently of other ciphers or digits, to represent zero as itself a number; they were simply part of the representations of positive integers and fractions. Hence the adoption of such symbolisms does not necessarily refute the Hindu claims of priority to the use of zero in this independent sense. Nevertheless, the numerational concept of zero—as distinct from a symbol for empty places—was also familiar to the ancient Greeks. This idea frequently is attributed to Plato,† but such ascription rests largely on gratuitous inferences derived from his sybilline phraseology. In the works of his pupil Aristotle, however, there appears a clear-cut reference not only to the numerical notion of zero but also to the result of division by zero. Historians of mathematics seem generally to have overlooked‡ this significant passage in the *Physics* which anticipates the earliest extant Hindu evidence in this connection by almost a millenium.

2. Aristotle's law of motion. A study of moving bodies had led Aristotle to conclude§ that for a given force or impulse the velocity was in all cases inversely

* Bibhutibhusan Datta, "Early history of the arithmetic of zero and infinity in India," Bulletin of the Calcutta Mathematical Society, XVIII, 1927, 165–176. See articles by the same author in this MONTHLY for 1926 and 1931.

† See John Burnet, Greek Philosophy. Thales to Plato, London, 1932, pp. 320–322, 330; A. E. Taylor, Plato. The Man and his Work, new ed., New York, 1936, pp. 505–506.

‡ The passage is noted quite incidentally in a philosophical treatment by Albert Görland, Aristoteles und die Arithmetik, Marburg, 1898, p. 18, with no indication of its mathematical significance. There are numerous other works on Aristotle's general mathematical thought, and more particularly on his concept of number and his notion of infinity; but these appear to contain no reference to his statement on division by zero. See, for example, Max Dehn, "Raum, Zeit, Zahl bei Aristoteles, vom mathematischen Standpunkt aus," Scientia, LX, 1936, 12–21, 69–74, supp. pp. 1–9, 32–36; Abel Burja, "Sur les connoissances mathématiques d'Aristote," Preussische Akademie der Wissenschaften, Berlin, Mémoires de l'académie royale des sciences et belles lettres, 1790–1791, pp. 257–276; J. L. Heiberg, "Mathematisches zu Aristoteles," Abhandlungen zur Geschichte der mathematischen Wissenschaften, 18, 1904, 1–49; Gaston Milhaud, "Aristote et les mathématiques," Archiv für Geschichte der Philosophie, 16, 1903, 367–392; Paul Mansion, Aristote et les mathématiques," Revue de Philosophie, 3, 1903, 832–834; Abraham Edel, Aristotle's Theory of the Infinite, New York, 1934. I have not had an opportunity to examine the following: Josephus Blancanus, Aristotelis Loca Mathematica ex Universis Ipsius Operibus Collecta, et Explicata, Bononiae, 1615; P. J. Monzó, De Locis apud Aristotelem Mathematicis, Valencia, 1556; Leo Reiche, Das Problem des Unendlichen bei Aristoteles, Breslau, 1911; Julius Stenzel, Zahl und Gestalt bei Plato und Aristoteles, Leipzig, 1924; R. Stölze, Die Lehre vom Unendlichen bei Aristoteles, Würzburg, 1882. The excellent source book by Ivor Thomas, Selections Illustrating the History of Greek Mathematics, 2 vols., Cambridge, Mass., 1939–1941, devotes more than a dozen pages to Aristotle, but does not include this material. So far as I am aware, there is no reference to this point in any history or textbook of mathematics.

§ That this notorious inference is incorrect need not concern us here; but it should perhaps be noted that only in the time of Newton was a more accurate study made of the difficult problem

proportional to the density of the medium—such as air or water—through which the object moved. In modern symbolism this principle may be represented in the form of an equation, $v=k/\delta$, where v, the speed, is a function of δ, the density of the medium, and k is a factor of proportionality. Aristotle then raised the question as to what would be the speed of a body in a vacuum—that is, when $\delta=0$. In answer he wrote:

> Now there is no ratio in which the void is exceeded by body, as there is no ratio of 0 to a number. For if 4 exceeds 3 by 1, and 2 by more than 1, and 1 by still more than it exceeds 2, still there is no ratio by which it exceeds 0; for that which exceeds must be divisible into the excess + that which is exceeded, so that 4 will be what it exceeds 0 by +0. For this reason, too, a line does not exceed a point—unless it is composed of points!*

3. Impossibility of division by zero. In this quotation it is quite evident that Aristotle had the arithmetical zero in mind. It is clearly distinguished from the philosophical void, and it is regarded as akin to, although not actually one of, the numbers. Zero is looked upon as bearing to number the same relationship as does a point to a line. Moreover, the impossibility of division by zero is here explicitly stated almost fifteen hundred years before the time of Bhaskara. The argument by which Aristotle excluded division by zero is based largely on the traditional meanings of words, and hence it differs from the modern point of view. But there can be no doubt as to his correct understanding of the situation. If division by zero were possible, then the result would exceed every integer. That this is Aristotle's position is clear from his inference with respect to motion in a vacuum:

> But if a thing moves through the thickest medium such and such a distance in such and such a time, it moves through the void with a speed beyond any ratio.†

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of the influence of a resisting medium. Indeed, the assumption of Aristotle was applied to the velocity of light both by Descartes in illustrating the law of refraction and by Fermat in demonstrating that the principle of "least time" satisfies this law.

* Physica IV. 8. 215b. The translation is taken from The Works of Aristotle, ed. by W. D. Ross and J. A. Smith, 11 vols., Oxford, 1908–1931, vol. II. The same translation is found in The Basic Works of Aristotle, ed. by Richard McKeon, 1941. This rendering of the Greek is misleading, however, in the use of symbols for numbers and operations. Aristotle did not here use the Greek symbol *o* for zero, but rather the word *οὐδεν*, corresponding to the Hindu *sunya*, "void," and the Arabic *sifra* (whence our *cipher*). In this respect the translation given by P. H. Wicksteed and F. M. Cornford, Aristotle, The Physics, 2 vols., London, 1929–1934 is somewhat more cautious, if also more diffuse: "But the nonexistent substantiality of vacuity cannot bear any ratio whatever to the substantiality of any material substance, any more than zero can bear a ratio to a number. For if we divide a constant quantity c (that which exceeds) into two variable parts, a (the excess) and b (the exceeded), then, as a increases, b will decrease and the ratio $a:b$ will increase; but when the whole of c is in section a there will be none of c for section b; and it is absurd to speak of 'none of c' as 'a part of c.' So the ratio $a:b$ will cease to exist, because b has ceased to exist and only a is left, and there is no proportion between something and nothing. (And in the same way there is no such thing as the proportion between a line and a point, because, since a point is no part of a line, taking a point is not taking any of the line.)"

† *Ibid.* The translation is again that of Ross and Smith. Wicksteed and Cornford render it:

He easily establishes this fact through a reductio ad absurdum: If there were a a ratio, it would follow that a body in a given time would traverse a certain distance whatever the density of the medium, a result which contradicts the principle of the premise,* $v = k/\delta$.

4. Aristotle vs. Bhaskara. It is illuminating to compare these passages from Aristotle with the much vaunted statement on the subject of division by zero given by Bhaskara:

> Statement: Dividend 3. Divisor 0. Quotient the fraction 3/0.
>
> This fraction, of which the denominator is cipher, is termed an infinite quantity.
>
> In this quantity consisting of that which has cipher for its divisor, there is no alteration, though many be inserted or extracted; as no change takes place in the infinite and immutable God, at the period of the destruction or creation of worlds, though numerous orders of beings are absorbed or put forth.†

A comparison of the above quotations from Aristotle and Bhaskara will show that the view of the former comes closer to the modern attitude than does the much more recent dictum of the latter. In contemporary mathematics it is indeed customary to say that the quotient of any number (other than zero) divided by zero is infinity. This language, however, is conventionally interpreted as signifying two things: first, that in this case the quotient is not defined and hence the indicated division is impossible; second, that as a variable divisor tends indefinitely closely toward zero, the (non-zero) dividend remaining constant, the quotient increases without limit. Both of these aspects are present in Aristotle's statement, the former categorically stated and the latter clearly implied.‡ Bhaskara, on the other hand, did not assert the impossibility of division by zero; nor is there in his form of expression any indication of the indefinite increase of $3 \div x$ as x tends towards 0. He looked upon $3 \div 0$ as a number having unusual properties, but very definitely a fixed quantity, much as Wallis in 1665 boldly but uncritically wrote $1/0 = \infty$. Bhaskara's further remark that 3/0 remains unchanged, whatever be added to or taken from it, is sometimes interpreted as a significant anticipation of the properties of the modern infinite; but it must be remarked that the contemporary viewpoint of infinity is con-

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"But if movement through the finest medium covers a given distance in a given time, movement through the void is out of all proportion."

* It was this obvious contradiction which led Aristotle to deny the existence of the void. This fact indicates that the Peripatetic doctrine, "Nature abhors a vacuum," was not based on animistic or even teleological notions, but was instead a logical consequence of a physical principle which science ultimately found cause to reject.

† Colebrooke, *op. cit.*, pp. 137–138.

‡ The implication in this connection is more clearly expressed in the Wicksteed-Cornford translation than in that of Ross and Smith. See note above.

cerned with infinite sets and one-to-one correspondence, or with improper geometrical elements, rather than with magnitudes and equality in the ordinary sense. Hence Bhaskara's naïve assumption (typical of Hindu arithmetic) that division by zero results in a quasi-theological ineffable fixed infinity is less to be admired than Aristotle's critical recognition (characteristic of Greek thought) of the impossibility of division by zero.*

5. **Judgment suspended.** That Aristotle was the first person to discuss this question is not unlikely, but this cannot be held as established. One may perhaps be permitted to conjecture from the very casualness of Aristotle's language in this connection that the arithmetic of zero was more or less well known in his day. However, there seems to be no other extant reference to division by zero before the time of Brahmagupta. As a matter of fact, the Greeks, Aristotle included, did not look upon zero as a number in the strict sense of the word, for this status was reserved exclusively for the natural numbers.† Statements on the arithmetic of zero are rare in ancient times,‡ and indeed zero was not fully accepted in algebra until the sixteenth and seventeenth centuries.§ Aristotle himself seems never to have had occasion to consider the quotient of zero divided by zero. The earliest, but quite inadequate, consideration of this problem traditionally has been ascribed to Brahmagupta:

> Positive, divided by positive, or negative by negative, is affirmative. Cipher, divided by cipher, is nought. Positive, divided by negative, is negative. Negative, divided by affirmative, is negative. Positive, or negative, divided by cipher, is a fraction with that for denominator: or cipher divided by negative or affirmative.||

Tradition in this particular respect may prove to be trustworthy, but it necessarily must be rejected with respect to the more general problem. Historical evidence points to Aristotle, rather than to Brahmagupta, as the one who first considered division by zero.

* Incidentally, one finds in Aristotle's Physica a discussion of the infinite which, while differing markedly from the modern treatment, shows an admirable grasp of the problems.

† The integers one and two frequently were excluded, together with zero, from the realm of number. This situation, however, does not alter the argument of this paper. That the Greeks had a thorough knowledge of the arithmetic of ratios and of the integers one and two—even though these were not strictly defined as numbers—indicates that a corresponding arithmetic of zero was quite possible, notwithstanding the nice distinctions which placed zero beyond the scope of the word number.

‡ Nicomachus (c. 100 A.D.) in his Arithmetica referred only once, and then somewhat equivocally, to zero in saying that the sum of nothing added to nothing is nothing. See Nicomachus of Gerasa, Introduction to Arithmetic, transl. by M. L. D'Ooge, with Studies in Greek Arithmetic by F. E. Robbins and L. C. Karpinski, New York, 1926, pp. 48, 120, 237–238. Cf. Johannes Tropfke, Geschichte der Elementar-Mathematik, Vol. II, Leipzig, 1921, pp. 56–59.

§ See Tropfke, *loc. cit. Cf.* Encyclopédie des sciences mathématiques, I, 1^1, p. 133, note 147; also G. Eneström, "Über die Anfänge der Benutzung von Null als eine wirkliche Grösse," Bibliotheca Mathematica (3), VII, 1906–1907, 309.

|| Colebrooke, *op. cit.*, pp. 339–340.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1942-43

Mathematics Club, Brown University

In each of the three semesters two of the meetings of the Club were devoted to a program of two half-hour talks by student speakers, followed by light refreshments. The following twelve papers were presented:

Paper folding, by Rose Mary Canning.

Solutions of the problem of Snellius and Pothenot, by F. J. Hunt, Jr.

Solutions of cubic equations by carpenter squares and by hydrostatics, by Q. B Leonard.

The Fibonacci series, by J. E. Cook, Jr.

Flatland, by Barbara Cotter.

A problem in life insurance, by W. W. Keffer.

The game of Nim, by D. G. Fernald.

Mathematical fallacies, by B. F. Taylor, Jr.

War problems in mathematics, by Frances Weeden.

Adjustment on the target for artillery fire, by S. L. Ehrlich.

Military and Naval maps, by Ruth Pearson.

Professor Gilman's new method in cryptanalysis, by L. W. Lees, Jr.

Two special meetings were addressed by visiting speakers:

Some properties of systems of circles, by Professor Virgil Snyder.

Third dimension air photographs in this war, by Dr. Richard Goldthwait.

Dr. Goldthwait spoke at the last meeting of the third semester. After this meeting, and after the December meeting at which the annual club picture was taken, some informal dancing was provided in the Crystal Room. Picnics were held in September and May. Chairmen of the Subcommittee on Arrangements (in the semesters indicated) were: J. H. Alger (1, 2 sem.), Q. B. Leonard (3 sem.). Other student members of the Committee on Program and Arrangements were: R. P. Breeding (3), Rose Mary Canning (2, 3), J. E. Cook, Jr. (1), Barbara Cotter (3), Doris Dunlap (1, 2), D. G. Fernald (3), Helen Hooper (1), Arline Major (1, 2), G. A. Levine (1, 2), J. R. Lombardo (2), B. F. Taylor, Jr. (1, 2). Professor R. C. Archibald was Faculty Representative.

Pi Mu Epsilon, Duke University

Four meetings of the fraternity were held this year, at which the following papers were presented:

Paths and Euler's theorem, by Dr. P. T. Maker, Mathematics Department.

Radio and sound, a lecture accompanied by demonstrations by Dr. W. J. Seeley of the School of Engineering.

Sound ranging, by Lt. Col. W. C. Bullock, F. A., Fort Bragg, N. C.

Practical applications of mathematics to astronomy, by Professor K. B. Patterson.

Initiation was held at the third meeting of the year on April 8, and the election of officers was held before Professor Patterson's lecture at the final meeting. Officers for 1943–44 are: President, Bessie Alston Cox; Vice-President, E. H. Smith; Treasurer, William Freeze; Faculty Adviser, Dr. W. W. Elliott.

Mathematics Club, Allegheny College

The club was newly organized in November 1942, and monthly meetings were planned. The topics presented were:

Continued fractions, by Gilbert Michel.

The game of Nim, by James Brown.

Hits and misses, by Professor J. S. Frame.

Diophantine equations, by Joseph Lepore.

Refreshments were served after each meeting. The aftermath of the second meeting, which was held at the home of Professor Frame, was featured by a Nim contest in which the speaker of the evening took on all comers. The evening ended with a showing of some colored movies. Officers for the year were: President, John Caughey; Secretary-Treasurer, Marcella Cooper; Faculty Adviser, Professor J. S. Frame.

Mathematics-Physics Club, College of Saint Teresa

The club meetings of the current year emphasized the role of mathematics in the war program. Reports were given on the following topics:

Mathematics and the Navy

Mathematics and the Army

Geometric design in peace and war

The elements of aeronautics.

A guest speaker, Brother Gerald of Saint Mary's College, Winona, Minnesota, presented this last topic. Other papers presented were:

Oddities in number

Magic squares

Pentagon building

Weather forecasting

Biographies of *Reginald Joseph Mitchell*, *Nicholas Saunderson*, *Newton*, *Steinmetz*, *Heaviside*, *Pupin*, *Karapetoff*, and *Jefferson.*

The Square versus the Circle: a mock debate.

The social meetings consisted of a Valentine party and a mathematical military picnic carried out on the plan of an induction into the *MATHS.* Officers were: President, Mary Brown; Vice-President, Mary Greisch; Secretary, Mary Louise Linehan; Treasurer, Mary Thornton; Faculty Advisers, Sisters Thomas à Kempis and Sister Leontius.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

FOUR POINTS

R. C. YATES, U. S. Military Academy

A conic may be determined by five coplanar points, a rectangular hyperbola by four. No generality is sacrificed if the hyperbola be taken as $xy=1$. Thus consider the four points $(\lambda_i, 1/\lambda_i)$ on $xy=1$ and let $\lambda_1\lambda_2\lambda_3\lambda_4=\sigma$. The orthocenter of the triangle $(i=1, 2, 3)$ is the intersection of the altitudes

$$\lambda_2\lambda_3 x - y = \lambda_1\lambda_2\lambda_3 - 1/\lambda_1$$

$$\lambda_3\lambda_1 x - y = \lambda_1\lambda_2\lambda_3 - 1/\lambda_2$$

that is, the point $(-1/\lambda_1\lambda_2\lambda_3, -\lambda_1\lambda_2\lambda_3)$. Accordingly, three points on a rectangular hyperbola have their orthocenter also on the curve. This is a well known property. The four chosen points are orthocentric if

$$\sigma = -1.$$

Any circle $x^2+y^2-2ax-2by+c=0$ cuts the hyperbola $xy=1$ in points determined by $x^4-2ax^3+cx^2-2bx+1=0$. Thus if the four points are cyclic,

$$\sigma = +1.$$

This is also sufficient since the determinant $|\lambda_i^2+1/\lambda_i^2, \lambda_i, 1/\lambda_i, 1|$, $(i=1, 2, 3, 4)$, vanishes if $\sigma=+1$.

We return now to the original set of four points $(\lambda_i, 1/\lambda_i)$. Taken in sets of three, their orthocenters are $(\mu_i, 1/\mu_i)$, where

$$\mu_i = -\lambda_i/\sigma.$$

We ask that this set of orthocenters $(\mu_i, 1/\mu_i)$ fall coincident in the order (1234) on (1234) with the selected set $(\lambda_i, 1/\lambda_i)$. Then $\sigma=-1$, which is the condition that $(\lambda_i, 1/\lambda_i)$ be orthocentric.

If, instead, the μ's fall on the λ's in the order (1234) on (2143), we have

$$\mu_1 = \lambda_2 = -\lambda_1/\sigma; \quad \mu_2 = \lambda_1 = -\lambda_2/\sigma; \quad \mu_3 = \lambda_4 = -\lambda_3/\sigma; \quad \mu_4 = \mu_3 = -\lambda_4/\sigma.$$

This is possible in either of two ways: first, if $\sigma=-1$ and $\lambda_1=\lambda_2$, $\lambda_3=\lambda_4$; second, if $\sigma=+1$ and $\lambda_1=-\lambda_2$, $\lambda_3=-\lambda_4$. This latter arrangement is of interest. Here the points $(\lambda_i, 1/\lambda_i)$ lie on a circle with center at the origin. In the cyclic order (1423) they form a rectangle with area

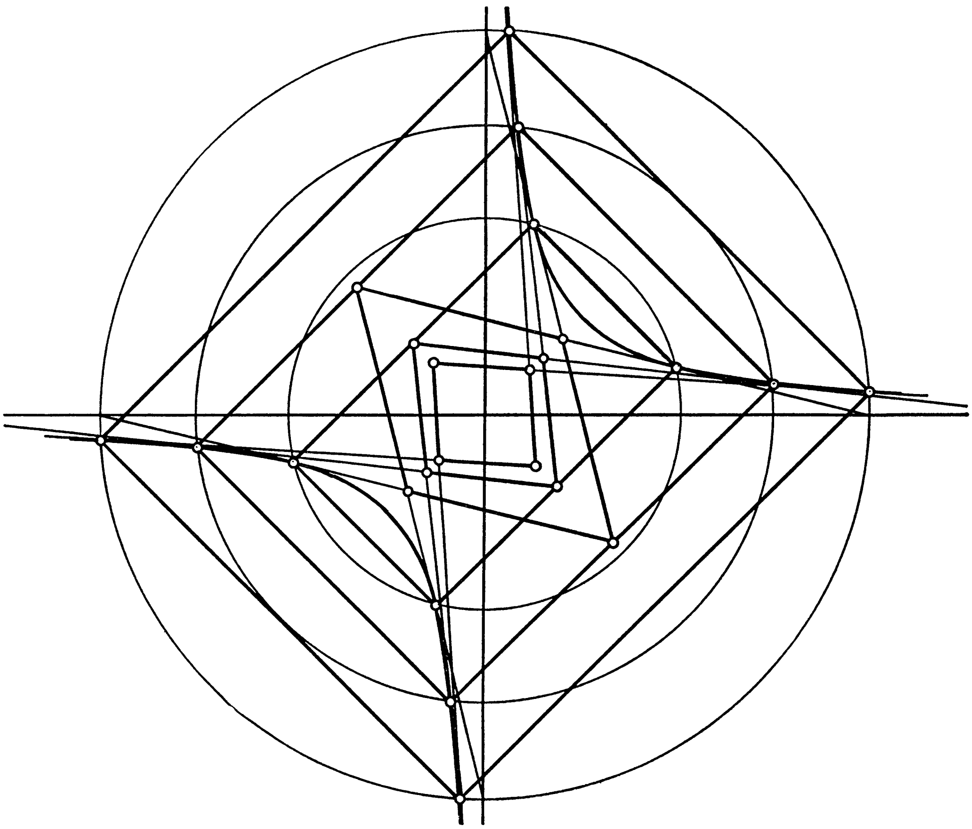
$$A = 2\cdot|\lambda_1^2 - \lambda_3^2|.$$

The tangents to the hyperbola at these points form a rhombus with area

$$B = 16 \cdot \left| \lambda_1\lambda_3/(\lambda_1^2 - \lambda_3^2) \right|.$$

Thus, since $\sigma = \lambda_1^2\lambda_3^2 = 1$,

$$A \cdot B = 32.$$



The geometrical aspects are now apparent. A fixed rectangular hyperbola intersects a concentric variable circle in four points. The product of the areas of the rectangle on these four points and the rhombus formed by their tangents to the hyperbola is constant.

The following problems to which the preceding notation is particularly well adapted were brought to my attention by Colonel Harris Jones.

If a circle and rectangular hyperbola (not necessarily concentric) meet in four real points: (1) the centroid of the quadrilateral on the four points is the midpoint of the line segment joining the center of the circle and the center of the hyperbola; (2) the sum of the squares of the distances of the four points from the center of the circle equals the sum of the squares of their distances from the center of the hyperbola.

AN ELEMENTARY LIMIT

M. S. Knebelman, State College of Washington

The proof of the fundamental theorem

$$\lim_{\theta\to 0} \frac{\sin \theta}{\theta} = 1,$$

as ordinarily given in elementary books, usually depends on two unproved theorems. The following proof is at least simpler, if not more rigorous.

If P is the perimeter of a regular n-gon inscribed in a circle of radius r, then $P=2nr \sin \pi/n$ and we know from plane geometry that $\lim_{n\to\infty} P=2\pi r$. Hence $\lim_{n\to\infty} (n/\pi) \sin \pi/n=1$ and if we let $\pi/n=\theta$ then $\theta\to 0$ as $n\to\infty$ and conversely. Hence $\lim_{\theta\to 0} \sin \theta/\theta=1$.

QUADRATIC AND CUBIC EQUATIONS*

O. J. Ramler, Catholic University of America

1. Introduction. When an algebraic equation having real coefficients is subjected to a bilinear transformation which transforms the axis of reals in the complex plane into the unit circle, the new equation will generally have complex coefficients and the roots of the original equation will either lie on the unit circle or pair off as points inverse to the unit circle. It would therefore seem reasonable to expect that the nature of the roots with respect to the unit circle of an algebraic equation with complex coefficients should be readily obtained by applying the bilinear transformation mentioned. However, even in the simple case of the quadratic equation this method of approach to the problem becomes long and involved. The following treatment is simpler and so far as the writer is aware does not appear in the literature.

2. The quadratic equation. Consider the quadratic equation

$$z^2 + pz + q = 0. \tag{1}$$

Let p and q have moduli r_1 and r_2 and amplitudes θ_1 and θ_2 respectively. Then we may write (1) in the form

$$z^2 + r_1t_1z + r_2t_2 = 0,$$

where $t_1=e^{i\theta_1}$ and $t_2=e^{i\theta_2}$. The equations whose roots are the conjugates and the reciprocals of the roots of (1) are, respectively,

$$t_1t_2z^2 + r_1t_2z + r_2t_1 = 0$$

and

$$r_2t_2z^2 + r_1t_1z + 1 = 0.$$

* Presented to the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America at Ashland, Virginia, May 2, 1942.

The equivalence of these two equations is the necessary condition that the roots of (1) shall lie on the unit circle. This condition leads to $r_2=1$ and $t_2=t_1^2$. Then equation (1) takes the form

$$z^2 + rtz + t^2 = 0,$$

whose roots will be denoted by z_1 and z_2. When $r>2$, $z_1\bar{z}_2=1$, and the roots are inverse points with respect to the unit circle. When $r<2$, $z_1\bar{z}_1=1$, and $z_2\bar{z}_2=1$, showing that z_1 and z_2 lie on the unit circle. Summarizing, we have

THEOREM 1. *The roots of $z^2+pz+q=0$ lie on the unit circle when the clinant $p/\bar{p}$ of p is q and $|p|<2$. The roots are inverse to the unit circle when the clinant of p is q and $|p|>2$. The roots are coincident points on the unit circle when $|p|=2$ and the clinant of p is q.*

3. The cubic equation. Let the cubic equation be

$$z^3 + at_1z^2 + bt_2z + ct_3 = 0, \tag{2}$$

where a, b, and c are positive and real and $|t_1|=|t_2|=|t_3|=1$. Comparing the equations whose roots are the conjugates and the reciprocals of the roots of equation (2), we find the necessary conditions that the roots of (2) shall lie on the unit circle to be $a=b$, $c=1$, and $t_1t_2=t_3$. Hence (2) may be written

$$z^3 + at_1z^2 + at_2z + t_1t_2 = 0. \tag{3}$$

Let the roots of (3) be x_1, x_2, x_3. If $x_1\bar{x}_1=x_2\bar{x}_2=x_3\bar{x}_3=1$, the roots are all obviously on the unit circle and $a\leqq 3$. If $x_1\bar{x}_2=x_2\bar{x}_3=x_3\bar{x}_1=1$, the roots are inverse to the unit circle in pairs. Hence $x_1=x_2=x_3$, $a=3$, and $t_2=t_1^2$. If $x_1\bar{x}_1=x_2\bar{x}_3=\bar{x}_2x_3=1$, x_1 will lie on the unit circle and x_2 and x_3 will be inverse with respect to it. Thus we find that at least one of the roots of (3) must lie on the unit circle and the other two lie on it or are inverse to it.

The roots of the Hessian of the cubic are the Isodynamic or Hessian points of the triangle determined by the cubic [1]. The Hessian points lie on the Apollonian circles of the triangle [2]. When the triangle determined by the cubic (3) is inscribed in the unit circle the Hessian points are inverse to the unit circle. When the cubic represents one point on the unit circle and two points inverse with respect to it the circumcircle of the triangle of roots will be orthogonal to the unit circle. Moreover in this case, the unit circle is an Apollonian circle of the triangle. Hence when the roots of the cubic represent one point on the unit circle and a pair of points inverse to it, the roots of the Hessian must lie on the unit circle.

The Hessian of (3) is

$$H = a(3t_2 - at_1^2)z^2 + (9 - a^2)t_1t_2z + at_2(3t_1 - at_2).$$

Identifying $H=0$ with $z^2+pz+q=0$ we readily find that $p/\bar{p}=q$ which insures

its roots to be either on the unit circle or inverse points with respect to it. From $H=0$ we have

$$|p| = \frac{|9 - a^2|}{a|3t_2 - at_1^2|}.$$

Hence

$$|3 - a|/a \leqq |p| \leqq (3 + a)/a,$$

and the restrictions on a and p may be analyzed.

The results for the cubic equation (3) may then be summarized in

THEOREM 2. *One root lies on the unit circle and two roots are inverse to it if* $a>1$ *and* $|p|<2$ *unless* $a=3$, $p=0$, *and* $t_2=t_1^2$ *when all three roots are coincident on the unit circle and the Hessian vanishes identically. All roots lie on the unit circle forming*

an equilateral triangle if $a=0$, $p=\infty$;
an acute-angled triangle if $0<a<1$, $|p|>2$;
an isosceles right triangle if $a=1$, $|p|=4$;
a scalene right triangle if $a=1$, $2<|p|<4$;
an obtuse triangle if $1<a<3$, $|p|>2$.

If $a=1$ *and* $|p|=2$, *two roots are coincident at one end of a diameter and the third root is at the other end of the diameter. If* $1<a<3$ *and* $|p|=2$, *the cubic and its Hessian have the same root repeated; the third root is distinct but lies on the unit circle.*

REFERENCES

1. F. V. and F. Morley, Inversive Geometry, p. 81 and p. 209.
2. R. A. Johnson, Modern Geometry, §492.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

A Source Book of Mathematical Applications. By E. G. Olds, L. E. Boyer, Ruth O. Lane, Nathan Lazar and F. L. Wren. (Seventeenth Yearbook of the National Council of Teachers of Mathematics.) New York, Bureau of Publications, Teachers College, Columbia University, 1942. 16+291 pages. $2.00.

This book is intended to be a reference book for those teaching junior and senior high school mathematics. Its four sections contain a wide variety of illus-

trations of the application of Arithmetic, Algebra, Geometry and Trigonometry to problems in daily life, in the professions and in various trades. These illustrations range from discussions or mere suggestions to the actual statement and solution of problems. The book is well arranged and indexed so that quick reference can be made to illustrations requiring a particular mathematical skill or to those pertaining to a given occupation. Most professions and trades are well represented, but it is surprising to find that the field of Chemistry is hardly mentioned.

Many unusual sources are drawn upon; yet the principal sources listed, especially in the Geometry section, are other mathematics text-books. It seems superfluous to acknowledge the permission of a well-known motor company for the use of Euler's formula for the lowest critical load of a column, and for the ordinary formula for the deflection of a cantilever.

One would expect from a mathematics book a far higher standard of precision than is found here. How can mathematical accuracy be represented by a discussion of astronomical data which begins with numbers given to two or four significant figures and derives from them numbers containing nine and 21 significant figures? The last number mentioned undoubtedly makes a very impressive addition to the book. We meet statements such as "$60\times 10\times 6=\$0.036$," "$240\times .0013=.312$ ft.," or "a gun has an initial velocity of 1500 feet per second." We are told that "the earth, all planets, and all satellites except a very few have their revolution about the sun and their rotation on their axis in a counter-clockwise direction."

Often units to be used are not specified, and numerous technical details must be supplied before such examples are used. Perhaps the authors hoped to make up for this by carefully specifying units in nearly all of their problems on variation—where they do no good at all because not a single set of data is given from which a constant of variation can be found. Many quantities such as work and power, force and pressure, velocity and acceleration are sometimes confused. Because of this careless inattention to units, a large portion of the Algebra section is of questionable value as reference material. This book should not be used by any teacher who lacks a working knowledge of physics.

R. E. GASKELL

An Introduction to Analytic Geometry. Volume I. By A. Robson. Cambridge, University Press, 1940. 14+409 pages. \$2.50.

This book starts at the beginning, but really presupposes a knowledge of graphs and of the elementary processes of the calculus, as the new ideas are introduced rapidly. Vectors and scalars, polar coordinates, determinants, homogeneous coordinates are treated at once; graphs and loci, points of intersection, change of axes are all discussed before the equation of a straight line is presented. The concepts of duality and of envelopes appear very early, together with those of composite loci and composite envelopes. The chapter on the circle

includes a discussion of pencils of circles, power of a point, radical axis, and limiting points.

True to the English tradition, generous lists of exercises for the reader to solve are found throughout the book. The explanations are concise and brief, but it is supposed that the added exercises will clarify the meaning of the text. A chapter on parametric representation follows immediately that on the circle. The discussion includes both point and line coordinates, with conditions for collinearity and concurrence. A short chapter on algebraic curves and one on abstract geometry introduce geometry of the complex domain. The presentation is interesting and is carefully discussed. It seems to the reviewer to be too brief to be of greatest service.

Now follow a number of concepts from pure geometry, though clothed for the most part in algebraic form. These are conical projection, cross-ratio, harmonic section, Desargues's theorem on perspective triangles, pole and polar, the general conic in point and in line coordinates, and the self-polar triangle.

Finally, after all these there is a series of chapters on the parabola, ellipse, and hyperbola, in which the ordinary topics usually associated with these names are treated in great detail. The book closes with four long lists of further exercises arranged in order of increasing difficulty. Answers to all the exercises are provided. There is a full index. The press-work is strikingly clear and legible. The figures are excellent. They are unnumbered.

A reader who has mastered this text and has solved a considerable number of the exercises has an adequate knowledge of both analytic and projective plane geometry.

VIRGIL SNYDER

NEW BOOKS RECEIVED

First Year College Mathematics. By W. L. Hart, W. A. Wilson, and J. I. Tracy. Fourth Edition. Boston, D. C. Heath and Co., 1943. 7+279+2+124 pages. $4.00.

Differential and Integral Calculus. Fourth Edition. By C. E. Love. New York, The Macmillan Co., 1943. 15+483 pages. $3.25.

Lecciones de Geometria Analitica. By Guido Castelnuovo. Translated by Andrea Levialdi and Manuel Sadosky, from the 7th Italian edition. LaPlata, Editorial Mundo Cientifico, 1943. 7+657 pages.

Plane and Spherical Trigonometry. By A. L. Nelson and K. W. Folley. With Mathematical Tables. Revised edition. New York, Harper and Brothers, 1943. 14+247+135 pages. $2.40.

Student's Handbook of Elementary Physics. By R. B. Lindsay. New York, The Dryden Press, 1943. 15+382 pages. $2.25.

Business Mathematics. By C. C. Richtmeyer and J. D. Foust. Second edition. New York and London, McGraw-Hill Co., 1943. 15+401 pages. $2.75.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 586. *Proposed by Frank Hawthorne, Allegheny College*

What is the path of an airplane "homing" from A to B by directional radio from B only, if acted on by a constant wind?

E 587. *Proposed by V. Thébault, San Sebastián, Spain*

Let AA', BB', CC', DD' be the altitudes of an orthocentric tetrahedron $ABCD$, with orthocenter H. Show that

$$\frac{BC\cdot DA}{B'C'\cdot D'A'}=\frac{CA\cdot DB}{C'A'\cdot D'B'}=\frac{AB\cdot DC}{A'B'\cdot D'C'}=\frac{HA\cdot HB}{HC'\cdot HD'}=\frac{HC\cdot HD}{HA'\cdot HB'}=\cdots.$$

E 588. *Proposed by J. Rosenbaum, Bloomfield, Connecticut*

Solve the equation

$$x^4-2cx^3+(2c^2-r^2)x^2+2cr^2x-c^2r^2=0.$$

When will two of the roots be rational? Can all four roots be rational?

E 589. *Proposed by Lloyd Dulmage, University of Manitoba*

Let ${}_2H_n$ denote the number of ways of putting n letters and n checks into n envelopes (respectively) so that every envelope contains either a wrong letter or a wrong check (or both), and let ${}_2J_n$ denote the number of ways when each envelope contains both a wrong letter and a wrong check. Show that

$${}_2H_n=\Delta^n(0!)^2,\qquad {}_2J_n=(\Delta^n0!)^2,$$

and generalize to ${}_sH_n$, ${}_sJ_n$.

E 590. *Proposed by H. W. Becker, Mare Island Training Division*

Let ${}_sK_n$ denote the number of Latin rectangles of n columns and $s+1$ rows, the first row consisting of the n letters in their natural order. Also let L_n denote the total number of $n\times n$ Latin squares (so that $L_2=2$, $L_3=12$, $L_4=576$,

$L_5=161280$, $L_6=812851200$, and $L_7 \geqq 61428210278400$). Show that, for a given s,

$$_sK_n \sim (n!)^s e^{-s(s+1)/2} \quad \text{as} \quad n \to \infty,$$

and that

$$L_n \sim (n!)^n e^{-n(n+1)/2}.$$

SOLUTIONS

The Meigs Hall Problem

E 531 [1942, 475; 1943, 202]. *Proposed by P. R. Hill, University of Georgia*

Suppose six students be standing an examination in a row of seats with an aisle at each end. If they finish in random order, what is the probability that a student will have to pass over one or more other students in order to reach an aisle? (The proposer intended "a student" to mean a particular student whose seat is not specified. Previous solvers took it to mean "some student.")

III. *Solution by W. H. Markham, John McNeese Junior College of the Louisiana State University.* The probability of any student finishing first is $1/6$; the probability that the student finishing first will have to pass one or more students is $2/3$; therefore the probability of any student finishing first and having to pass one or more students is $(1/6)(2/3)=1/9$. Similarly, the probability of any student finishing second and having to pass one or more students is $(1/6)(3/5)=1/10$; and the probability of any student finishing third or fourth and having to pass is $(1/6)(1/2)=1/12$ or $(1/6)(1/3)=1/18$, respectively. There will be no occasion for a student finishing fifth or sixth to pass any others. Hence the probability that any student will have to pass one or more students in order to reach an aisle is

$$\tfrac{1}{9}+\tfrac{1}{10}+\tfrac{1}{12}+\tfrac{1}{18}+0+0=\tfrac{7}{20}.$$

Also solved by E. P. Starke and the proposer.

Perfect Squares with Even Digits

E 548 [1942, 683]. *Proposed by R. V. Heath, Wall St., New York City*

Find a perfect square of seven digits with all digits even and positive. Show that the digits of a perfect square (>9) cannot be all *odd*.

Solution by Alan Wayne, Flushing, New York. The last two digits of N determine the last two digits of N^2. If e, o, a denote even, odd, any integers, respectively, then it is easily seen that

$$(eo)^2 = aeo, \qquad (oo)^2 = aeo, \qquad (ee)^2 = aae, \qquad (oe)^2 = aae,$$

whence the ultimate and/or penultimate digit must be even.

A scrutiny of Barlow's Tables gives

$$1692^2 = 2862864, \qquad 2878^2 = 8282884, \qquad 2978^2 = 8868484.$$

Incidentally, we have found (for any positive even value of m) the expression

$$f(n) = n^m + \frac{(n + 2B)^{m+1}}{2(m + 1)}$$

(in which, after expanding by the binomial theorem, we replace the symbols $B, B^2, B^3, B^4, \cdots$ by $-\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, \cdots$).

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4093. *Proposed by B. M. Stewart, Michigan State College*

A store conducted the following game in which the player paid 10¢, and if he won received $1 in merchandise. Out of five throws, each time throwing ten dice, a win was declared if the player had *fourteen* or more appearances of the numbered face named by him before playing; the player had the additional advantage that on the *first* throw of ten dice he could count the face occurring the greatest number of times as if it were the special face he had selected. The problem is to compare the theoretical probability with the odds offered by the store.

4094. *Proposed by N. A. Court, University of Oklahoma*

A. If a line moves so that the segment intercepted on it by two fixed skew lines subtends a right angle at a fixed point in space, the line generates a ruled quadric (This proposition is due to E. Bobillier, 1797–1832. It was proved analytically in *Nouvelles Annales de Mathématiques*, 1862, pp. 318, 320 and 379, 380).

B. The segment intercepted by two elements of the same system of a ruled quadric on a variable element of the complementary system subtends a right angle at each of two fixed points in space.

4095. *Proposed by H. D. Ruderman, Cooper Jr. H. S., Manhattan, N. Y.*

If x_1 and x_2 are any two real roots of the equation

$$a \sin x + \sin (bx + c) = 0,$$

with a, b, c real and positive, such that no real root lies between x_1 and x_2, determine the maximum and minimum values of $|x_1 - x_2|$ in terms of a, b, c.

4096. *Proposed by V. Thébault, San Sebastián, Spain*

An octahedron $ABCDEF$ with triangular faces is such that the vertices A, B, C, D are coplanar. Show that: (1) The intersections of the planes of the opposite faces (BEC, DFA), (BEA, DFC) lie in the same plane (P); the intersections of the planes of the opposite faces (AED, CFB), (DEC, BFA) lie in the same plane (Q); and the intersections of the planes (AEC, DFB), (BED, CAF) lie in the same plane (R). (2) The planes (P), (Q), (R) intersect in the same straight line which meets the diagonal EF.

SOLUTIONS

Artzt Parabolas

4019 [1942, 64]. *Proposed by Robin Robinson, Dartmouth College*

Given a triangle ABC. Prove that the bisectors of the interior and exterior angles at C, the side AB and its perpendicular bisector, and the perpendiculars to AC at A and to BC at B, are all tangent to a parabola. Locate its focus.

II. *Solution by L. M. Kelly, Coast Guard Academy, New London, Conn.* Let angle A of the triangle ABC be less than angle B; let the perpendicular bisector of AB meet the latter in M and the circumcircle (O) in D and H so that CD and CH are the interior and exterior bisectors of angle C; let CO meet (O) again in E; and let G be the projection of D on CA. We show first that GM is parallel to HC. Since G, A, D, M, are concyclic, $\angle MGC = \angle MDA = \angle HCA$, and this proves the statement. Consider now the hexagon $AMDC\infty_1\infty_2$, where the fifth and sixth points are at infinity on HC and GM, and on AE and GD. The diagonals are CA, GM, GD, and hence the sides of the hexagon are tangent to a parabola. A similar proof shows that BE is tangent to the same parabola. The directrix is CM, since tangents from C and M are perpendicular. Reflect CM in CD and then in MB; the intersection of these two reflections is the focus.

Solved also by E. P. Starke using the two bisectors of angle C as axes of coordinates. Solution I by the proposer is given [1943, p. 128].

Editorial Note. There exists a parabola tangent to the sides of the quadrilateral formed by the straight lines of HC, CD, HD, AB, and also a parabola tangent to the sides of the one formed by AE, BE, HD, AB. It is known that the focus in each case lies on (O); and, assuming as above that angle A of ABC is less than angle B, the focus in each case lies on the arc $\widehat{DB}$, where the order of points is D, B, C, H. In the second case, if C' is the orthocenter of AEB, the figure $AC'BC$ is a parallelogram and C' is on the extension of CM; and it follows that in the two cases the directrix is along CM. Let CM meet (O) again in M', and draw $M'F$ parallel to AB meeting the arc $\widehat{DB}$ in F. Then in each case the reflection of CM in AB (or HD) is along MF; thus F is the focus in each case. Hence we have a construction for the focus F and the directrix CM of the unique parabola tangent to the straight lines of the problem CD, HC, AB, HM, AE, BE.

The following information was supplied recently by J. R. Musselman. The parabola of the problem is related to one of the Artzt parabolas for a triangle ABC which is tangent to the sides of the complete quadrilateral formed by the internal and external bisectors of the angle C and the perpendicular bisectors of sides CA and CB. Its focus is the vertex of the second Brocard triangle for ABC corresponding to C and hence the focus is the midpoint of the chord of circumcircle (O) cut off from the symmedian for C; its directrix is the median for C.

We shall state a proof supposing that angle A is less than angle B. The quadrilateral contains two right triangles RSC, PQC, where the two straight lines QRC, PCS are the internal and external bisectors of angle C; and their circumcircles (RSC), (PQC) intersect again in the focus F. We easily show that these two circles are respectively tangent at C to CA, CB and pass through B, A. Hence F has all of the above properties, see Johnson's *Modern Geometry*, p. 279.

This Artzt parabola is identical with the parabola of the problem, as we see by extending the sides CA, CB of the problem to double their lengths forming the triangle $\bar{A}\bar{B}C$ for the Artzt parabola. Thus the solution of the problem adds two tangents to the Artzt parabola, the straight line joining the midpoints of the two sides through C and the perpendicular bisector of the segment between these two midpoints.

There is another Artzt parabola which is defined as tangent to CA and CB at A and B. It is easily seen that its axis is parallel to the median for C, and that the straight line joining the midpoints of CA and CB is tangent to the parabola at the midpoint of the segment between these two sides. The focus F lies on the circumcircle of the triangle formed by C and these two midpoints. Since triangles FCB and FAC are similar, the focus F must lie on the symmedian for C; hence it is the midpoint of the symmedian chord of the circumcircle (ABC).

Isogonol Conjugate Points

4038 [1942, 341]. *Proposed by V. Thébault, San Sebastián, Spain*

The point M is chosen arbitrarily on a bisector of angle A of the triangle ABC, and let M' be its isogonal conjugate with respect to ABC. Show that the two circles each through M and M' and tangent to the side BC are tangent also to the circumcircle of ABC.

I. *Solution by Howard W. Eves, Syracuse University.* Let the bisector of angle A meet BC in N. Inverting with respect to B as center of inversion, we obtain the triangle $A'B'C'$; its circumcircle $(A'B'C')$, the inverse of AC; and the circle $(B'N'M''M'A')$, where M and M' invert into M' and M'', and the latter circle bisects the angle formed by $A'B'$ and $(A'B'C')$. We are to prove that a circle through M'' and M' and touching the straight line $B'C'$ touches also the straight line $A'C'$, which is the inverse of the circumcircle (ABC). But this is evident since it is easy to show that the center of $(B'N'M''M'A')$ lies on the midpoint of one of the arcs $A'B'$, and hence on one of the bisectors of angle C'. Since the

arcs $N'M''$ and $M'A'$ are equal, it follows that M'' and M' are symmetric in the bisector of angle C', and the theorem follows.

II. *Solution by L. M. Kelly, U. S. Coast Guard Academy, New London, Conn.* The following equivalent theorem will be demonstrated:

If a circle tangent to the circumcircle of triangle ABC and to its side BC, cuts the bisector of angle A in real points, these points are isogonal conjugates.

The circle (Q) is tangent to the circumcircle (O) of ABC at D, tangent to BC at E, and meets the bisector of angle A in the points M, M'. Let the straight line DE meet (O) again in F; we shall show that F is the midpoint of arc BC of (O) and thus lies on AMM', the bisector. If the two tangents to (Q) at D and E meet in G, the angles GDE and DEG are equal, and arc DBF=arc DB +arc FC. Hence arc BF=arc FC, and the above statement is proved. We show next that triangles BFM' and MFB are similar. They have the angle at F in common and $FM'\cdot FM=FE\cdot FD$; also $FE\cdot FD=FB^2$, since triangles BFE and DFB are similar. Hence $FB^2=FM'\cdot FM$, and the statement is proved. It follows that angles FBM' and BMF are equal; then angle FBC+angle CBM'=angle BAF+angle MBA. Thus angles CBM' and MBA are equal, and M and M' are isogonal conjugate points.

Editorial Note. The proposer stated that the theorem of this problem is a variation of Bricard's Theorem II in the solution of 3852 [1941, 70]; that its proof is simple by an inversion; and that it is also a generalization of 3887 [1938, 482].

Omitting the parts not required here Bricard's proof of his Theorem II gives a simple and elegant solution of the present problem which we shall state. Let (O) be the circumcircle of ABC; ω, ω_a, ω_b, ω_c the incenter and excenters for angles A, B, C; the interior bisector of angle A contains ω and ω_a and meets BC and (O) in N and I; and the exterior bisector of A contains ω_b and ω_c and meets BC and (O) in N' and I'. It is known and it is easily proved that I is the center of a circle (I) through ω, ω_a, B, C; and similarly for I'. Let F, F' be isogonal conjugate points with respect to ABC on the interior bisector AI, then the straight line $C\omega$ bisects the angle $F'CF$; and it follows that F, ω, F', ω_a is a harmonic set. Any circle (Γ) through F and F' is orthogonal to (I) and is invariant in an inversion with respect to (I); and, if (Γ) is also tangent to BC at T, the inversion carries (Γ), BC, (O), T into (Γ), (O), BC, T', where T' must be the point of tangency of (Γ) with (O). Conversely, if (Γ) passes through F and F' and is tangent to (O) at T', it is tangent also to BC at T, where T', T, I are collinear. If F, F' are isogonal conjugates on AI', we have a similar theorem and proof.

Suppose now that (Γ) is any circle tangent to BC at T and to (O) at T'; then by the first part of Kelly's proof the straight line TT' meets (O) again at I, or at I', and it follows from the above inversion that (Γ) is invariant with respect to (I) in the first case or to (I') in the second. If then in the first case (Γ) meets AI in real points F, F', they are isogonal conjugates; and similarly for the sec-

ond case. We consider now the cases in which TT' passes through I giving possible isogonal conjugates F, F' on AI. The point T lies within the segment BC and T' lies on the arc CAB of (O), and F, F' lie within segment AN; the point T lies on BC outside the segment BC and T' lies on arc BIC giving possible points F, F' on the extension of AI. For the cases where TT' passes through I', we have T outside BC and T' on arc CAB giving possible points F, F' on the extension of AI', or on the extension of $I'A$. There are eight special cases of (Γ) where F, F' coincide at ω, ω_a, ω_b, ω_c which are easily constructed. These special circles (Γ) determine the regions for T on BC, and regions for T' on (O) for which there are no real corresponding points F, F'. For the case of circles (Γ) tangent to BC at T within that segment and to (O) at T' within the arc BIC, the straight line TT' must go through I'; and there are such circles which meet AI in real points, but which cannot be isogonal conjugates.

Tetrahedrons, Isogonal Conjugates

4039 [1942, 341]. *Proposed by N. A. Court, University of Oklahoma*

The circumcenter of a tetrahedron (T) and any point M are isogonal conjugates with respect to the tetrahedron formed by the centers of the four spheres passing through M and the circumcircles of the faces of (T).

Solution by Howard Eves, Syracuse University. The theorem is an obvious consequence of the sufficiency part of the following criterion for two points to be isogonal conjugates with respect to a given tetrahedron.

Criterion: A necessary and sufficient condition for two points to be isogonal conjugates with respect to a given tetrahedron is that one of the points be the circumcenter of the tetrahedron whose vertices are the reflections of the second point in the faces of the given tetrahedron.

The necessity is established in art. 750 of Altshiller-Court's *Modern Pure Solid Geometry*. We now prove the sufficiency. Let the two points be Q and P, and let the reflections of P in any two selected faces ABC and ABD of the given tetrahedron $ABCD$ be D' and C' respectively. Then we have

$$(1)\qquad D' - AB - C = C - AB - P, \quad P - AB - D = D - AB - C'.$$

Also, since Q and AB are each equidistant from C' and D',

$$(2)\qquad D' - AB - Q = Q - AB - C'.$$

But

$$D'-AB-C+C-AB-P+P-AB-D+D-AB-C'=D'-AB-Q+Q-AB-C',$$

whence, by (1) and (2),

$$C - AB - P = Q - AB - C' - D - AB - C' = Q - AB - D.$$

It may similarly be shown that P and Q form equal dihedral angles with every other pair of faces of $ABCD$. This proves the sufficiency of the criterion.

The corresponding theorem for the plane may be established in an analogous manner.

Solved also by L. M. Kelly and the proposer.

Editorial Note. The remaining two solutions are similar and consist of the argument for the part of the above solution which leads to its criterion, but using in place of it the theorem of art. 750 of the cited text. The converse of this latter theorem is easily proved by reversing the proof given in the text. The isogonal conjugate relation between two points ceases to be of one to one character if one of the points lies in the plane of a face of the given tetrahedron, and in this case one or both of the two tetrahedrons obtained by reflection degenerates, two or three of the vertices coinciding. However, the theorem of the problem remains true in this case of degeneracy.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Professor Alfonso Nápoles of the National University of Mexico has been elected president of the Mexican Mathematical Society.

Assistant Professor D. B. Ames of Rensselaer Polytechnic Institute has been promoted to an associate professorship.

Dr. K. J. Arnold of Massachusetts Institute of Technology has been appointed to an assistant professorship at the University of New Hampshire.

Assistant Professor Theodore Bennett of Marietta College has been promoted to a professorship.

Assistant Professor M. T. Bird of Utah State Agricultural College has been promoted to an associate professorship.

Dr. R. P. Boas, Jr., has been appointed visiting lecturer at Harvard University.

Assistant Professor W. M. Borgman, Jr., of Wayne University has been promoted to an associate professorship.

Assistant Professor Richard Brauer of the University of Toronto has been promoted to an associate professorship.

Assistant Professor F. J. H. Burkett of Union College has been promoted to an associate professorship.

Dr. Herbert Busemann of the Illinois Institute of Technology has been promoted to an assistant professorship.

Dr. Nathaniel Coburn of the University of Texas has been promoted to an assistant professorship.

Associate Professor A. H. Copeland of the University of Michigan has been promoted to a professorship.

Assistant Professor H. S. M. Coxeter of the University of Toronto has been promoted to an associate professorship.

Dr. John DeCicco of Illinois Institute of Technology has been promoted to an assistant professorship.

Assistant Professor D. G. Fulton of Ohio Northern University has been appointed to an assistant professorship at the University of New Hampshire.

Associate Professor W. H. Gage of the University of British Columbia has been promoted to a professorship.

Sister Agnes A. Green of the Immaculate Heart College, Los Angeles, California, has been promoted to an associate professorship.

Assistant Professor G. G. Harvey of the Massachusetts Institute of Technology has been promoted to an associate professorship.

Dr. F. B. Hildebrand of Massachusetts Institute of Technology has been promoted to an assistant professorship.

Dr. A. S. Householder of the University of Chicago has been promoted to an assistant professorship of mathematical biophysics.

Assistant Professor G. B. Huff of Southern Methodist University has been promoted to an associate professorship.

Professor R. D. James of the University of Saskatchewan has been appointed to a professorship at the University of British Columbia.

Assistant Professor Marie M. Johnson of Oberlin College has been promoted to an associate professorship.

Dr. Mark Kac of Cornell University has been promoted to an assistant professorship.

Associate Professor Claribel Kendall of the University of Colorado has been promoted to a professorship.

Professor C. C. MacDuffee of Hunter College has been appointed to a professorship at the University of Wisconsin.

Dr. Rhoda Manning of Oregon State College has been promoted to an assistant professorship.

Professor Deane Montgomery of Smith College is on leave at Princeton University.

Associate Professor C. A. Nelson of New Jersey College for Women, Rutgers University, has been promoted to a professorship.

Dr. H. E. Newell of the University of Maryland has been promoted to an assistant professorship.

Professor Abba V. Newton of Hartwick College, Oneonta, New York, has been appointed to an assistant professorship at Smith College.

Dr. E. P. Northrop of Hotchkiss School, Lakeville, Connecticut, has been appointed to an assistant professorship at the University of Chicago.

Assistant Professor Gordon Pall of McGill University has been promoted to an associate professorship.

Assistant Professor P. M. Pepper of the University of Notre Dame has been promoted to an associate professorship.

Dr. Harry Pollard of Harvard University has been appointed to an assistant professorship at Kenyon College, Gambier, Ohio.

Associate Professor G. B. Price of the University of Kansas has been promoted to a professorship.

Dr. J. K. Riess of Brown University has been appointed to an assistant professorship at Tulane University.

Professor L. D. Rodabaugh of Shurtleff College, Alton, Illinois, has been appointed visiting lecturer at Oberlin College.

Dr. Ralph Salem of the Massachusetts Institute of Technology has been promoted to an assistant professorship.

Dr. O. F. G. Schilling of the University of Chicago has been promoted to an assistant professorship.

Dr. Edith R. Schneckenburger of Michigan State Normal College has been promoted to an assistant professorship.

Assistant Professor A. J. Smith of the College of William and Mary has been appointed to an assistant professorship at Montana School of Mines.

Professor C. C. Spooner of the Northern Michigan College of Education has retired.

Assistant Professor B. M. Stewart of Michigan State College has been appointed to an assistant professorship at Denison University.

Assistant Professor I. D. Stewart of Whitman College, Walla Walla, Washington, has been promoted to an associate professorship.

Assistant Professor A. H. Taub of the University of Washington has been promoted to an associate professorship.

Mr. M. L. Vest of West Virginia University has been promoted to an assistant professorship.

Dr. Alexander Weinstein of the University of Toronto has been promoted to an assistant professorship.

Assistant Professor F. M. Wood of McGill University has been promoted to an associate professorship.

Dr. H. S. Zuckerman of the University of Washington has been promoted to an assistant professorship.

The following appointments to instructorships are announced:
Iowa State Teachers College: E. Marie Hove
Princton University: L. L. Rauch
Rice Institute: George Piranian
Syracuse University: F. L. Celauro
Vassar College: Janet C. Durand

Professor James McGiffert of Rensselaer Polytechnic Institute died June 18 1943, at the age of 75 years.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

SERVICE OPPORTUNITIES FOR WOMEN

From the Navy: The Navy is making an urgent search for women to perform important services in the Aerology and in the Electronic Programs. These Programs are under the sponsorship of the WAVES. In addition to the regular requirements for a candidate for officer training, the following qualifications are necessary. For Aerology, the applicant must have had college mathematics through the integral calculus, and also have had one year of physics. For Electronics, the requirement is one year of college mathematics and one year of college physics. College graduates who have majored in mathematics are especially urged to apply. After a period of training in the regular Officer Training Schools, persons enlisted in these Programs will continue with special work at various colleges and universities throughout the country. For further details, one should consult the nearest Office of Naval Officer Procurement.

From Civil Service: Persons who have completed a four-year college course with study in mathematics through the calculus may be eligible for Federal positions paying $2,533 a year. Mathematicians are needed in the Coast and

Geodetic Survey, in the Hydrographic Office and Naval Observatory, in the Navy Department, in the War Department arsenals, and by the National Advisory Committee for Aeronautics. Persons with one year of college and one course in mathematics may qualify for mathematical aid positions, paying $1,970 and $1,752 a year. Completion of a war training course in mathematics is qualifying for these latter positions. One year of appropriate experience is also qualifying. Higher training is required for positions as aids paying up to $3,163 a year. Mathematical aids are employed in the Department of Commerce and in the War and Navy Departments. Trainee aids in mathematics may be employed in positions existing in Washington, D. C., only. Persons who have completed a high school course in mathematics, biology, physics, chemistry, or general science are eligible for trainee positions, and may apply to take the written test. Appointees perform simple computations, and are paid at the rate of $1,560, $1,620, and $1,752 a year while learning the duties of the position. Junior astronomers in the Naval Observatory, Washington, D. C., assist in making astronomical observations or computations, preparing information for publication, and caring for instruments and equipment. For Junior Astronomer positions paying $2,433 a year, a four-year course in a recognized college, with a minimum of twelve semester hours in mathematics and twelve semester hours in astronomy, will qualify. Women are especially asked to apply for these various positions.

IMPORTANT BILLS BEFORE CONGRESS

The Victory Corps Act of 1943 (S. 875), sponsored by Senator Carl Hayden. This bill before the Congress of the United States has received the official support of the Army and the Navy, and has been favorably recommended by the Senate Committee on Education and Labor. The endorsement of the latter Committee resulted from such conclusions as the following: "The Army and the Navy have found glaring deficiencies in basic preparation in the fields of mathematics and science. They should not be called upon to take time out of the limited time available for training of soldiers to make up these deficiencies. The schools can be helped to do much in the way of revising courses and improving basic training in these fields."

In brief, the bill provides an authorization for an annual appropriation of $8,484,377 for the war service preinduction training of high-school students in mathematics, science, and preflight aeronautics, the selection of students to be prepared for war service, and the physical training of such students. The authorization is limited to the duration of the present war, and follows the usual pattern of Federal grants-in-aid to the States, the procedures for which have been developed over a period of some twenty-five years. This pattern gives great latitude to the States in the development of their own plans within the limitations of the purposes of the grants-in-aid, and is therefore flexible and adjustable to the different needs which exist in the several States. The bill does not set up a Federal system of preinduction education and training. It undertakes only to

encourage and to aid all the States in their own high-school programs of improvement to meet clearly defined and pressing war needs. A definite formula is provided for the allotments to the States of funds authorized.

The Science Mobilization Bill (S. 702), sponsored by Senator H. M. Kilgore. The necessity of the bill, as conceived by its sponsor, is explained in the preliminary part of the first section as follows: "The Congress hereby recognizes that the full development and application of the Nation's scientific and technical resources are necessary for the effective prosecution of the war and for peacetime progress and prosperity, and that serious impediments thereto consist in the unassembled and uncoordinated state of information concerning existing scientific and technical resources; the lack of an adequate appraisal, and the unplanned and improvident training, development, and use, of scientific and technical personnel, resources, and facilities in relation to the national need; the consequent delay and ineffectiveness in meeting the urgent scientific and technical problems of the national defense and essential civilian needs; the trend toward monopolized control of scientific and technical data and other resources with lack of access thereto in the public interest; and the absence of an effective Federal organization to promote and coordinate, in the national interest, scientific and technical developments."

To remove these "impediments," it is proposed to establish an "Office of Scientific and Technical Mobilization." The National Roster of Scientific and Specialized Personnel of the War Manpower Commission would be abolished as a separate entity, and its powers, personnel, and unexpended appropriation would be transferred to the new agency. The "Office" would be directed by an "Administrator," appointed by the President to serve at the pleasure of the President, with the assistance of a salaried "National Scientific and Technical Board," also appointed by the President, but whose duties are to be defined by the Administrator. This Board is to consist of six members besides the Administrator, one to represent industry, one to represent agriculture, one to represent labor, one to represent the consuming public, and two additional members at large who shall be scientists or technologists. The Administrator will appoint all other employees, and he may waive the provisions of the civil-service laws and regulations if he determines that it is necessary to do so. Provision is also made for a non-salaried "National Scientific and Technical Committee," to be appointed by the President to advise and consult with the Administrator at least once a month upon basic policies of the Board. This Committee consists of the Board previously mentioned, a representative from each of such Federal departments as the President shall designate, four additional representatives of the consuming public, six additional members representing labor, six representing management, and three additional scientists or technologists.

The bill vests the Office with broad powers over "scientific and technical personnel" and "scientific and technical facilities." By definition, "Scientific and technical facilities shall include all real and personal property, . . . programs, projects, . . . methods, processes, procedures, . . . patents, inventions, . . . in-

formation or knowledge of every description used or intended to be used for scientific or technical purposes in research and development or in the production or supply of war or civilian goods or services." "Scientific and technical personnel shall include all persons, excepting physicians and dentists, who have completed any course of study in any college or university in any branch of science or its practical application or who have had not less than an aggregate of six months' training or employment in any scientific or technical vocation."

During August, the Executive Committee of the American Association for the Advancement of Science passed the following resolution pertaining to the bill: "After careful consideration of the purposes and provisions of the Science Mobilization Bill (S.702), the American Association for the Advancement of Science, an organization having nearly 25,000 members (and having 187 associated and affiliated societies with a combined membership of over 500,000 persons whose interests cover broadly all the natural and social sciences) now, through its Council of about 250 members chosen from among the leaders of American science, respectfully recommends to the Senate and to the House of Representatives of the United States that the Kilgore Bill (S.702) be not passed either in its present form or in any other form containing similar provisions."

FROM SELECTIVE SERVICE

Student Deferment. The provisions of Bulletin No. 33-6 pertaining to the deferment of undergraduate students were revised on July 24. The revised statement follows.

"A student in undergraduate work in any of the scientific and specialized fields . . . should be considered for occupational classification if he is a full-time student in good standing in a recognized college or university and if it is certified by the institution as follows:

(a) That he is competent and gives promise of successful completion of such course of study, and

(b) That if he continues his progress he will graduate from such a course of study within 24 months from the date of certification."

New Occupational Deferment Policy. Local Board Memorandum No. 115, as amended August 16, 1943, sets forth a new occupational deferment policy through the medium of an extensive list of critical occupations (I: Production and Services Occupations; II: Professional and Scientific Occupations), and a list of nondeferrable activities and occupations. The occupation, "Mathematician (including Cryptanalyst)," is listed in Part II of the first list. The Memorandum specifically states that the titles appearing in this critical list "are also intended to cover those persons who are engaged in full-time teaching of these professions." The relation of the new memorandum to previous Activity and Occupation Bulletins is clarified by the sentence, "The Activity and Occupation Bulletins should be used by the agencies of the Selective Service System as a guide and should be considered in occupational classification matters along with all other available information."

COMMITTEE ON AVAILABLE TEACHERS OF COLLEGIATE MATHEMATICS

The Committee on Available Teachers of Collegiate Mathematics, established by the War Policy Committee of the American Mathematical Society and the Mathematical Association of America, has been in existence since the beginning of April, 1943. During this time it has received and answered numerous inquiries from colleges and universities needing teachers of mathematics, as well as from teachers who were free to accept appointments.

A total of 149 persons have registered with the Committee. Of this number, 129 have at least the master's degree. The Committee has answered requests for teachers of mathematics from 67 different institutions. To these institutions a total of 348 names, representing 111 different persons, have been suggested. According to the records of the Committee, 57 of the persons registered are no longer available. It should be pointed out that of the remaining candidates some have indicated that they are available only for summer or part-time teaching, others have had only secondary school teaching experience, while others are available for appointment in restricted geographical areas.

It is anticipated that the demand for teachers will increase considerably during the next two or three months. On the other hand, the number of available well-qualified candidates who have registered with the Committee for such appointments has been reduced to such an extent that the remaining supply has become quite inadequate to meet the expected demand.

For this reason, the committee requests that (a) individual teachers report their availability and (b) departments of mathematics inform the Committee at the earliest possible date of their needs during the next half year, giving as full details as they can concerning the qualifications expected, the salary offered and other pertinent facts relating to their vacancies, and of members who are free to fill temporary positions in other institutions.

The effectiveness of the work of this Committee will be enhanced if registrants will report positions which they have accepted and if inquiring institutions will report appointments that have been made.

Committee on Available Teachers,
W. D. Cairns
Arnold Dresden
J. R. Kline

110 Bennett Hall
University of Pennsylvania
September 2, 1943.

THE SPRING MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The nineteenth meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at Pennsylvania College for Women, Pittsburgh, Pennsylvania, on April 17, 1943. The chairman of the Section, Dr. R. G. Sturm of the Aluminum Research Laboratories, presided at both morning and afternoon sessions.

The attendance was twenty-four, including the following eight members of the Association: Helen Calkins, H. L. Dorwart, B. P. Hoover, David Moskovitz, E. G. Olds, J. B. Rosenbach, R. G. Sturm, E. D. Wells.

At a business meeting prior to the afternoon session, the present officers were reelected for another term. It was voted to hold the next meeting during the month of April, 1944. The invitation of Carnegie Institute of Technology to hold the meeting at that institution was accepted.

After an address of welcome by Dean Helen Marks of Pennsylvania College for Women, the following papers were presented:

1. *New theorems on ordinary linear differential equations*, by Dr. R. F. Clippinger, Carnegie Institute of Technology, introduced by Professor Dorwart.

The speaker presented an expository treatment of certain recent developments of his own generalizing Sturm-Liouville theorems in several directions.

2. *Application of mathematics to some geophysical problems*, by Dr. Sigmund Hammer, Gulf Research Laboratories, introduced by Dr. Sturm.

Dr. Hammer discussed applications of mathematics to problems in geophysical exploration. The subject was introduced by a description of the technique of prospecting for minerals by measuring exceedingly small irregularities ("anomalies") in the earth's gravitational field caused by buried mineral masses or deformed rocks. The method requires a precise knowledge of the normal gravitational field of the earth. The speaker discussed the effect upon the earth's gravitational field of the centripetal acceleration due to the rotation of the earth, and also the effect due to the spheroidal shape of the earth. He then considered the application of a theorem of Gauss to the problem of estimating the mass of a buried body of ore from its gravitational anomaly. The paper was concluded with a brief mention of important applications of other mathematical concepts to geophysical problems.

3. *Approximations for complex roots of algebraic equations*, by Professor H. C. Hicks, Carnegie Institute of Technology, introduced by Professor Dorwart.

This paper dealt with a practical method of approximating complex roots of algebraic equations. The method was compared with other standard devices such as that of Graeffe.

4. *A mathematical peculiarity of the plastic stress-strain relations*, by E. A. Davis, Westinghouse Research Laboratories, introduced by Dr. Sturm.

Mr. Davis stated that when a ductile metal flows under the influence of combined stresses, three mutually perpendicular directions can be found along which the deformations are pure extensions. There are two theories at present which deal with the values of the three principal strain rates at any given instant. The older is based upon the law of viscous flow which states that the shear rate on any plane of principal shear stress is proportional to the stress acting on that plane. The newer theory claims that the rates are not proportional to the stresses, but that they may be expressed by relations involving power functions of the stresses. This theory reduces to the older one when a certain exponent n has the value 1. The peculiarity pointed out is that in the newer theory the distribution of strain rates when $n=1$ is the same as the distribution when $n=3$. This is due to the fact that the expression $[a^n-b^n]/[(a+b)^n+b^n]$ has the same value for $n=3$ and for $n=1$.

H. L. DORWART, *Secretary*

THE ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION

The twenty-seventh annual meeting of the Rocky Mountain Section of the Mathematical Association of American was held on Friday and Saturday, April 16–17, 1943, at the University of Denver, Denver, Colorado. It was a joint meeting with the National Council of Teachers of Mathematics and the Eastern Division of the Colorado Education Association. Section meetings were held Friday afternoon and evening, at both of which the Vice-Chairman of the Section, Professor A. W. Recht, presided. Three additional sessions were held on Saturday in conjunction with the other organizations participating in the meeting.

The attendance was one hundred and thirty-five, including the following fifteen members of the Association: A. G. Clark, Sister Rose Margaret Cook, J. R. Everett, J. C. Fitterer, G. W. Gorrell, D. F. Gunder, J. O. Hassler, A. J. Kempner, Claribel Kendall, W. J. LeVeque, A. J. Lewis, A. E. Mallory, A. W. Recht, C. H. Sisam, H. W. Williams.

At the business meeting the following officers were elected for the coming year: Chairman, Professor A. E. Mallory, Colorado State College of Education; Vice-Chairman, Professor A. J. Kempner, University of Colorado.

The following papers were presented:

1. *On the place of mechanics in the system of sciences, and the training of mathematicians for work in applied mechanics*, by Dr. Paul Nemenyi, University of Colorado, introduced by Professor Kempner.

The speaker discussed the unity of the sciences and the place of mechanics in the scheme of scientific studies. He classified mechanics as a part of physics, and considered the relation of mechanics to other parts of physics and to mathe-

matics. In particular, the relation between mechanics and probability was brought out. A program for the training of mathematicians for research and teaching in mechanics was then outlined.

2. *A method of measuring effectiveness in the teaching of college mathematics*, by Professor J. O. Hassler, University of Oklahoma.

Professor Hassler investigated the grades in Calculus II of the students of eleven teachers of Calculus I. The records of the "A," "B," "C" and "D" students were examined separately, and the successes (by separate groups) of the students of the various teachers compared. This was a measure of the effectiveness of the teaching in the first course. The students in each of the four grade classifications were also divided into two groups, namely those who remained with the same teacher and those who had the second course with a different teacher. In this way was obtained an evaluation of the teachers' grading scales.

3. *The integral* $\int x^{-1}dx = \log x$ *as a limiting case of* $\int x^{n-1}dx = x^n/n$, by W. J. LeVeque, University of Colorado.

In this paper it was shown that the integral $\int x^{-1}dx$ can be studied by considering the limit of $\int x^{n-1}dx$ as n approaches zero. The geometric properties of the approximation curves were also investigated.

4. *On a continuous stochastic process*, by Professor A. G. Clark, Colorado State College.

Professor Clark stated that, in ballistics research, it has been the practice to measure the variability of an ordered succession of random variables $x_1, x_2, \cdots, x_n$ by using the mean square successive difference $\delta^2 = (N-1)^{-1}\sum_{i=1}^{N-1}(x_{i+1}-x_i)^2$ as a criterion for measuring variability, rather than the quantity $s^2 = N^{-1}\sum_{i=1}^{N}(x_i - \bar{x})^2$. He considered the problem of testing for determination of trend, and showed that δ/s is the proper criterion to use for this purpose.

5. *On the introduction of coördinates in an affine plane geometry*, by Mrs. Margaret S. Matchett, University of Denver, introduced by Professor Recht.

Using only axioms of connection for points and lines, the parallel axiom, and the configuration of Desargues, it is possible to introduce a system of coördinates into a geometry. These coördinates satisfy all the field properties except that of commutative multiplication. In order to define these coördinates one considers the group of those one-to-one transformations of the plane which map parallel lines into parallel lines. The translations form an invariant subgroup of this group. The inner automorphisms of this sub-group, with addition and multiplication suitably defined, form the division algebra from which the coördinates are taken.

6. *Graphical methods for representation of various types of functional relations*, by Professor A. J. Kempner, University of Colorado.

This paper dealt with a method of plotting the graph of an equation of the type $F[f(x, y), g(x, y)] = C$. The details are as follows: Let $X = f(x, y)$ and

$Y=g(x, y)$. Plot the curve $F(X, Y)=C$ with reference to X and Y axes, and plot on a separate chart the two families of curves $f(x, y)=\alpha$ and $g(x, y)=\beta$ where α and β are parameters. If (α, β) is any point on the curve $F(X, Y)=C$, a point of intersection of the two curves $f(x, y)=\alpha$ and $g(x, y)=\beta$ is a point of the curve $F[f(x, y), g(x, y)]=C$. The method can be extended to the representation of the equation $F[f(x, y), g(x, y)]=H(z)$, leading to a one parameter family of curves in the xy-plane.

7. *Teaching mathematics effectively for war or peace*, by Professor J. O. Hassler, University of Oklahoma.

Professor Hassler reviewed briefly the controversy over the transfer of training, and reported that eighty per cent of the psychological experiments up to date show clear evidence of transfer of training. He remarked that to teach consciously for transfer of training is a prime goal of effective teaching. It was also stated that this object can be achieved by relating subject-matter in every possible way to practical applications, by cultivating habits of independent thinking and generalization, and by exploiting the spirit of discovery in the pupil.

8. *Mathematics abridged has gone to war*, by Professor J. O. Hassler, University of Oklahoma.

The speaker reviewed the present situation wherein frantic efforts (by means of concentrated courses) are being made to make amends for deficient training in mathematics among the youths in the army or about to go into the army. He gave some facts concerning dilution of mathematics courses in the recent past which has contributed to this delinquency. He made a plea for teachers to equip themselves to fight against having a denatured, abridged, and diluted mathematics in the post-war curriculum of the high schools.

A. J. LEWIS, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The twenty-third regular meeting of the Southern California Section of the Mathematical Association of America was held at the University of Southern California, Los Angeles, California, on Saturday, March 13, 1943. Professor Morgan Ward, chairman of the Section, presided.

The attendance was fifty-five, including the following twenty-six members of the Association: O. W. Albert, C. K. Alexander, L. D. Ames, Clifford Bell, L. T. Black, Myrtie Collier, P. H. Daus, D. C. Duncan, W. H. Glenn, Jr., Frances C. Hinds, P. G. Hoel, C. G. Jaeger, G. R. Kaelin, Ada A. McClellan, G. F. McEwen, P. M. Niersbach, W. T. Puckett, Jr., H. R. Pyle, J. M. Robb, G. E. F. Sherwood, D. V. Steed, A. E. Taylor, S. E. Urner, Morgan Ward, W. M. Whyburn, Euphemia R. Worthington.

The following officers were elected for the coming year: Chairman, D. C. Duncan, Los Angeles City College; Vice-Chairman, P. G. Hoel, University of California at Los Angeles; Program Committee, C. K. Alexander, Chairman, P. M. Niersbach, and the Secretary. The next meeting was tentatively scheduled to be held March 11, 1944, at Los Angeles City College.

The following six papers were read:

1. *The slide rule—modern uses and devices*, by Dr. E. J. Hills, Los Angeles City College, introduced by Professor Urner.

Dr. Hills described the range of usefulness of the rule and the types of slide rules available, and showed how new scales can be and are developed. He considered the most natural ways to multiply, divide, obtain squares, and extract square roots. He stressed the importance of the principle of proportion and urged that it be used as much as possible. In conclusion, he discussed the use of the *LL* scales, and included an analysis of the natural functions.

2. *Training of the naval aviation cadet*, by Lieutenant J. L. Brader, U. S. N., introduced by Professor Ward.

3. *Report of the subcommittee on mathematics of the California committee on education*, by Professor W. M. Whyburn, University of California at Los Angeles.

Professor Whyburn, chairman of the California subcommittee on mathematics, gave a brief discussion of the organization of this subcommittee and of the chief problems which it has under study. He then read a portion of a preliminary report made by the subcommittee in October, and this portion of the report was endorsed by the Section. This same portion of the report had been presented to the Northern California Section, and was endorsed by that body. That part of the report which was read by Professor Whyburn was printed in this MONTHLY, vol. 50, 1943, p. 337.

4. *The application of geometrical probability to sampling problems in ecology*, by Professor P. G. Hoel, University of California at Los Angeles.

The speaker treated the problem of sampling a region for coverage or abundance of a given species of plants from a geometrical point of view. He applied statistical distribution theory and geometrical probability to obtain formulas which measure the relative accuracy of various common sampling techniques in ecology.

5. *Arithmetical properties of recurring series*, by Professor Morgan Ward, California Institute of Technology.

This paper dealt with the sequence of integers (u) defined recursively by the equality

$$u_{n+k} = p_1 u_{n+k-1} + p_2 u_{n+k-2} + \cdots + p_k u_n$$

where $p_1, p_2, \cdots, p_k$ are given integers, and the initial values of (u) are also integers. If $u_0 = 0$, $u = 1$, and if u_n divides u_m whenever n divides m, then (u) is

called a Lucasian sequence in honor of Lucas, who studied the case $k=2$. Professor Ward obtained new results for the cases $k=4$, 5, and 6. In particular, he showed that if $k=5$, the polynomial $f(x)$ associated with the sequence (u) must have an integral root; and that if $k=4$, and $f(x)$ is irreducible, its Galois group is of order 2, 4, or 8.

6. *Certain generalizations of the formulae for the range of normally distributed variates*, by Professor G. F. McEwen, Scripps Institution of Oceanography.

By substitution of logarithms of the variates in existing formulae for the range R of a normally distributed variate, the speaker obtained more general formulae for this special type of asymmetry. One result was

$$R = \sinh \left[\frac{a_n}{2} \sigma_{\ln x} \right]$$

where a_n is a known function of variates. Similarly, from a formula relating the range R_1 to R_{rn}, the mean difference between the rn greatest and the rn smallest variates, and the assumption of a normal distribution, he derived general formulae for the asymmetrical case.

P. H. Daus, *Secretary*

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-seventh Annual Meeting, Chicago, Illinois, November 27–28, 1943.

The following is a list of the Sections of the Associations with dates of future meetings so far as they have been reported to the Secretary.

Allegheny Mountain, Pittsburgh, Pa., April, 1944
Illinois
Indiana, Indianapolis, Oct. 29–30, 1943
Iowa
Kansas
Kentucky
Louisiana-Mississippi, Ruston, La., 1943
Maryland-District of Columbia-Virginia
Metropolitan New York
Michigan
Minnesota
Missouri
Nebraska
Northern California, Berkeley, Jan. 29, 1944
Ohio, Columbus, April 6, 1944
Oklahoma
Philadelphia, Philadelphia, Nov. 27, 1943
Rocky Mountain
Southeastern
Southern California, Los Angeles, March 11, 1944
Southwestern
Texas
Upper New York State
Wisconsin

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ALGEBRAIC IDENTITIES IN THE THEORY OF NUMBERS

E. T. BELL, California Institute of Technology

1. Two identities. Elementary algebraic identities have frequently been used to establish arithmetical theorems whose proofs are not evident otherwise. Fermat, for example, noted that an odd integer is prime if and only if it is the difference of two squares in only one way, from which an interesting two-dimensional representation of the sieve of Eratosthenes is obtainable. Typical of another kind of result, Oltramare stated (1894) that every integer is a sum of five integer cubes, and this was seen to be an immediate consequence of the identity

$$6m = (m+1)^3 + 2(-m)^3 + (m-1)^3.$$

Similar cubic identities have been used in attempts to prove the conjecture that four cubes suffice.

Several such identities are special cases of formulas of the types

$$ncN = c_1 u_1^n + \cdots + c_{n+1} u_{n+1}^n, \tag{1}$$

$$ncN + c' = c_1 v_1^n + \cdots + c_n v_n^n, \tag{2}$$

in which n is any constant integer >1, N is an arbitrary integer, the u's, v's are variable integers (depending on N), and the constant coefficients $c, c', c_1, \cdots, c_{n+1}$ (independent of N) are polynomials, none identically zero, with integer coefficients, in $2n-2$ integer parameters. The degree in the parameters of c, c' is $n(n-1)$; that of each of the remaining constants, $n(n-2)$. For all N with the possible exception of certain divisors of c, the u's, v's are integers >0 if n is even, and may be chosen >0 if n is odd by changing the signs of some of the constants. In the exceptional case some of the u's, v's may be zero.

2. Explicit forms. To state explicit forms of (1), (2), we note that any integer N may be written $x^{n-1}y$, where x, $|x| \geqq 1$, is any $(n-1)$th-power divisor of N. Identically in x, y,

$$\sum_{r=1}^{n-1} (n, r) a_i^{n-r} b_i^r x^{n-r} y^r \equiv w_i,$$

$$(n, r) \equiv n!/r!(n-r)!, \qquad w_i \equiv (a_i x + b_i y)^n - a_i^n x^n - b_i^n y^n.$$

Provided the determinant of the system obtained from this identity by taking $i = 1, \cdots, n-1$ does not vanish, the system may be solved for the $x^{n-r}y^r$, $r = 1, \cdots, n-1$. The determinant is readily evaluated and the solution appears in a simple form. Write

$$A \equiv a_1 \cdots a_{n-1}, \qquad B \equiv b_1 \cdots b_{n-1},$$

$$d_{i,j} \equiv a_i b_j - a_j b_i, \qquad D \equiv \prod_{r=1, s>r}^{n-1} d_{r,s},$$

$$A_i \equiv A/a_i, \qquad B_i \equiv B/b_i, \qquad D_i \equiv D/D^{(i)},$$

where $D^{(i)}$ is the product of all $d_{r,s}$ in which one of r, s is i, and $a_1, \cdots, a_{n-1}$, $b_1, \cdots, b_{n-1}$ are such that $ABD \neq 0$. Then

$$nABDx^{n-1}y = \sum_{i=1}^{n-1} (-1)^{i-1} A_i B_i^2 D_i w_i,$$

for (1) with $N = x^{n-1}y$; and

$$nABDN + \sum_{i=1}^{n-1} (-1)^{i-1} a_i^n A_i B_i^2 D_i$$
$$= \sum_{i=1}^{n-1} (-1)^{i-1} A_i B_i^2 D_i (a_i + b_i N)^n - \left(\sum_{i=1}^{n-1} (-1)^{i-1} A_i B_i^2 D_i b_i^n \right) N^n$$

for (2), obtained from the preceding formula by $x=1$, $y=N$.

For $n=3$ the formula (2) is

$$a_1 a_2 d_{1,2}(3b_1 b_2 N + a_1 b_2 + a_2 b_1) = a_2 b_2^2 (a_1 + b_1 N)^3 - a_1 b_1^2 (a_2 + b_2 N)^3 + b_1^2 b_2^2 d_{1,2} N^3,$$

which gives Oltramare's identity when $a_1 = 2$, $a_2 = b_1 = b_2 = 1$, $N = m-1$. The choice $a_1 = a_2 = b_1 = 1$, $b_2 = 2$, $N = m$ gives

$$6m + 3 = 4(m+1)^3 - (2m+1)^3 + 4m^3;$$

whence every integer is of the form

$$x^3 + y^3 + 4(z^3 + w^3)$$

with x, y, z, w integers. This follows by reducing $6m+3+t^3$, $t=0, \cdots, 5$, modulo 6. Thus if $3+t^3 = 6r+s$, $0 \leqq s < 6$, the identity gives

$$6(m+r) + s = 4(m+1)^3 - (2m+1)^3 + 4m^3 + t^3,$$

or, with m replaced by $m-r$,

$$6m + s = t^3 + (-2m + 2r - 1)^3 + 4(m - r + 1)^3 + 4(m-r)^3;$$

and since the residues s are a permutation of the integers t, the result follows. From the last with $s = 0, \cdots, 5$ it is seen that x, y, z, w may all be chosen different from zero with the following possible exceptions:

$$\begin{array}{ll} s = 0, \quad m = 4, 5; & s = 1, \quad m = 10, 11; \\ s = 2, \quad m = 20, 21; & s = 3, \quad m = 0, -1; \\ s = 4, \quad m = 0, -1; & s = 5, \quad m = 0, 1. \end{array}$$

For $n=4$, $a_1 = a_2 = b_2 = b_3 = 1$, $a_3 = b_1 = 2$, $N = m-1$ we have

$$24m - 3 = (2m-1)^4 + 2(m-1)^4 - 12m^4 - 6(m-1)^4.$$

Whence, as in the preceding example, all integers of the forms $24m+6$, 13, 21, 22 are represented in

$$x^4 + y^4 + 2z^4 - 6w^4 - 12u^4$$

in which the integers $x, \cdots, u$ may all be chosen greater than zero with the possible exceptions $24m+6$, $m=2, 3, 4$; $24m+13$, $m=-1, 0, 1$; $24m+21$, $m=-2, -1, 0$; $24m+22$, $m=-2, -1, 0$.

As an example for $n=5$, we take $a_1=a_2=a_3=a_4=-b_1=b_4=1$, $b_2=-b_3=2$.

$$60y = 8(y-1)^5 + 8(y+1)^5 - (2y-1)^5 - (2y+1)^5 + 2(2y)^5;$$

whence it follows (by referring to a table of fifth powers for residues of x^5+2w^5 mod 60) that every integer is of the form

$$x^5 + y^5 + z^5 + 2w^5 + 2u^5 + 8v^5 + 8t^5,$$

with $x, \cdots, t$ integers.

3. Derived identities. Several further identities may be obtained from (1) by differentiations and use of the relations $a_iA_i=A$, $b_iB_i=B$. For example,

$$ADx^{n-1} = \sum_{i=1}^{n-1} (-1)^{i-1} A_iB_iD_i[(a_ix + b_iy)^{n-1} - b_i^{n-1}y^{n-1}];$$

$$0 = \sum_{i=1}^{n-1} (-1)^{i-1} A_iD_i[(a_ix + b_iy)^{n-2} - b_i^{n-2}y^{n-2}];$$

$$Dx^{n-r} = \sum_{i=1}^{n-1} (-1)^{i-1} a_i^{r-2} B_iD_i(a_ix + b_iy)^{n-r}, \quad 1 < r \leqq n;$$

$$0 = \sum_{i=1}^{n-1} (-1)^{i-1} D_ia_i^{n-t-3}b_i^t, \quad 0 \leqq t < n-3.$$

Identities similar to (1) may be constructed from the expansion of

$$\left(\sum_{i=1}^{n} a_ix_i\right)^m - \sum_{i=1}^{n} a_i^m x_i^m.$$

Examples of the use of differentiation are given incidentally in the next section.

4. Rational identities. Writing y for N in the second identity, we set the left member equal to the arbitrary t,

$$nABDy + A\sum_{i=1}^{n-1} (-1)^{i-1} a_i^{n-1} B_i^2 D_i = t,$$

solve for y, and substitute into the original identity. Then

$$t = \sum_{i=1}^{n-1} (-1)^{i-1} A_iB_i^2D_ix_i^n - B\left(\sum_{j=1}^{n-1} (-1)^{j-1} A_jB_jD_jb_j^{n-1}\right)x^n, \tag{3}$$

in which

$$nABDx_i = b_i t + AB\left(nDa_i - \sum_{j=1}^{n-1} (-1)^{j-1} a_j^{n-1} B_j D_j\right),$$

$$nABDx_n = t - A \sum_{j=1}^{n-1} (-1)^{j-1} a_j^{n-1} B_j^2 D_j,$$

with $ABD \neq 0$.

For $b_j = 1, j = 1, \cdots, n-1$, the identity (3) takes a remarkably simple form. After some straightforward reduction, we find

$$t = \sum_{i=1}^{n-1} (-1)^{i-1} A_i D_i' y_i^n - (-1)^n D' y_n^n, \tag{4}$$

in which

$$nAD'y_i = t + AD'[na_i - (a_1 + \cdots + a_{n-1})],$$
$$nAD'y_n = t - AD'(a_1 + \cdots + a_{n-1}),$$
$$D' \equiv \prod_{r<s} (a_r - a_s), \qquad A \equiv a_1 \cdots a_{n-1},$$

with $AD' \neq 0$, and D_i' is obtained from D' by deleting all $a_r - a_s$ in which one of r, s is i.

From (4) it follows that every rational number t is of the form

$$\sum_{i=1}^{n} c_i y_i^n, \tag{5}$$

in which $c_i = (-1)^{i-1} A_i D_i'$, $c_n = (-1)^{n-1} D'$, $AD' \neq 0$, $(i \neq n)$. If $a_1, \cdots, a_{n-1}$ are rational integers, the c_j are rational integers all different from zero, and the y_i are n rational numbers, all different. (If $y_i = y_j$, $i \neq j$, then $AD' = 0$; a contradiction.) In numerical examples, any common factor k of the coefficients c_i in (5) may be suppressed, with a corresponding change in the values of the y_i. For if t be replaced by kt, we may divide out k in (4) and in the expressions for the y_i, y_n. This device is used in obtaining the identities (6), (7).

If t in (4) is (temporarily) a continuous real variable, we may differentiate successively with respect to t. Replacing t by $AD't$ in the results, we get

$$\sum_{i=1}^{n-1} (-1)^{i-1} A_i D_i y^s + (-1)^{n-1} D' y_n^s = 0, \qquad n > 1, \tag{6}$$
$$s = 0, 1, \cdots, n-2,$$

in which

$$y_i = t + na_i - (a_1 + \cdots + a_{n-1}),$$
$$y_n = t - (a_1 + \cdots + a_{n-1}).$$

Here we have dropped the denominator n appearing in the expressions for

y_i, y_n on dividing throughout by AD', as is permissible since the $n-1$ equations (6) are homogeneous in the y's. The constants and variables in (6) are unrestricted.

For the special case of (6) in which all the letters denote rational integers, let g be the G.C.D. of

$$a_1 + \cdots + a_{n-1}, \qquad na_i - (a_1 + \cdots + a_{n-1}), \qquad i = 1, \cdots, n-1,$$

and h the G.C.D. of D', A_iD_i', $i=1, \cdots, n-1$; and define the g_i, h_i by

$$a_1 + \cdots + a_{n-1} = -gg_n, \; na_i - (a_1 + \cdots + a_{n-1}) = gg_i, \; (-1)^{i-1}A_iD_i' = hh_i,$$
$$(-1)^{n-1}D' = hh_n.$$

Then, replacing t by gt in (6) and dropping the denominator g, we get

$$\sum_{i=0}^{n} h_i(t+g_i)^s = 0, \qquad s = 0, 1, \cdots, n-2. \tag{7}$$

In this the h_i are non-zero rational integers and the g_i are n different rational integers.

We give some examples of the preceding formulas. From (7) with $n=6$, $a_1=1$, $a_2=2$, $a_3=3$, $a_4=-1$, $a_5=-2$ we get

$$(t+5)^p + 5(t-3)^p + 10(t+1)^p = (t-5)^p + 5(t-3)^p + 10(t-1)^p,$$
$$p = 0, 1, 2, 3, 4.$$

The choice $n=5$, $a_i=i$ gives

$$4(t+1)^p + 4(t-1)^p = 6t^p + (t+2)^p + (t-2)^p, \qquad p = 0, 1, 2, 3.$$

For $n=4$ the identity (4) may be written

$$t = bc(b-c)x^4 + ca(c-a)y^4 + ab(a-b)z^4 + dw^4,$$

in which $d \equiv (a-b)(b-c)(c-a)$, $abcd \equiv k \neq 0$,

$$-4kx = t - k(3a - b - c), \qquad -4ky = t - k(-a + 3b - c),$$
$$-4kz = t - k(-a - b + 3c), \qquad -4kw = t + k(a + b + c).$$

By an obvious transformation this is equivalent to

$$t = fx^4 + gy^4 + hz^4 - (f+g+h)w^4:$$
$$d \equiv r^3 fgh(1+f)(1-g)(1+fg) \neq 0;$$
$$-4dx = t - rd(1 - f - 3g - fg),$$
$$-4dy = t - rd(1 + 3f + g - fg),$$
$$-4dz = t - rd(1 - f + g + 3fg),$$
$$-4dw = t + rd(3 + f - g + fg),$$

with the condition $fgh+f+g+h=0$. These are of interest in connection with

the statement (1832) of Libri* that each of the forms

$$2x^4-20y^4+30z^4-12w^4, \qquad x^4-y^4+3z^4-3w^4$$

represents all rational numbers with x, y, z, w rational numbers. The first of these is given by the choice $a=5$, $b=2$, $c=1$; the second by $f=1$, $g=-1$, $h=3$. As in the derivation of (7), it follows from the first that all rational numbers are also represented in $x^4-10y^4+15z^4-6w^4$ with rational x, y, z, w.

The formulas have numerous applications to diophantine analysis. To illustrate one of them, we transform the identity (4) for $n=3$, which may be written

$$\begin{aligned} t &= bx^3-ay^3+(a-b)z^3, \qquad ab(a-b)\neq 0: \\ 3ab(a-b)x &= t+ab(a-b)(2a-b), \\ 3ab(a-b)y &= t+ab(a-b)(-a+2b), \\ 3ab(a-b)y &= t-ab(a-b)(a+b). \end{aligned}$$

If the identity is multiplied throughout by $[3ab(a-b)]^3$, and in the result, after expansion of the left side, a, b, t are replaced by a^{3n}, b^{3n}, $3^{3n-3}t^{3n}$ we get the indicated three-parameter solution of

$$x_1^3+x_2^3+x_3^{3n}+3x_4^{3n}=y_1^3+y_2^3+y_3^{3n}+3y_4^{3n}:$$

$$\begin{aligned} x_1 &= b^{3n}[3^{3n-3}t^{3n}+a^{3n}b^{3n}(a^{3n}-b^{3n})(2a^{3n}-b^{3n})], & x_3 &= 3a^3b^6t, \\ x_2 &= a^{3n}[3^{3n-3}t^{3n}-a^{3n}b^{3n}(a^{3n}-b^{3n})(a^{3n}+b^{3n})], & x_4 &= 3a^5b^4t, \\ y_1 &= a^{3n}[3^{3n-3}t^{3n}+a^{3n}b^{3n}(a^{3n}-b^{3n})(-a^{3n}+2b^{3n})], & y_3 &= 3a^6b^3t, \\ y_2 &= b^{3n}[3^{3n-3}t^{3n}-a^{3n}b^{3n}(a^{3n}-b^{3n})(a^{3n}+b^{3n})], & y_4 &= 3a^4b^5t. \end{aligned}$$

Thus for n any integer >0, the equation has a three-parameter solution in integers $x_1, \cdots, y_4$. If in the result of differentiating the equation with respect to t we replace n by $2n$, and in the resulting equation t, a, b by a^{6n-1}, b^{6n-1}, $3^{6n}t^{6n-1}$, and make the same successive substitutions in the values of $x_1, \cdots, y_4$, we get a three-parameter integer solution of

$$x_1^2+x_2^2+x_3^{6n-1}+3x_4^{6n-1}=y_1^2+y_2^2+y_3^{6n-1}+3y_4^{6n-1}.$$

If $x_1', \cdots, y_4'$ are the values of $x_1, \cdots, y_4$ in the case $n=1$ of the former equation, derivation with respect to $t(n=1)$ gives

* See L. E. Dickson, History of the Theory of Numbers, vol. 2, 1920, p. 727, where only one of Libri's forms is cited. Both are given in Journal für Mathematik (Crelle's Journal), vol. 9, 1832, p. 292, with two evident misprints in the exponents. Libri's method in this paper is Lagrange's, based on the product-theorem for norms of algebraic numbers. He does not give the values of the variables; these may be found by substituting in the formulas above. Libri notes that since each form represents all rational numbers, the product of two forms of either kind is a form of the same kind; and he infers that the forms repeat under multiplication, as do Lagrange's norms. But the variables in Lagrange's forms are independent; in all forms such as Libri's and those of the present paper they are polynomials in a parameter, and hence are not independent. The forms therefore do not repeat under multiplication as the term is now commonly understood.

$$bX_1^2 + aX_2^2 + aX_3^2 + 3aX_4^2 = aY_1^2 + bY_2^2 + bY_3^2 + 3bY_4^2:$$
$$X_1 = tx_1', \qquad X_2 = tx_2', \qquad X_3 = ab^3x_3', \qquad X_4 = a^2b^2x_4',$$
$$Y_1 = ty_1', \qquad Y_2 = ty_2', \qquad Y_3 = a^3by_3', \qquad Y_4 = a^2b^2y_4'.$$

As a last application, we may consider the constants in the identities as numbers in any algebraic number field, and the variables as numbers in the same or another field. Expression of the numbers concerned in the canonical (basis) form then gives, on equating coefficients of linearly independent numbers, a system of equations (or identities) equivalent to the original, the constants and variables in the system being rational numbers. A parametric solution of the system, in rational numbers, is given by equating coefficients in the solution of the original equation. Without this reduction we see for example that any number of an algebraic number field A of degree m is of the form (5), with the c_i rational integers and the y_i numbers in A.

PARTITION OF SPACE

R. C. BUCK, Harvard University

1. The problem. It is well known that n straight lines divide the plane into at most $(n^2+n+2)/2$ regions. This often appears in a more familiar guise as the answer to the question: *how many pieces can be obtained from a round flat cheese by exactly n straight cuts?* We will generalize this to both Euclidean r-space —an r dimensional cheese—and projective r-space, obtaining in each case the number of p dimensional regions into which the space is partitioned, for $p=0, 1, \cdots, r$.

2. A geometrical lemma. Consider the configuration formed by n hyperplanes in Euclidean r-space (E_r) having general intersection.* Let $F_r(n)$ be the number of r dimensional regions in the configuration. We will show that

$$F_r(n) = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{r}.$$

Let us examine our configuration more closely; it consists of a finite number of 0-cells (points), 1-cells (edges), $\cdots$, r-cells (polyhedroids). Let $M_r(p, n)$ be the number of p-cells in the configuration; thus, $M_r(r, n) = F_r(n)$. We define $F_0(n)$ to be 1 for all n.

LEMMA.

$$M_r(p, n) = \binom{n}{r-p} F_p(n + p - r).$$

* By this we mean that every set of k hyperplanes ($r-1$ dimensional subspaces) intersect in an $r-k$ dimensional plane; if $r-k$ is negative, the intersection is void. In the plane, general intersection means that no two lines are parallel and no three are concurrent.

Consider an arbitrary p-plane (p dimensional plane) of the configuration. It is formed by the intersection of $r-p$ of the n hyperplanes; this leaves $n-(r-p)$ hyperplanes that cut the p-plane, forming in turn $n+p-r$ $(p-1)$-planes. These too have general intersection in the p-plane—which is itself an E_p—and therefore divide it into $F_p(n+p-r)$ p-cells. Every set of $r-p$ hyperplanes intersect in a p-plane, so that there are precisely $\binom{n}{r-p}$ p-planes in the configuration; this gives the value of $M_r(p, n)$ as above.

3. An application of combinatorial topology. We now make use of a result from elementary topology. If a configuration drawn on an r-sphere* consists of α_0 points, α_1 edges, $\cdots$, then, independent of the configuration drawn,

$$\alpha_0 - \alpha_1 + \alpha_2 - \alpha_3 + \cdots = \begin{cases} 2 & \text{if } r \text{ is even} \\ 0 & \text{if } r \text{ is odd.} \end{cases}$$

The value of this sum is known as the Euler Characteristic; if $r=2$, this reduces to the Euler Polyhedral Formula.

A corresponding formula holds for unbounded Euclidean r-space, namely:

$$\sum_{k=0}^{r} (-1)^k \alpha_k = (-1)^r$$

To prove this, add the point at infinity to E_r—producing a configuration for which $\alpha_0 = 1$, $\alpha_k = 0$ if $k > 0$—and map E_r on to the r-sphere with the point at infinity going into the north pole. Since the Euler Characteristic of the r-sphere is $1+(-1)^r$, we remove our added point, leaving $(-1)^r$ as the Euler number of E_r.

To apply this to our present problem, set $\alpha_p = M_r(p, n)$, whence

$$\sum_{p=0}^{r} (-1)^p M_r(p, n) = (-1)^r,$$

or, by the lemma,

$$\sum_{p=0}^{r} (-1)^p \binom{n}{r-p} F_p(n+p-r) = (-1)^r,$$

and replacing $r-p$ by k,

$$\sum_{k=0}^{r} (-1)^k \binom{n}{k} F_{r-k}(n-k) = 1. \tag{1}$$

THEOREM 1.

$$F_r(n) = \sum_{\lambda=0}^{r} \binom{n}{\lambda} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{r}$$

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* An r-sphere is the r dimensional surface given by the equation

$$x_0^2 + x_1^2 + x_2^2 + \cdots + x_r^2 = 1.$$

Proof. Since (1) is a recurrence relation with initial conditions $F_0(n)=1$, there is a unique solution. We need only show that the stated form of $F_r(n)$ satisfies (1) for all r and n.

$$\begin{aligned}\sum_{k=0}^{r}(-1)^k\binom{n}{k}\left[\sum_{\lambda=0}^{r-k}\binom{n-k}{\lambda}\right] &= \sum_{k=0}^{r}\sum_{\lambda=0}^{r-k}(-1)^k\binom{n}{k}\binom{n-k}{\lambda}\\ &= \sum_{k=0}^{r}\sum_{\lambda=0}^{r-k}(-1)^k\binom{n}{k+\lambda}\binom{k+\lambda}{k}\\ &= \sum_{j=0}^{r}\sum_{k=0}^{j}(-1)^k\binom{n}{j}\binom{j}{k}\\ &= \sum_{j=0}^{r}\binom{n}{j}\delta_0^j = \binom{n}{0} = 1.\end{aligned}$$

COROLLARY 1. If $r \geqq n$, $F_r(n)=2^n$.

This means that if the dimension of the embedding space is high enough, every additional hyperplane merely halves each of the regions previously formed.

COROLLARY 2.

$$M_r(p, n) = \sum_{k=r-p}^{r} C_k\binom{n}{k} \quad \text{where} \quad C_k = \binom{k}{r-p}.$$

For $r=1, 2, 3, 4$, we have:*

$$\begin{aligned}F_1(n) &= 1 + n\\ F_2(n) &= \binom{n}{0}+\binom{n}{1}+\binom{n}{2} = (n^2+n+2)/2\\ F_3(n) &= (n^3+5n+6)/6\\ F_4(n) &= (n^4-2n^3+11n^2+14n+24)/24.\end{aligned}$$

Thus we see that a large spherical Edam cheese can be divided into exactly $(n^3+5n+6)/6$ pieces by n straight cuts.†

4. Projective r-space. The same technique can be applied to the problem of determining the number of r-cells obtained by slicing up *projective* r-space by n hyperplanes. The only difference encountered is that we must now use the Euler number of projective r-space, $[1+(-1)^r]/2$—*i.e.*, 1 if r is even, 0 if r is odd. If we denote by $M_r'(p, n)$ the number of p-cells obtained, and by $F_r'(n)$ the number of r-cells, the lemma still holds and we have:

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* It might be observed that an arbitrary polynomial $f(x)$ can be uniquely expressed as $f(x) \equiv \sum a_k\binom{x}{k}$; this seems to be the natural form for the functions with which we are here concerned.

† See problem E554, this MONTHLY, vol. 50, p. 59.

THEOREM 2.

$$F'_r(n) = \sum_{\lambda=0}^{r} \left\{ \frac{1 + (-1)^{r+\lambda}}{2} \right\} \binom{n}{\lambda} = \sum_{\lambda=0}^{[r/2]} \binom{n}{r - 2\lambda}.$$

COROLLARY.

$$M'_r(p, n) = \sum_{k=r-p}^{r} C_k \left\{ \frac{1 + (-1)^{r+k}}{2} \right\} \binom{n}{k} = \sum_{\lambda=0}^{[r-p/2]} C_{r-2\lambda} \binom{n}{r - 2\lambda},$$

where C_k is defined as in corollary 2, namely $C_k = \binom{k}{r-p}$.

For example*

$$F'_1(n) = n, \qquad F'_2(n) = \binom{n}{2} + \binom{n}{0} = (n^2 - n + 2)/2.$$

5. Bounded regions of E_r. Returning again to the configuration in E_r, we obtain the number of *bounded* r-cells, $F_r^*(n)$, by a repetition of the process this time using the Euler number 1.

THEOREM 3.

$$F_r^*(n) = \sum_{\lambda=0}^{r} (-1)^{r+\lambda} \binom{n}{\lambda} = \binom{n-1}{r}.$$

COROLLARY.

$$M_r^*(p, n) = \sum_{k=r-p}^{r} (-1)^{r+k} C_k \binom{n}{k} = \frac{r+1}{n+p-r} \binom{r}{p} \binom{n}{r+1}.$$

The simplifications are obtained by using the identity

$$\binom{n}{r+1} - \binom{n-1}{r} = \binom{n-1}{r+1}.$$

In conclusion, we observe that from any two of the theorems, the third may be inferred. For we have the relation

$$F'_r(n) = \frac{F_r^*(n) + F_r(n)}{2} = F_r^*(n) + \frac{F_r(n) - F_r^*(n)}{2}.$$

Geometrically, this means that in passing from Euclidean space to projective space, we count only half of the unbounded regions as being distinct.

* See W. B. Carver, The polygonal regions into which a plane is divided by n straight lines, this MONTHLY, vol. 48, p. 668.

THE RATE OF INTEREST IN INSTALMENT CONTRACTS

C. H. FISCHER, University of Michigan

1. Introduction. In recent years there has been a growing tendency to devote more attention in courses in the mathematics of finance to the subject of instalment buying and, in particular, to the determination of the rate of interest involved in an instalment contract. Instalment contracts generally contain no mention of a rate of interest—instead, it is stated that a certain flat sum called a *carrying charge* is to be added to the principal and this sum divided by the number of payments yields the payment size. The purpose, then, in computing an index such as the rate of interest must be merely to shed more light upon the transaction, perhaps to point out how intrinsically costly it is to buy "on time." We shall try to show here that the interest rate as generally computed is not the proper one to measure the real burden of the debt to the debtor but that it, instead, usually overstates it. A more suitable index for this purpose is then developed.

2. Typical solution. To clarify the issue and for the sake of concreteness. let us consider a typical instalment contract. Suppose that a debt of \$100 is to be discharged by means of twelve monthly payments, the size of each being determined by adding a carrying charge of \$8 to the original debt and dividing the sum by 12. The usual method of finding the interest rate is to equate the principal of \$100 to the present value, at the unknown rate of interest, of the twelve payments of \$9 each. The resulting equation, solved by interpolation in an interest table, yields an interest rate of 15.45%, effective.

3. Defect in the usual method. The above method of solution involves the implicit assumption that "money is worth" the unknown interest rate, that is, that all sums of money of any size can be immediately invested for any length of time at this uniform rate. This assumption is closely realized in fact as far as the finance company is concerned, and hence this is a very reasonable determination of the rate of interest earned by a company engaged in the issuance of such loans. From the point of view of the debtor, however, this assumption does not fit the facts. The ordinary individual does not have an opportunity to invest money at such a high rate of interest—in fact, he may find it difficult to earn any interest on small sums invested for short periods. Since in this application the basic assumption is invalid, the computed interest rate which rests upon it cannot be a proper measure to use in the case of the debtor.

To get a better idea of the true burden placed upon the debtor by his instalment contract, the possible alternative investments of his payments should be considered. Thus, if he had the opportunity to invest each \$9 payment at, say, 2% simple interest up to the termination date of the contract, he would be able to accumulate a total of \$108.99. This is the ultimate sum which he foregoes in making his monthly payments directly to the finance company, and this sum

suggests itself as the measure for which we are seeking. Since in this illustration the contract period is exactly one year and the principal is \$100, it is reasonable to state that the interest rate paid by the debtor is 8.99%; that is, he sacrifices the equivalent of \$108.99 one year hence in payment for a loan of \$100 contracted at the present time. Similarly, had the debtor been able to invest his payments at 4% instead of 2%, he would have been able to accumulate \$109.98 at the end of the year and it might well be argued that the true interest rate he paid on this \$100 loan was 9.98%.

An individual might find it almost impossible to obtain any interest on such small sums invested for such short periods. In this case it would be immaterial to the debtor whether he were required to pay over each \$9 to the finance company or were permitted to keep it on hand to the end of the year and pay all twelve instalments at that time. In any event his payments would amount to only \$108.00 at the end of the year and it would be fair to state that he paid only 8% interest.

4. New definition of the rate of interest. Let us, then, define *the rate of interest paid by the debtor as that rate of interest at which the original principal would accumulate, over the contract period, to an amount equal to the sum of the instalment payments, each accumulated to the termination date of the contract at the highest investment rate available to the debtor.*

This definition permits the use of either simple or compound interest for either or both of the interest rates involved. In practice, the type of interest used for the debtor's investment rate would be that which is actually available to him, while the choice of the type of interest paid by the debtor could be made arbitrarily.

5. Examples of resulting formulas. Three examples of formulas which would follow from the above definition are given below.

A. Both rates at compound interest. Let us define the symbols:

r = nominal interest rate, compounded m times per year, available to the debtor as an investor;

n = length of the contract in years;

k = carrying charge per unit of original principal;

m = number of instalment payments per year;

i = effective rate of interest paid by the debtor.

It is then easy to show that i is determined by the following equation:

$$(1 + i)^n = \frac{1 + k}{mn} s_{\overline{mn}|r/m}.$$

B. Debtor's investment rate at simple interest. If, in the above definitions of symbols, the meaning of r be altered so that

r = simple interest rate per year available to the debtor as an investor,

we find that i is determined by the equation

$$(1 + i)^n = (1 + k)\cdot[1 + r(mn - 1)/2m].$$

C. Both rates at simple interest. If we now alter the definition of i so that

i = simple interest rate per year paid by the debtor,

i is determined by

$$1 + ni = (1 + k)\cdot[1 + r(mn - 1)/2m].$$

In this latter case we note that there is a linear relationship between i and r. In general, the interest rate paid by the debtor as defined above is a monotonic increasing function of the effective rate available to the debtor in the role of investor.

6. Numerical illustration. The following table shows how in two numerical examples the interest rate paid by the debtor varies with the rate available to him as an investor. In both cases it is assumed that the debt will be discharged by means of twelve monthly payments. In the first example the carrying charge is \$8 per \$100 of principal; in the second example it is \$20 per \$100.

Debtor's investment rate compounded semi-annually	Effective rate paid by debtor	
	Case I	Case II
	Carrying charge \$8	Carrying charge \$20
0%	8.00%	20.00%
1	8.50	20.55
2	8.99	21.10
3	9.49	21.65
5	10.48	22.76
10	12.98	25.54
15	15.50	28.34

By the usual methods, assuming that the debtor's investment possibilities were equal to those of the finance company, his effective interest rate would be 15.45% for Case I and 41.30% for Case II.

ITERATION OF THE ϕ FUNCTION

W. H. MILLS, Aberdeen Proving Ground

1. Introduction. In the January 1943 number of this MONTHLY Dr. Harold Shapiro divided all positive integers into classes by means of the iteration of the Euler ϕ function as follows: Let

$$\phi^0(x) = x,$$
$$\phi^n(x) = \phi[\phi^{n-1}(x)], \tag{1}$$

where $\phi(x)$ represents the number of integers less than x and relatively prime to x. If $x=\Pi_i p_i^{\alpha_i}$, where the p_i are distinct prime factors of x, then

$$\phi(x) = \prod_i p_i^{\alpha_i-1}(p_i - 1). \tag{2}$$

From (2) it follows that $\phi(x)$ is even and less than x if $x>2$. Therefore if $x\geqq 2$ there exists a number n such that $\phi^n(x)=2$. Then x is said to be of class n.

Dr. Shapiro observed that the smallest number in every class up to the eighth was a prime and raised the question whether the smallest number in each class was a prime. It will be shown that this is not the case.

2. The smallest number. The simplest case in which the smallest number of a class is composite is the class 32.

THEOREM: $(2^{16}+1)^2$ *is the smallest number in class* 32.*

Proof: Let $N=(2^{16}+1)^2$. From (2) and the fact that $2^{16}+1$ is a prime it follows that

$$\begin{aligned}\phi(N) &= 2^{16}(2^{16}+1),\\ \phi^k(N) &= 2^{33-k} \quad \text{for} \quad 2 \leqq k \leqq 32,\\ \phi^{32}(N) &= 2.\end{aligned}$$

Therefore N is of class 32.

Let P be a number of class 32 smaller than N. Let $\phi^m(P)=2^{\alpha_m}A_m=2^{\alpha_m}\Pi_j p_{jm}^{\beta_{jm}}$ where A_m is an odd integer, and the p_{jm} are the distinct prime factors of A_m. Let $r_m=A_m/\phi(A_m)$ if $m\geqq 1$ and $r_0=P/\phi(P)$. $r_m\geqq 1$ for all values of m.

If $1\leqq m\leqq 31$, $\alpha_m\geqq 1$ since $\phi(x)$ is even for $x>2$, and

$$\phi^{m+1}(P) = \phi[\phi^m(P)] = \phi(2^{\alpha_m}A_m) = 2^{\alpha_m-1}\phi(A_m),$$
$$\frac{\phi^m(P)}{\phi^{m+1}(P)} = \frac{2A_m}{\phi(A_m)} = 2r_m.$$

Furthermore

$$\frac{\phi^0(P)}{\phi^1(P)} = \frac{P}{\phi(P)} = r_0.$$

Now

$$2^{32} + 2^{17} = N - 1 \geqq P = \phi^{32}(P)\prod_{i=0}^{31} \frac{\phi^i(P)}{\phi^{i+1}(P)} = 2^{32}\prod_{i=0}^{31} r_i.$$

Therefore

* The smallest numbers in classes 11, 12, and 24 are $(2^4+1)\cdot 137$, $(2^4+1)(2^8+1)$, and $(2^8+1)(2^{16}+1)$ respectively. Furthermore the numbers $(2^{16}+1)^k$ where $1\leqq k\leqq 15$ are the smallest numbers in their respective classes.

$$1 + 2^{-15} \geqq \prod_{i=0}^{31} r_i, \tag{3}$$

and

$$1 + 2^{-15} \geqq r_m \quad \text{if} \quad 0 \leqq m \leqq 31. \tag{4}$$

From (2) it follows that

$$r_m = \frac{A_m}{\phi(A_m)} = \prod_j \frac{p_{jm}}{p_{jm} - 1} \geqq \frac{p_{jm}}{p_{jm} - 1}.$$

Therefore

$$1 + 2^{-15} \geqq \frac{p_{jm}}{p_{jm} - 1},$$

$$p_{jm} \geqq 2^{15} + 1. \tag{5}$$

Let a be the largest number for which $\phi^a(P)$ is not a power of 2. $0<a<32$ since $\phi^{32}(P)=2$ by our assumption that P is of class 32, and $\phi^0(P)=P$ cannot be a power of 2 since the only power of 2 in class 32 is 2^{33} which is greater than N. Then

$$\phi^{a+1}(P) = \phi[\phi^a(P)] = 2^{\alpha_a - 1} \prod_j p_{ja}^{\beta_{ja}-1}(p_{ja} - 1)$$

which must be a power of 2. Therefore $\beta_{ja}=1$ for all values of j, and all the p_{ja} are of the form 2^n+1. But, since $2^{32}+1$ is known to be composite, the only prime of this form less than N that satisfies (5) is the Fermat prime $2^{16}+1$. Therefore $A_a=2^{16}+1$.

The only number of the form $2^s(2^{16}+1)$ in class 32 is $2^{17}(2^{16}+1)>N$, and therefore $a\neq 0$.

Now $\phi^a(P)=\phi[\phi^{a-1}(P)]=2^{\alpha_{a-1}-1}\prod_j p_{ja-1}^{\beta_{ja-1}-1}(p_{ja-1}-1)=2^{\alpha_a}(2^{16}+1)$. It is impossible for $\phi^a(P)$ to contain $(2^{16}+1)^2=N$ as a factor since in that case $P\geqq N$. Therefore one of the p_{ja-1} is a prime of the form $2^s(2^{16}+1)+1$ where $1\leqq s\leqq 16$. Let $R_s=2^s(2^{16}+1)+1$. Then

R_s is divisible by 3 if $s\equiv 0 \pmod 2$,
R_s is divisible by 5 if $s\equiv 1 \pmod 4$,
R_s is divisible by 17 if $s\equiv 3 \pmod 8$,
R_s is divisible by 257 if $s\equiv 7 \pmod{16}$,

and R_{15} is divisible by 23. Therefore R_s is not a prime for $1\leqq s\leqq 16$. It follows that N is the smallest number in class 32.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The department of Discussions and Notes is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A NOTE ON DETERMINANTS AND HADAMARD'S INEQUALITY

RICHARD BELLMAN, University of Wisconsin

It is easy to verify directly the familiar theorem that the product of two numbers, each the sum of two squares, can be represented as the sum of two squares, namely,

$$(1)\qquad (a_1^2 + a_2^2)(b_1^2 + b_2^2) = (a_1b_2 - a_2b_1)^2 + (a_1b_1 + a_2b_2)^2.$$

This has an interesting interpretation. Let (a_1, a_2), (b_1, b_2) be two points A, B, and O the origin. Then $(-a_2, a_1)$ is a point C, and OC is perpendicular to OA. The area of triangle OAB is $\frac{1}{2}OA\cdot OB \sin\theta$, where θ is angle (OA, OB), or $\frac{1}{2}\sqrt{(a_1^2+a_2^2)(b_1^2+b_2^2)}\sin\theta$; the area of triangle OBC is $\frac{1}{2}OB\cdot OC\cos\theta$ or $\frac{1}{2}\sqrt{(a_1^2+a_2^2)(b_1^2+b_2^2)}\cos\theta$. Adding the squares of their areas and expressing these areas by determinants we have

$$(2)\qquad (a_1^2 + a_2^2)(b_1^2 + b_2^2) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & b_2 \\ -a_2 & a_1 \end{vmatrix}^2.$$

We thus obtain equality (1) and in addition we have the case $n=2$ of Hadamard's inequality

$$D^2 \leqq \prod_{j=1}^{n}\left(\sum_{i=1}^{n} a_{ij}^2\right),$$

where D is the determinant $|a_{ij}|$, $i, j=1, \cdots, n$.

We wish to show that this process can be extended to higher spaces and similar equalities can be obtained, giving as a by-product Hadamard's inequality for the general case. For simplicity we shall restrict ourselves to the case $n=3$.

Consider a tetrahedron $OA_1B_1C_1$, O the origin. Denote the coordinates of A_1, B_1, C_1 by (a_1^1, a_2^1, a_3^1), (b_1^1, b_2^1, b_3^1), and (c_1^1, c_2^1, c_3^1), respectively, and the lengths OA_1, OB_1, OC_1 by a, b, c, respectively. The volume V_1 of the tetrahedron is $\frac{1}{6}h_1ab\sin\theta_1$, where h_1 is the altitude from C_1 to the plane OA_1B_1 and θ_1 is the angle (OA_1, OB_1) in the triangle OA_1B_1. In determinant form the volume

$$V_1 = \pm\frac{1}{6}\begin{vmatrix} a_1^1 & a_2^1 & a_3^1 \\ b_1^1 & b_2^1 & b_3^1 \\ c_1^1 & c_2^1 & c_3^1 \end{vmatrix}.$$

Now consider two other points $C_2(c_1^2, c_2^2, c_3^2)$ and $C_3(c_1^3, c_2^3, c_2^3)$ such that $OC_1 = OC_2 = OC_3$ and OC_1, OC_2, OC_3 are mutually perpendicular. Let V_2 be the volume of the tetrahedron $OA_1B_1C_2$, V_3 the volume of $OA_1B_1C_3$, h_2 the altitude from C_2, and h_3 the altitude from C_3. It is easy to show that

$$h_1^2 + h_2^2 + h_3^2 = OC_1^2 = {c_1^1}^2 + {c_2^1}^2 + {c_3^1}^2 = c^2,$$

and thus

$$V_1^2 + V_2^2 + V_3^2 = \frac{1}{36}(h_1^2 + h_2^2 + h_3^2)a^2b^2 \sin^2\theta_1 = \frac{1}{36}a^2b^2c^2 \sin^2\theta_1.$$

Also

$$V_1^2 + V_2^2 + V_3^2 = \frac{1}{36}\begin{vmatrix} a_1^1 & a_2^1 & a_3^1 \\ b_1^1 & b_2^1 & b_3^1 \\ c_1^1 & c_2^1 & c_3^1 \end{vmatrix}^2 + \frac{1}{36}\begin{vmatrix} a_1^1 & a_2^1 & a_3^1 \\ b_1^1 & b_2^1 & b_3^1 \\ c_1^2 & c_2^2 & c_3^2 \end{vmatrix}^2 + \frac{1}{36}\begin{vmatrix} a_1^1 & a_2^1 & a_3^1 \\ b_1^1 & b_2^1 & b_3^1 \\ c_1^3 & c_2^3 & c_3^3 \end{vmatrix}^2.$$

This is sufficient to show Hadamard's inequality for three dimensions, but we are interested in an equality of the form (2).

Now for any position C_1, C_2, C_3 of C, let B vary so that $OB_1 = OB_2 = OB_3$ and OB_1, OB_2, OB_3 are mutually perpendicular, and denote the coordinates of B_2 and B_3 by (b_1^2, b_2^2, b_3^2) and (b_1^3, b_2^3, b_3^3), respectively.

Considering the nine possible volumes resulting from both B and C varying in the prescribed manner, with A_1 fixed, we see that

$$\sum_1^9 V_i^2 = \frac{1}{36}a^2b^2c^2(\sin^2\theta_1 + \sin^2\theta_2 + \sin^2\theta_3),$$

where θ_k is the angle (OA_1, OB_k), $k = 1, 2, 3$. But the lines OB_1, OB_2, OB_3 constitute a set of axes and the θ_k are direction angles; thus

$$\sin^2\theta_1 + \sin^2\theta_2 + \sin^2\theta_3 = 2.$$

Therefore we obtain

$$\sum_{i,j}\begin{vmatrix} a_1^1 & a_2^1 & a_3^1 \\ b_1^i & b_2^i & b_3^i \\ c_1^j & c_2^j & c_3^j \end{vmatrix}^2 = 2a^2b^2c^2.$$

We can remove the asymmetry with respect to A_1 by forming the corresponding identities with B and C fixed and then adding the three resulting equations, obtaining

$$\sum_{i,j,k}\begin{vmatrix} a_1^i & a_2^i & a_3^i \\ b_1^j & b_2^j & b_3^j \\ c_1^k & c_2^k & c_3^k \end{vmatrix}^2 = 6a^2b^2c^2.$$

This process can clearly be continued to higher dimensions.

THE MORLEY TRIANGLE: A NEW GEOMETRIC PROOF

H. D. GROSSMAN, New York City

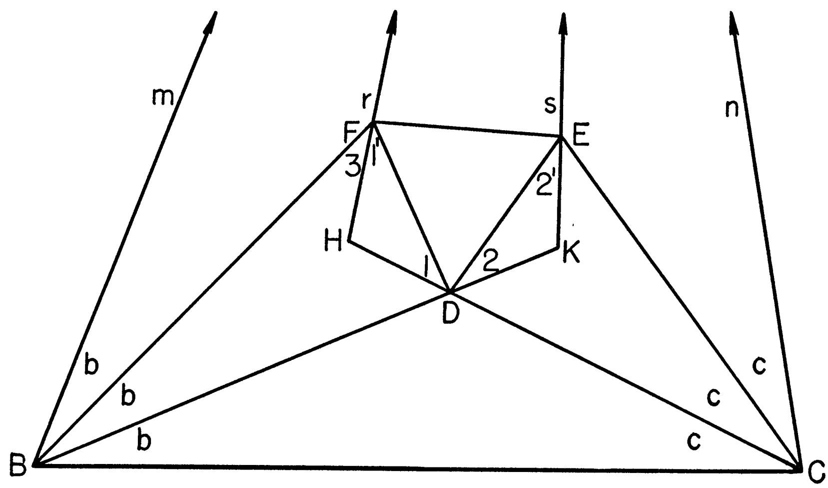
THEOREM. *If the angles of a triangle be trisected, the intersections of the pair of trisectors adjacent to each side determine an equilateral triangle.*

Proof. Let the triangle have base BC and angles $3a$, $3b$, $3c$. Let BDK, BF, CDH, CE be angle-trisectors. E is determined by making angle $CDE=60°+b$ and F by making angle $BDF=60°+c$. Then

$$\text{angle } EDF = 360° - (180° - b - c) - (60° + b) - (60° + c) = 60°.$$

Also

$$\text{angle } BFD = 180° - (60° + b + c) = 60° + a = \text{angle } CED.$$



Since D is equidistant from BF and CE, $DF=DE$ and triangle DEF is equilateral. Angle $1=(60°+c)-(b+c)=60°-b$. Similarly angle $2=60°-c$. Through F draw line r making angle $1'=$angle 1. Through E draw line s making angle $2'=$angle 2. Angle $3=(60°+a)-(60°-b)=a+b$. Angle $mr=(a+b)-b=a$. Similarly angle $sn=(a+c)-c=a$. Angle $mn=180°-3b-3c=3a$.

It remains only to prove that the lines m, n, r, and s converge to a point. The line KF joins the vertices of two isosceles triangles and therefore bisects angle K. Then in triangle $mBKs$ the bisector of angle ms passes through F and being parallel to r, coincides with it. Similarly in triangle $rHCn$ the bisector of angle rn passes through E and being parallel to s, coincides with it.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1942-43

Kappa Mu Epsilon, Louisiana State University

We have been rather active this semester. The following topics were presented at our meetings:

Games of Chance, by K. L. Nielsen.

A proof that "conic sections" are conic sections, by F. A. Rickey.

Beta and Gamma functions, by Sam Cunningham.

The pendulum problem, by Carson Jeffries.

The brachistochrone problem in the calculus of variations, by Joseph Pryor.

Sam Cunningham won the Senior Award this year, and Robert Anding won the Freshman Honors Award. Sixty-four members have been initiated this year. Officers for next year are: President, Leland Morgan; Vice-President, Nina Nichols; Secretary, Gloria McCarthy; Treasurer, Julia Weil; Historian, William Wray; Faculty Adviser, Marelena White; Corresponding Secretary, Houston T. Karnes.

Kappa Mu Epsilon, Central Michigan College

The *Pythagorean Club* of Central Michigan College of Education became the *Michigan Beta Chapter of Kappa Mu Epsilon* on April 25, 1942. Regular business meetings were held once a month. The topic at one of the meetings was *Mathematics and the Navy*, by Mr. L. H. Serier. Several other meetings were devoted to the solving of mathematical puzzles. Two social gatherings were held in the College Den, at which time new members were initiated. The annual picnic was held June 5 at the city park. Officers for 1942–43 and 1943–44 were elected as follows: President, Jennie Master (42–43), Paul Brown (43–44); Vice-President, Paul Brown (42–43), Zelda Montague (43–44); Recording Secretary, Eleanore Mucynski (42–43), Robert Mark (43–44); Corresponding Secretary, Nicoline Bye (42–44); Treasurer, Jack Fiebing (42–43), Kenneth Miller (43–44). Dr. C. C. Richtmeyer acted as Faculty Adviser.

Pi Mu Epsilon, Lehigh University

Program meetings were held once a month throughout the year, and one open lecture by a speaker from outside the University was sponsored. This lecture was entitled *The motion of the spinning projectile*, by E. J. McShane, from the Aberdeen Proving Grounds and the University of Virginia.

There was a final initiation banquet at which the newly elected president,

Mr. Wright, presided. Officers for the past year and officers elect are as follows: President, Maynard Arsove (42–43), Robert Wright, Jr. (43–44); Secretary, Stanley Caplan (42–43), Ralph Evans (43–44); Treasurer, C. S. Bennett (42–43), Robert W. Logan (43–44); Faculty Adviser, Tomlinson Fort.

Mathematics Club, Wellesley College

The Mathematics Club of Wellesley College held three meetings during the year. At the first meeting, held in October, talks were given by those students who had held mathematical jobs during the summer. The subjects of these talks were as follows:

The application of mathematics to astronomy, by Jean North.

A draftswoman in a locomotive designing plant, by Elizabeth Bird.

A mathematical job at the Quonset Naval Base, by Elizabeth Weibel.

Testing bullets dimensionally, by Elizabeth Ann Wilson.

The topics presented at the November and April meetings were:

Probability and its applications, by Professor E. B. Mode of Boston University.

The algebra of logic, by Professor G. D. Birkhoff of Harvard, who showed how propositions of logic could be turned into algebraic expressions which are maneuverable according to certain rules.

A dinner was given before Professor Mode's talk. The third meeting, held at Wellesley in April, was a joint meeting with other clubs of the Greater Boston Intercollegiate Mathematics Clubs Association. The colleges represented were Harvard University, Regis College, Boston University, and Boston College. The officers of the club for 1942–43 were: President, June Nesbitt; Vice-President, Elizabeth Ann Wilson; Treasurer, Martha Adams; Junior Executive, Elizabeth Bird; Sophomore Executive, Ann Pettingell; Secretary, Phyllis Fox; Faculty Adviser, Professor L. P. Copeland.

Mathematics Club, Kansas State College

The Mathematics Club of Kansas State College is organized each year by *Pi Mu Epsilon*. After that the Club carries on with the advice of a faculty member who acts as program chairman. Meetings are held once a month. During the past year, students in the Civilian Pilot Training Program gave several talks on air navigation. Other meetings were taken up with talks by undergraduates in mathematics and engineering. Most of the talks came under one of the two general headings:

Applications of calculus, analytic geometry, and higher algebra to problems in engineering.

Solution of problems posted on the mathematics bulletin board.

The Club's officers this year were: President, Virginia Bell; Vice-President, Jean Burnette; Secretary, Aileen Hostinsky; Treasurer, Ann Dueser; Program Chairman, Dr. D. T. Sigley. (Report submitted by H. C. Fryer, Director, *Pi Mu Epsilon.*)

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

Concise Spherical Trigonometry with Application and Reviews of Solid Geometry and Plane Trigonometry. By J. R. Hammond. Boston, Houghton Mifflin Co., 1943. 13+256 pages. $2.00.

The present crisis has produced a large number of texts on spherical trigonometry, ranging from brief syllabi to extensive treatises. The volume under review is original in many respects. The aim of the author is two fold: to acquaint the reader with a dependable guide, with a minimum of trustworthy preparation. The subject is introduced through Solid Geometry, no previous knowledge of which is presupposed. Theorems involving incidence properties are given, followed by considerable detail on spherical triangles and polar relations on the sphere. Knowledge of logarithms and of elementary plane trigonometry are presupposed.

The entire discussion of spherical trigonometry is based on one method, that is, of dividing every triangle into one or more right triangles. Napier's Rules are treated in great detail; only these elementary formulas are employed in each case.

A second feature is a prescribed scheme of arrangement of the details of computation, to be followed in detail in each case. The text proper is later amplified by a readable explanation of the means employed to provide the data for each problem, including a generous fund of celestial and terrestrial applications, mostly utilitarian, based on the author's extensive experience with technical problems. Some of these demand considerable general preparedness, a fairly high grade of wholesome interested maturity. Two chapters are added to provide more discussion of the ambiguous cases, first from a geometric point of view then by a more analytic treatment of the formulas, now derived. This part follows more closely the traditional treatment, including the Law of Sines, Law of Cosines, and their various corollaries. This Alternative Method is the only case in which any formula more complicated than those included in Napier's Rules is employed.

An appendix describes the sextant, the compass, the chronometer, transit, *etc.*, and a fairly full discussion of various kinds of time.

The style of the book is uniformly consistent; an effort is made to acquaint the reader with the elements of spherical trigonometry. It demands a receptive attitude on the part of the reader, and assumes a reasonable effort is being made to achieve that result.

VIRGIL SNYDER

Elementary Mathematics for the Machine Trades. By J. J. Weir. New York and London, McGraw-Hill Book Co., 1943. 8+193 pages. $1.60.

This is a text-book covering simple arithmetic, trigonometry and miscellaneous subjects useful to machine-shop workers. A machinist, however, would find most of the material too elementary, but the thousands of unskilled men and women who have been drawn into the war plants should find help in the simplified presentation of everyday problems provided they have an instructor's assistance. For the man who needs the simple arithmetic of the first part of the book, the range is too great. Probably with the aid of a good teacher, the pupil may be drawn from the fog of illiteracy of arithmetic, through decimals, simple algebra, geometry, pulleys, belts, verniers and gear trains to the use of trigonometric tables and functions. The treatment of the micrometer, height gage, vernier, caliper and graduated dials is very well done and amply illustrated. The books may be particularly useful to those who have covered all these subjects some time in the past, and now, when the prospect of practical use for bits of information presents itself, have motivation to relearn.

SAMUEL LERNER

College Mathematics. By W. L. Hart, W. A. Wilson and J. I. Tracey. Boston, Heath, 1943. 7+875 pp. $4.00.

This text, designed to meet the curricular requirements of the college programs of the army and navy below the level of calculus, is in three parts: 1. algebra; 2. trigonometry; 3. analytic geometry. Part 1 is a selection made by W. L. Hart from his *College Algebra, Revised Edition* giving a review of elementary algebra and the material in advanced algebra most useful in calculus. Part 2 is a selection with slight revision from *Trigonometry, Solid Geometry and Spherical Trigonometry* by W. W. Hart and W. L. Hart which omits solid geometry but includes tables. Part 3 is a reprinting of W. A. Wilson and J. I. Tracey: *Analytic Geometry*; *Alternate Edition.*

The requirements of the armed forces are here efficiently met in a single volume. Because of the great number of topics which must be touched, especially in the navy V-12 courses, a further selection and compression will probably have to be made by those who use this book in that program. It occurs to the reviewer that in spherical trigonometry, which is required by the navy but for which very little time is allowed in the standard curriculum, it might have been wiser to treat the oblique triangle by breaking it up into two right triangles (except when three sides or angles are given where, if the case is treated at all, the law of cosines may be used). Modern navigation requires exclusively the solution of the case: given two sides and the included angle. Several of the most popular methods in actual use solve it by means of two right triangles (*Hydrographic Office Publications* 208, by Dreisenstock, and 211, by Ageton; Weems, *Line of Position Book*; Hughes, *Sea and Air Navigation* (British)).

W. W. FLEXNER

Students' Handbook of Elementary Physics. By R. B. Lindsay. New York, The Dryden Press, 1943. 15+382 pages. $2.25.

R. B. Lindsay's "Handbook of Elementary Physics" is of unique design. It consists of three main parts, first a brief text of general elementary physics (about 100 pages), then a dictionary of physical terms and finally a collection of useful physical formulas, physical constants and mathematical tables. The text part is unusual in that it contains no mathematical formulas whatsoever. These are all relegated to the last two sections. The author feels that the student will get a better initial grasp of the principles and laws of physics if he is encouraged to state them in everyday English, rather than the abbreviated symbolism of mathematics. The idea of requiring the student to know how to state the laws of physics in his native tongue is fundamentally sound. However, the important thing is for him to be able to give a clear and accurate statement of the conditions under which the laws are valid and the operational meaning of the terms involved rather than just a long hand quotation of the mathematical formulas. This is not accomplished by simply omitting the symbolic mathematical formulas.

The style of writing in the text is clear and concise and will probably appeal to the student. Inasmuch as the handbook will usually be used to supplement a regular textbook, its brevity is not an objection. An interesting feature of the text is the frequent use of simple experiments which the student can perform himself to illustrate the points being discussed.

The dictionary part of the handbook contains the definitions of about 1000 physical terms together with appropriate diagrams and defining relations. The definitions might not all receive a perfect mark if found on an examination paper, but they do provide a useful source of first aid to the student who has difficulty in extracting bits of information from long winded texts. The table of physical formulas will be very popular with the students, but its pedagogical value is doubtful. The author warns that the formulas are not to be used without being completely understood. Yet in many cases it is difficult to see how a student can have a complete grasp of the meaning of the particular formula without being able to write it down instantly or derive it for himself in a few seconds.

The mathematical background needed by the student in using the handbook is negligible. To understand the text he need only know what "proportional to" and "average" means. If he is ambitious enough to attempt the problems he should know a little algebra. He is encouraged by the author to use the powers of 10 method for obtaining rough numerical answers. The formulas in the last two sections of the handbook contain trigonometric and exponential functions only occasionally, and integrals and derivatives only in one or two instances where they cannot be avoided.

H. HURWITZ, JR.

Learning to Navigate. By P. W. H. Weems and W. C. Eberle. The Pitman Publishing Co., 1943. 135 pp., 73 Figs. $2.00.

This is a second edition enlarged by more than sixty pages and forty illustrations.

The war has, of course, created an urgent demand for thousands of sea and air nagivators, many of whom will have little time for preparation. This book is intended to provide a foundation of the essentials of navigation for both sea and air.

After a bit of history of an early voyage, an account of the development of the compass and some elementary definitions, there follow a discussion of charts and how to use them and chapters on the magnetic and gyro compass, piloting and dead reckoning, celestial navigation, time, almanacs, sextants, line of position, log-book and air navigation.

This last is a new chapter. It discusses the various types of aeronautical charts, contact flying, dead reckoning, the use of mechanical computers, the graphical solution of wind-drift problems and celestial air navigation.

Radio navigation and meteorology are considered beyond the scope of the book.

Throughout the text only the shortest and simplest processes are explained. Dead reckoning is carried out directly on the chart instead of by logarithms or traverse tables. The line of position is derived by H.O. 214 or Weems' *Line of Position Book.* The latter uses cosecant and secants, but, to avoid frightening the beginner with trigonometrical terms, these are called simply A and B.

For the quickest determination of latitude and longitude in the air the use of Weems' *Star Altitude Curves* is recommended. For all other methods either the Nautical Almanac or Air Almanac is required. The authors advise the use of the Air Almanac for both sea and air.

In the process of condensation from larger works to the limited scope of this book there have resulted some inaccuracies of statement and inadequacies of explanation. However, these are not serious and the beginner may be assured that all the fundamental principles are sound.

FREDERICK SLOCUM

1000 Pre-flight Problems. By W. H. Thompson and M. L. Aiken. New York, Harper and Brothers, 1943. 15+160 pages. $0.88, paper; $1.20, cloth.

This pamphlet provides systematic drill in the solution of numerical problems connected with fundamental concepts in physics and mathematics that are directly or indirectly used in aviation. The scheme is to give a statement explaining briefly what the problem is, then follows the formula concerned, with the unknown appearing explicitly. Functions may contain radicals and simple trigonometric ratios. The derivation of the formulas are not given, nor are any answers to the problems. Readers are encouraged to use logarithms and the slide rule, but neither is required.

VIRGIL SNYDER

Solution of the Trisection Problem. By E. Vennigerholz. Moscow, Idaho, E. Vennigerholz, 1943. 16 pages. $1.25.

This little pamphlet offers not only what is claimed to be a solution of the problem of trisection, but also of various related problems. After a few pages of remarks, two constructions for the trisectors of a given angle are presented.

(a) Given two lines intersecting at O, at any given angle. With O as center and any given radius k describe a circle, meeting the sides of the given angle in AA' and BB', and the internal bisector of the angle in CC'. Draw the lines $A'C$, $B'C$, and the circle $O(3k)$. Call S, S' the points of intersection of the lines and the circle on the same side of O as A and B. Draw OS and OS'. These are the alleged trisectors of the given angle.

(b) In the same construction as in (a), draw OD, bisecting the angle AOC. Through C draw a line parallel to OD, meeting the circle $O(3k)$ in S, defined above. Similarly for S'.

No proofs are given. Every construction is incorrect, even when the given angle is a right angle and the true positions of the trisectors are easily found.

The message contained in this pamphlet is not only valueless, but is actually harmful.

VIRGIL SNYDER

NEW BOOKS RECEIVED

Mathematics of Flight. By J. Naidlich. New York and London, McGraw-Hill Book Co., 1943. 10+409 pages. $2.00.

1000 Pre-flight Problems. By W. H. Thompson and M. L. Aiken. New York, Harper and Brothers, 1943. 15+169 pages. $0.88.

Solution of the Trisection Problem. By E. Vennigerholz. Moscow, Idaho. E. Vennigerholz, 1943. 16 pages. $1.25.

The Problem of Moments. By J. A. Shohat and J. D. Tamarkin. (Mathematical Surveys, No. 1.) New York, American Mathematical Society, 1943. 14+140 pages. $2.50.

Gallileo. By C. de Losada y Puga. Lima, Universidad Catolica del Peru. 54 pages.

Meromorphic Functions and Analytic Curves. By H. Weyl and F. J. Weyl. (Annals of Mathematics Studies, No. 13.) Princeton, University Press; London, Humphrey Milford and Oxford University Press, 1943. 9+269 pages. $3.50.

Plane Trigonometry with Tables. By H. D. Ballou and F. H. Steen. Boston, Ginn and Co., 1943. 6+123+84 pages. $2.50.

Elements of Statistical Method. By A. E. Waugh. Second edition. New York and London, McGraw-Hill Book Co., 1943. 21+532 pages. $4.00.

Commercial Algebra. By H. E. Stelson and H. F. Rogers. New York, The Macmillan Co., 1943. 11+283 pages. $2.50.

The Theory of Rings. By N. Jacobson. (Mathematical Surveys, No. 2.) New York. American Mathematical Society, 1943. 6+150 pages. $2.00.

SOLUTIONS

The Euler Lines of Three Triangles

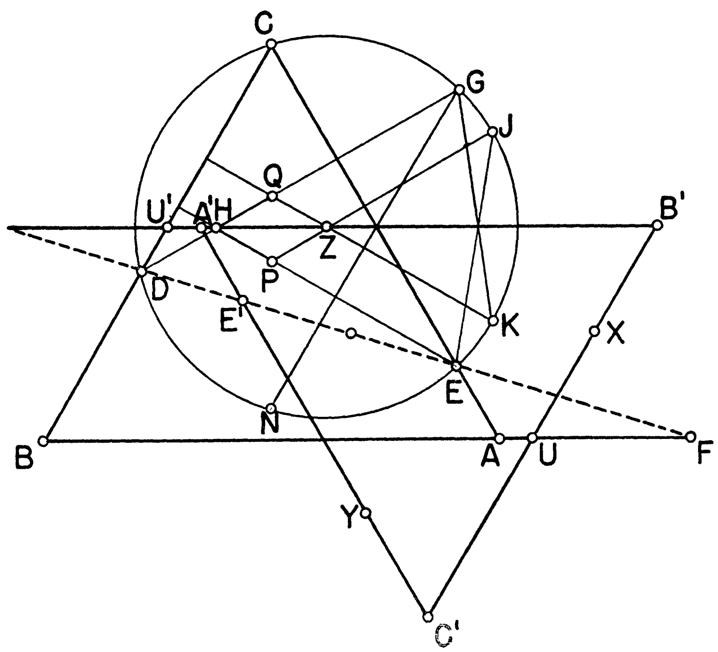
E 547 [1942, 683]. *Proposed by V. Thébault, San Sebastián, Spain*

A diameter d of the circumcircle of an equilateral triangle ABC cuts the sides BC, CA, AB in points D, E, F. Prove that the Euler lines of the three triangles AEF, BFD, CDE form a triangle symmetrically equal to ABC, the center of symmetry lying on d.

Solution by W. B. Clarke, San Jose, California.

LEMMA. *If a triangle CDE has angle* $C=60°$ *or* $120°$, *its Euler line forms an equilateral triangle with the lines CD and CE.*

Proof of Lemma. Let Z and H be the circumcenter and orthocenter of the special triangle CDE. Let DH meet the circumcircle again in G, and let the line through G parallel to CD meet the circumcircle again in N. Let J and K be the midpoints of the arcs EC and GN. Then the lines DHG, JZ (perpendicular to CE) and EH, KZ (perpendicular to CD) form a parallelogram $HPZQ$, with P on JZ and Q on KZ.



Since both CG and JK are arcs of $60°$, we have $CJ=GK=KN$, whence

$$\angle\, CEJ = \angle\, KGN.$$

But $\angle PEC = 30° = \angle QGN$. Hence, by addition or subtraction,

$$\angle PEJ = \angle QGK.$$

Also $\angle EPJ = 60° = \angle GQK$, and $EJ = JC = GK$. Hence the triangles EJP and GKQ are congruent, $JP = KQ$, $ZP = ZQ$, the parallelogram $HPZQ$ is a rhombus, and its diagonal HZ makes an equilateral triangle with CD and CE, as desired.

Proof of Theorem. By the lemma, the Euler lines of AEF, BFD, CDE form a triangle $A'B'C'$ whose sides are the parallels to BC, CA, AB through the respective circumcenters X, Y, Z. Let $B'C'$ meet AB in U, and let BC meet $A'B'$ in U'. Then we can compute

$$A'U' = AF^2/(AB + 3AF) = AU,$$

whence $A'B' = AB$, and the triangles ABC, $A'B'C'$ are not merely similar but congruent. Moreover, if DEF meets $A'C'$ in E', we find $A'E' = AE$. It follows that the center of symmetry is the midpoint of EE'.

Remarks. The triangle XYZ is equilateral, and its circumcircle passes through the center of ABC. More generally, it can be proved that the triangles ABC, $A'B'C'$ are congruent even if they are not equilateral, and even if the transversal d does not pass through the circumcenter of ABC; but in the latter case the center of symmetry will no longer lie on d.

Parabolic Sections of a Degenerate Quadric

E 549 [1942, 683]. *Proposed by L. M. Kelly, U. S. Coast Guard Academy*

The face planes of a proper tetrahedron intersect a circumscribed quadric cone in four parabolas. What conditions are thus imposed on the cone?

Solution by the Proposer. The quadric cone under consideration cannot have an accessible vertex, but must reduce to a parabolic cylinder. For, let its equation, referred to the given inscribed tetrahedron, be

$$a_{12}x_1x_2 + a_{13}x_1x_3 + a_{14}x_1x_4 + a_{23}x_2x_3 + a_{24}x_2x_4 + a_{34}x_3x_4 = 0.$$

The condition for degeneracy is

$$\det |a_{ij}|_4 = 0,$$

where $a_{ji} = a_{ij}$ and $a_{ii} = 0$. If the plane at infinity has the equation

$$a_{01}x_1 + a_{02}x_2 + a_{03}x_3 + a_{04}x_4 = 0,$$

the parabolic sections require the vanishing of the remaining fourth-order principal minors of $\det |a_{ij}|_5$. But it has been proved by V. W. Parker (Bull. Amer. Math. Soc., vol. 38, 1932, pp. 259–262) that if the real symmetric determinant $D = \det |a_{ij}|_5$ with $a_{ii} = 0$ has all five of its fourth-order principal minors zero, then $D = 0$. It follows that our quadric touches the plane at infinity and hence is a parabolic cylinder.

Cylinder and Helicoid

E 551 [1943, 59]. *Proposed by J. H. Butchart, Grinnell College*

Show that the arcs common to a circular cylinder and a right helicoid whose axis lies on the cylinder are pieces of a circular helix.

Solution by C. E. Springer, University of Oklahoma. On a common arc of the right helicoid

$$x = u \cos v, \qquad y = u \sin v, \qquad z = av$$

and the cylinder $x^2+y^2=2rx$, we have $u=2r \cos v$. Hence the common arc has the parametric equations

$$x = 2r \cos^2 v, \qquad y = r \sin 2v, \qquad z = av,$$

which by a change of origin become

$$x = r \cos 2v, \qquad y = r \sin 2v, \qquad z = av.$$

Also solved by Howard Eves, Jerzy Szmojsz, and the proposer.

Squares with Repeated Digits

E 552 [1943, 59]. *Proposed by V. Thébault, Sàn Sebastiân, Spain*

In certain scales of notation, a number of the form $aabb$ can be the square of a number of the form cc, where c is a multiple of b. Show that (1) a and b are relatively prime; (2) b and $c^2/b-a$ are perfect squares.

Solution by Free Jamison, U. S. Navy Air Navigation School. From the multiplication worked out on the right, with radix r, we see that

$$\begin{array}{cccc} & & c & c\\ & & c & c\\ \hline & & q & b\\ & q & b & \\ & q & b & \\ q & b & & \\ \hline a & a & b & b\end{array}$$

$$c^2 = qr + b, \qquad r = q + b, \qquad q = a - 1.$$

Thus

$$c^2 = q^2 + (q + 1)b = (a - 1)^2 + ab.$$

Hence b, which was given to be a divisor of c, is also a divisor of $(a-1)^2$. This makes a and b relatively prime. Suppose that $c=mb$ and $(a-1)^2=nb$; then

$$m^2b = c^2/b = n + a.$$

Therefore n and b are relatively prime; and since their product is a perfect square, each is a perfect square. So also is

$$(a - 1)^2/b = (c^2 - ab)/b = c^2/b - a.$$

In the simplest instances, we have $a=r-3$, $b=4$, $c=r-2$. It seems that the smallest values with $c>b>4$ are

$$a = 26391, \qquad b = 16900, \qquad c = 33800, \qquad r = 43290.$$

Also solved by D. H. Browne and E. P. Starke, though the former was not aware that b could be greater than 4.

Cospherical Circles

E 553 [1943, 59]. *Proposed by N. A. Court, University of Oklahoma.*

If two of the four circles of intersection of two spheres with two planes are cospherical, prove that the remaining two circles are likewise cospherical.

Solution by E. P. Starke, Rutgers University. Let $S_1=0$, $S_2=0$, $P_1=0$, $P_2=0$ be the equations of the two spheres and the two planes, and assume that in S_1 and S_2 the coefficient of $x^2+y^2+z^2$ is unity. Then a value of k exists for which $S_1+kP_1=0$ represents any sphere through the circle determined by $S_1=0$ and $P_1=0$. A similar remark applies to $S_2+k'P_2=0$. By hypothesis, there exist values of k and k' such that

$$S_1 + kP_1 \equiv S_2 + k'P_2$$

(since a common sphere passes through the two circles). But this same identity in the form

$$S_1 - k'P_2 \equiv S_2 - kP_1$$

expresses the fact that a certain sphere passes through the other two circles, $S_1=P_2=0$ and $S_2=P_1=0$.

Also solved by C. E. Springer.

The proposer remarks that the two planes may be inverted into two further spheres, with the following result. If it is possible, in one way, to group four given spheres into two pairs so that the circle of intersection of the first pair (real or imaginary) and the circle of intersection of the second pair are cospherical, then the grouping can be done in two other ways. Moreover, the centers of the four spheres are coplanar, and the four spheres belong to the same coaxal net.

Cutting the Cheese

E 554 [1943, 59]. *Proposed by J. L. Woodbridge, Philadelphia*

Show that n cuts can divide a cheese into as many as $(n+1)(n^2-n+6)/6$ pieces.

Solution by Free Jamison, Pittsburgh. Since n straight lines can divide a plane into $(n^2+n+2)/2$ areas, the $(n+1)$st plane can be divided by the first n planes into that number of areas. For each of these areas the $(n+1)$st plane divides a piece of cheese already formed into two, and increases the total number of pieces by $(n^2+n+2)/2$. Since $(n^3+5n+6)/6$ gives the number of pieces for $n=1$ or 2, and since

$$\frac{n^3+5n+6}{6}+\frac{n^2+n+2}{2}\equiv\frac{(n+1)^3+5(n+1)+6}{6},$$

the expression $(n^3+5n+6)/6$ holds for every n.

It is interesting to note that

(1) n points can divide a line into $1+n$ parts,

(2) n lines can divide a plane into $1+n+\binom{n}{2}$ parts,

(3) n planes can divide space into $1+n+\binom{n}{2}+\binom{n}{3}$ parts.

Also solved by D. H. Browne, W. E. Buker, and the proposer.

Editorial Note. The general formula

$$1+n+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{m},$$

for the case of an m-dimensional cheese, was obtained by L. Schläfli on page 39 of his great posthumous work, *Theorie der vielfachen Kontinuität* (Denkschriften der Schweizerischen naturforschenden Gesellschaft, vol. 38, 1901).

The Genus of a Graph

E 555 [1943, 59]. *Proposed by Howard Eves, Syracuse University*

Consider a rectangular parallelepiped of dimensions $a\times b\times c$, made up of abc unit cubes. Imagine the edges of all these unit cubes replaced by material wires, common edges sharing the same wire. Prove that the exterior surface of the resulting wire network is of genus $p=2abc+bc+ca+ab$ (*i.e.*, that it is topologically equivalent to a sphere with p handles).

I. *Solution by C. E. Springer, University of Oklahoma.* Five cuts will render one unit cube topologically equivalent to a sphere. The desired genus is then $5abc$ less the number of common faces of the unit cubes, or

$$p = 5abc - \{(a-1)bc + (b-1)ca + (c-1)ab\} = 2abc + bc + ca + ab.$$

II. *Solution by R. C. Buck, Harvard University.* The wires form a "graph" with V vertices and E edges, where

$$\begin{aligned} V &= (a+1)(b+1)(c+1) = abc + (bc+ca+ab) + (a+b+c) + 1, \\ E &= a(b+1)(c+1) + b(c+1)(a+1) + c(a+1)(b+1) \\ &= 3abc + 2(bc+ca+ab) + (a+b+c). \end{aligned}$$

If we cut each of the E edges, the resulting array is homeomorphic to V spheres. Joining $V-1$ of these to the remaining one, we obtain a single sphere. Hence the genus of the original surface is

$$p = E - (V-1) = 2abc + (bc+ca+ab).$$

Editorial Note. The principle involved in the latter solution may be expressed as follows. (Cf. Kurt Reidemeister, *Einführung in die kombinatorische Topologie*, Braunschweig, 1932, p. 106, or Dénes König, *Theorie der endlichen und unendlichen Graphen*, Leipzig, 1936, pp. 53–57.) A finite, connected graph, with V nodes and E branches, has for its "skeleton" or "scaffolding" (Gerüst) a tree with the same nodes and $V-1$ of the branches. The connectivity (Zusammenhangszahl) of the graph is defined as the number of its branches that have to be removed to make the tree, namely $E-V+1$. This is the same as the genus of the surface obtained by "blowing up" the graph (*i.e.*, replacing its edges by material wires); for the blown-up tree is homeomorphic to a sphere, and each extra branch can be regarded as a handle. The notion of the connectivity of a graph occurred simultaneously to Kirchhoff (*Annalen der Physik und Chemie*, vol. 72, pp. 498, 506) and von Staudt (*Geometrie der Lage*, Nürnberg, 1847, pp. 20–21).

The Turning Point of a Quadratic Function

E 556 [1943, 119]. *Proposed by C. W. Bruce, Wesleyan College, Macon, Georgia*

The graph of a quadratic function passes through three given points (x_i, y_i). Show that the abscissa of its maximum or minimum is

$$\begin{vmatrix} x_1^2 & y_1 & 1 \\ x_2^2 & y_2 & 1 \\ x_3^2 & y_3 & 1 \end{vmatrix} \div 2 \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Solution by Virginia Carlton, Wesleyan College, Macon, Georgia. Since the curve

$$ax^2 + bx + c = y \tag{1}$$

passes through the three given points, we have

$$ax_i^2 + bx_i + c = y_i \qquad (i = 1, 2, 3). \tag{2}$$

Eliminating a, b, c from the four equations (1) and (2), we obtain the same curve in the alternative form

$$\begin{vmatrix} x^2 & x & 1 & y \\ x_1^2 & x_1 & 1 & y_1 \\ x_2^2 & x_2 & 1 & y_2 \\ x_3^2 & x_3 & 1 & y_3 \end{vmatrix} = 0.$$

When we differentiate, the first row becomes

$$2x \quad 1 \quad 0 \quad dy/dx.$$

Setting the derivative equal to zero, and expanding, we obtain

$$2x\begin{vmatrix} x_1 & 1 & y_1 \\ x_2 & 1 & y_2 \\ x_3 & 1 & y_3 \end{vmatrix} - \begin{vmatrix} x_1^2 & 1 & y_1 \\ x_2^2 & 1 & y_2 \\ x_3^2 & 1 & y_3 \end{vmatrix} = 0,$$

and it remains to transpose the second and third columns of each determinant.

Also solved by R. K. Allen, Howard Eves, P. G. Smith, E. P. Starke, Eugene Titus, C. W. Topp, Alan Wayne, Hazel S. Wilson, and the proposer.

A Sphere Rolling on a Fixed Sphere

E 557 [1943, 120]. *Proposed by V. Thébault, San Sebastián, Spain*

A sphere (S) of constant radius rolls on a fixed sphere (O) in such a way as to pass through a fixed point A. Determine the loci of the centers of similitude of the spheres (S) and (O).

Solution by Howard Eves, Syracuse University. Let M be the internal, N the external, center of similitude, and let S and O be the centers of the spheres. Since length OM, length ON, line OA, and angle AOM are all fixed, it is evident that the loci of M and N are circles having OA as an axis. The locus of M is also, of course, on the sphere (O).

Also solved by the proposer.

Areas and Distances

E 558 [1943, 120]. *Proposed by V. V. Nákladem, Philadelphia*

Let P be any point in the plane of a triangle ABC. Show that the sum of the squares of the areas of the three triangles PBC, PCA, PAB cannot exceed

$$(PA^2 + PB^2 + PC^2)^2/16.$$

I. *Solution by J. B. Kelly, Brown University.* Let a, b, c, α, β, γ denote the lengths PA, PB, PC and angles BPC, CPA, APB. Then the sum of the squares of the three areas is given by

$$S = (b^2c^2 \sin^2 \alpha + c^2a^2 \sin^2 \beta + a^2b^2 \sin^2 \gamma)/4.$$

We maximize S, regarding a, b, c fixed, and α, β, γ variable, with the relation

$$\alpha + \beta + \gamma = 2\pi \tag{1}$$

connecting them. Applying the method of Lagrange, we have

$$\sin 2\alpha = \lambda/b^2c^2, \qquad \sin 2\beta = \lambda/c^2a^2, \qquad \sin 2\gamma = \lambda/a^2b^2. \tag{2}$$

Then, since $2 \sin^2 x = 1 - \sqrt{1 - \sin^2 2x}$, the maximum value sought is given by

$$8S = b^2c^2 + c^2a^2 + a^2b^2 - \sqrt{b^4c^4 - \lambda^2} - \sqrt{c^4a^4 - \lambda^2} - \sqrt{a^4b^4 - \lambda^2}. \tag{3}$$

From (1) we have

$$\sin 2\alpha = - \sin (2\beta + 2\gamma) = - \sin 2\beta \cos 2\gamma - \sin 2\gamma \cos 2\beta,$$

or, by virtue of (2),

$$\frac{\lambda}{b^2c^2} = -\frac{\lambda}{c^2a^2}\sqrt{1-\frac{\lambda^2}{a^4b^4}} - \frac{\lambda}{a^2b^2}\sqrt{1-\frac{\lambda^2}{c^4a^4}},$$

which reduces to

$$2\sqrt{b^4c^4-\lambda^2} = a^4 - b^4 - c^4.$$

Substitution of this and two similar expressions in (3) yields

$$16S = (a^2+b^2+c^2)^2.$$

It is evident that this value of S is a maximum. Thus the theorem is established.

II. *Solution by the Proposer.* Let P, A, B, C have Cartesian coordinates $(0, 0)$, (x_i, y_i) $(i=1, 2, 3)$, respectively. Using a different interpretation for the same coordinates, consider vectors $X=(x_1, x_2, x_3)$ and $Y=(y_1, y_2, y_3)$. Then we have

$$\begin{aligned} 4(PBC^2+PCA^2+PAB^2) &= \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}^2 + \begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix}^2 + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}^2 \\ &= |X\times Y|^2 \leqq X^2Y^2 \\ &\leqq (X^2+Y^2)^2/4 = (PA^2+PB^2+PC^2)^2/4. \end{aligned}$$

A Question from Plato

E 560 [1943, 120]. *Proposed by S. H. Gould, University of Toronto*

In the fifth book of his *Laws*, the philosopher Plato, discussing the distribution of land in a colony, seeks a number divisible by every integer from 1 through 10 and chooses 5040. Show in general that if m and n are positive integers with $n<p$, where p is the smallest prime greater than m, then $m!$ is divisible by n except when $m=3$.

Solution by Arthur Rosenthal, University of New Mexico. If $n\leqq m$ there is no question; so let us suppose that $m<n<p$. Then n is composite, say $n=n_1n_2$ (with $2\leqq n_1\leqq n_2$).

Suppose first that $n_1<n_2$. By Tschebyscheff's Theorem, there is at least one prime between m and $2m$. Hence $n<p<2m$ and

$$n_1 < n_2 \leqq n/2 < m.$$

Thus $m!$ is divisible by n_1 and n_2, and hence by n also.

If $m>3$, a refinement of Tschebyscheff's Theorem gives $m<p<2m-2$, whence

$$n \leqq 2m-4 = m(2-4/m) \leqq m\cdot m/4 = (m/2)^2.$$

Thus in the remaining case $n_1=n_2\leqq m/2$. Hence $m!$ is divisible by $n_1\cdot 2n_1$, and so also by $n_1^2=n$. Among the cases $m=1, 2, 3$, only $m=3$ gives an exception.

Also solved by D. H. Browne, Walter Penney, E. P. Starke, and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4097. *Proposed by R. Bellman, Madison, Wisconsin*

Let

$$f(x) = 1 + \sum_{n=1}^{\infty} a_{1n}x^n, \qquad [f(x)]^k = 1 + \sum_{n=1}^{\infty} a_{kn}x^n.$$

What is the necessary and sufficient condition that the following system of equations have a solution

$$x_0 + \sum_{n=1}^{N} a_{kn}x_n = u_k; \qquad k = 1, 2, \cdots, N+1; N \geqq 1.$$

4098. *Proposed by P. R. Halmos, Urbana, Illinois*

Evaluate the determinant

$$\begin{vmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ x_1^4 & x_2^4 & x_3^4 & \cdots & x_n^4 \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ x_1^{2^{n-1}} & x_2^{2^{n-1}} & x_3^{2^{n-1}} & \cdots & x_n^{2^{n-1}} \end{vmatrix}$$

4099. *Proposed by J. H. Butchart, Grinnell College*

If P is any point of a curve and Q is the corresponding point of the pedal with respect to the point O, then OQ makes the same angle with the pedal that OP makes with the curve.

4100. *Proposed by V. Thébault, San Sebastián, Spain*

Let $N=1, 2, 3, \cdots, n$ be a number in the system of base $n+1$ where N is written with the consecutive increasing digits omitting O. The product $P=N\cdot L$ is formed where $L=\alpha\beta$ is a number of two digits such that $\gamma=\alpha+\beta$ is a number less than n and such that γ and n have δ as the greatest common divisor. Show that

$$P = N\cdot L = ab\cdots pqab\cdots pq\cdots ab\cdots p(n-\gamma)q,$$

the $\delta-1$ periods $ab, \cdots, pq$ being formed by n/δ distinct digits and the number of $(n/\delta)+1$ digits on the right contains the digit $(n-\gamma)$ between the digits p and q.

Dedicated to E. P. Starke.

SOLUTIONS

Differential Equation, Curvature

4037 [1942, 340]. *Proposed by Cezar Coşniţă, Focşani, Roumania.*

Integrate

$$(x^{n+1} + y^n)y' - x^n y = 0;$$

calculate and examine the radius of curvature of the integral curves at the origin.

Solution by Paul D. Thomas, U. S. Coast and Geodetic Survey, Sherburne, N. Y. Writing the given differential equation thus

$$\left(\frac{x}{y}\right)^n \frac{ydx - xdy}{y^2} - \frac{dy}{y^2} = 0,$$

the integral is seen to be

$$\frac{1}{n+1}\left(\frac{x}{y}\right)^{n+1} + \frac{1}{y} = C, \quad \text{or} \quad x^{n+1} - C(n+1)y^{n+1} + (n+1)y^n = 0. \tag{1}$$

From (1) it is seen that for $n \geqq 2$, the curves have a cusp at the origin, and the radius of curvature at this point is therefore 0. For $n=1$, the integral curves are the conics $x^2-2Cy^2+2y=0$, the radius of curvature at the origin being $+1$. For the computation write the given differential equation in the form $M\,dx+N\,dy=0$. Then the radius of curvature will be given by

$$\begin{gathered} R = \frac{(M^2+N^2)^{3/2}}{MN(M_y+N_x) - N^2M_x - M^2N_y}, \\ M = x^n y, \qquad M_x = nyx^{n-1}, \qquad M_y = x^n, \qquad N = -x^{n+1} - y^n, \\ N_x = -(n+1)x^n, \qquad N_y = -ny^{n-1}. \end{gathered} \tag{2}$$

Let $x=r\cos A$, $y=r\sin A$. With $C=0$, (1) may be written $r=-(n+1)\tan^n A \sec A$. With these and the above values of M, N, M_x, M_y, N_x, N_y (2) becomes

$$R = -\frac{\tan^{n-1} A}{n} \cdot [(n+1)^2 \cdot \tan^2 A + n^2]^{3/2}.$$

From this last with $A=0$, $n=1$, $R=-1$. For $n \geqq 2$, $R=0$.

Editorial Note. The computations may be made directly from the equation in polar coordinates, and it will be seen that for the approximate formulas needed here for the origin neighborhood the term with the arbitrary constant coefficient may be neglected. Thus for $n \geqq 2$ we have

$$r \to -(n+1)\theta^n, \quad dr/d\theta \to -n(n+1)\theta^{n-1}, \quad d^2r/d\theta^2 \to -n(n^2-1)\theta^{n-2},$$
$$R \to -n^2\theta^{n-1}.$$

Thus the integral curve is tangent to the x-axis at the origin; and, if n is also an even positive integer, there is a cusp at the origin, the two branches lying respectively in the second and third quadrants, whereas for n odd the curve passes from the third to the fourth quadrant and tangent to the x-axis at the origin.

Powers of a Point

4040 [1942, 408]. *Proposed by N. A. Court, University of Oklahoma*

Given n points by $A_1, A_2, \cdots, A_n$ in space, let G denote their centroid, k^2 the sum of the squares of the $n(n-1)/2$ segments determined by the given points, and P_{ij} the power of a given point P with respect to the sphere having for diameter the segment A_iA_j; show that

$$\sum P_{ij} = n(n-1)PG^2/2 - k^2/2n.$$

Solution by Paul D. Thomas, U. S. Coast & Geodetic Survey, Little Falls, N. Y. Consider the points A_i, A_j the midpoint R of A_iA_j, and the point P. If P_{ij} is the power of P with respect to the sphere upon A_iA_j as diameter,

$$2P_{ij} = 2\overline{PR}^2 - \overline{A_iA_j}^2/2. \tag{1}$$

Since PR is a median of triangle PA_iA_j, $2\overline{PR}^2 = \overline{PA}_i^2 + \overline{PA}_j^2 - A_iA_j^2/2$, and substituting in (1) gives $2P_{ij} = \overline{PA}_i^2 + \overline{PA}_j^2 - \overline{A_iA}_j^2$.

Thus it is easily seen (or readily proved by mathematical induction) that

$$2\sum P_{ij} = (n-1)\sum_{i=1}^{n} \overline{PA_i}^2 - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{A_iA}^2_{i_{i \neq j}},$$

or

$$2\sum P_{ij} = (n-1)\sum_{i=1}^{n} \overline{PA_i}^2 - k^2. \tag{2}$$

But

$$\sum_{i=1}^{n} \overline{PA_i}^2 = n\overline{PG}^2 + k^2/n, \tag{3}$$

See Court's article in the *National Mathematics Magazine*, 15, 1941, p. 273, art. 4. Substituting (3) in (2) gives the announced relation.

Solved also by Howard Eves, C. E. Springer, and the proposer.

Editorial Note. The solutions of Eves and Springer derived the relation (3), and Eves remarked that the theorem of the problem and its proof applies to any euclidean space.

First Order Ordinary Differential Equations

4041 [1942, 408]. *Proposed by Cezar Coşniţă, Focşani, Roumania*

Integrate the following differential equations

$$y' = \frac{x^2 - y^2 + 1}{x^2 - y^2 - 1}, \qquad y' = -\frac{y(2x + y - 1)}{x(x + 2y - 1)}.$$

Determine the integral curves passing through the origin.

Solution by B. Z. Linfield, University of Virginia. Writing the equations in the form $M\,dx+N\,dy=0$ (or sums of such forms), they become

$$(x^2 - y^2)(dx - dy) + dx + dy = 0,$$
$$(x + y - 1)(y\,dx + x\,dy) + xy(dx + dy) = 0.*$$

The first has $(x+y)^{-1}$ for an obvious integrating factor, yielding

$$\frac{(x - y)^2}{2} + \log\frac{x + y}{C} = 0, \quad \text{or} \quad (x + y)e^{(x-y)^2/2} = C.$$

The second is exact, yielding

$$xy(x + y - 1) = C.$$

The integral curves of the first form therefore a continuous family of probability curves having the line $x+y=0$ for their common asymptote when $C\neq 0$, together with this asymptote when $C=0$, and thus cover the entire plane. In other words, for any (finite) point in the plane this equation has one, and only one, integral curve passing through it. And, according as this point *is* or *is not* on the line $x+y=0$, the integral curve through it is this line or a probability curve. For this equation, in the form $M\,dx+N\,dy=0$, there is no (finite) point that makes its M and N both zero.

On the other hand, for the second equation, in the same form, there are four "singular points" that make its M and N both zero, namely the vertices of the triangle

$$xy(x + y - 1) = 0, \quad \text{and its centroid} \quad (1/3, 1/3).$$

For any point in the plane, except these four "singular points," there is one, and only one, integral curve passing through it. If the point is not on a "side" (line) of the last triangle, the corresponding integral curve is a cubic of the family $xy(x+y-1)=C$, having the "sides" (lines) of this triangle for asymptotes, and having its maxima and minima on a median of this triangle.

According as this point is *inside* or *outside* the triangle, the part of the integral curve through it *is* or *is not* an oval. In the 1st case $0>27C>-1$. In the 2nd

* The arrangement of these sums is clearly determined by symmetry (elegance) and exactness. All the integral curves here are beautiful to look at, but are left for the reader to draw.

case, either the point lies in an "alternate" angle, and $27C<-1$; or it doesn't, and $C>0$. If the point is on a "side" (line) of the triangle (but not one of the vertices), the corresponding integral curve is that line. However, corresponding to a vertex of the triangle, there are two integral curves through it, namely the lines of the triangle through it.

Corresponding to the point (1/3, 1/3), the integral curve

$$27xy(x+y-1)=-1$$

is a solution of the differential equation, and this point satisfies its equation. But (1/3, 1/3) is an isolated point of this curve. (The point and the rest of the curve are separated by the triangle as in the case of the oval.) Hence, in the strict sense of the terms, there is *no* integral curve through the point (1/3, 1/3). Still, all integral curves not through the "singular points" are asymptotic to those through them, and are non-degenerate and non-singular.

It would seem, therefore, that for neither of these two differential equations has the origin any properties that other points (finite or infinite in number) do not have. The point (1/3, 1/3) is, however, "singular" (exceptional) for the second equation, and is different from the remaining three "singular points," the vertices of the triangle $xy(x+y-1)=0$.

The *moral* in these differential equations seems to be, therefore, to watch the "singular points" that make M and N both zero in $M\,dx+N\,dy=0$, or that makes $F(x, y)$ "indeterminate" in $y'=F(x, y)$; and the integral curves through them.† Having never seen a Roumanian calculus, and having no desire to cause any foreign entanglements, the writer cannot help speculating that had all authors of the calculus called explicit attention to the last *moral*, without fail, this problem might not have been proposed at all. Also, students of the calculus would then be better prepared to continue with mathematics.

This solution of the problem is clearly related to the work of Henri Poincaré as discussed by Marston Morse (*What is Analysis in the Large?*) in the same number of the MONTHLY where this problem appeared.

Solved also by G. A. Baker, Cecil Hastings, Jr., C. E. Springer, and P. D. Thomas.

Minimum of a Determinant

4042 [1942, 408]. *Proposed by Henry Scheffé, Princeton University*

Prove that if A is a fixed positive definite hermitian matrix and X is a variable non-negative hermitian matrix (rank = index), then the minimum value of the determinant $|A+X|$ is $|A|$ and is attained if and only if $X=0$.

Solution by the Proposer. There exists a square matrix P such that

$$P^*AP=I. \tag{1}$$

Then

† And hence the points that make the integrating factor infinite or zero.

$$P^*(A+X)P = I + P^*XP. \tag{2}$$

Since P^*XP is non-negative hermitian, there exists a unitary matrix S_X (depending on X),

$$S_X^* S_X = I, \tag{3}$$

such that

$$S_X^*(P^*XP)S_X = (y_i\delta_{ij}),$$

a diagonal matrix with y_i real and $\geqq 0$. Applying S_X^* on the left and S_X on the right of (2) yields

$$S_X^* P^*(A+X)PS_X = I + (y_i\delta_{ij}). \tag{4}$$

Taking determinants in (1), (3), (4), we get

$$|S_X^*| \cdot |S_X| \cdot |P^*| \cdot |P| \cdot |A+X| = \prod_i (1+y_i),$$

$$|P^*| \cdot |P| \cdot |A| = 1, \qquad |S_X^*| \cdot |S_X| = 1,$$

whence

$$|A+X| = |A| \prod_i (1+y_i).$$

The minimum is obviously $|A|$ and is obtained if and only if all $y_i=0$, that is if and only if $X=0$. The theorem was stated originally for real matrices but was altered for hermitian matrices at the suggestion of Orrin Frink, Jr.

Pedal Spheres of a Tetrahedron

4044 [1942, 408]. *Proposed by V. Thébault, San Sebastián, Spain*

Determine the straight lines such that the circumsphere of the pedal tetrahedron of each of its points with respect to any given tetrahedron $ABCD$ passes through a fixed point P.

Examine the case for which $ABCD$ is orthocentric and P is the foot of one of its altitudes.

Editorial Note. The proposer remarked that this problem was suggested by the following theorem, a verification of which he had requested of Cl. Servais:

The pedal spheres of points P of a given straight line relative to the tetrahedron $ABCD$ cut a certain sphere orthogonally.

Servais made this the starting point for important studies published in the *Bulletins de l'Académie royale de Belgique*, 1922, 52. This theorem is true when the sphere reduces to a point. The planes perpendicular to AP, BP, CP, DP at P meet the faces BCD, CDA, DAB, ABC in the straight lines ($\alpha\alpha_1$, $\beta\beta_1$, $\gamma\gamma_1$, $\delta\delta_1$) which are generators of the same system of a ruled surface for which four directrices (d_1, d_2, d_3, d_4) are also on the surface Ω of the 4th order, the locus of

the points for which the pedal spheres with respect to $ABCD$ pass through P. The desired lines are the above eight straight lines and only these.

The case where $ABCD$ is orthocentric and P is one of the feet of its altitudes is very simple and curious. This case was treated by the proposer before proposing to Servais the more general problem.

Euler's Constant

4045 [1942, 479]. *Proposed by A. M. Glicksman, The Bronx High School of Science*

Show that γ, Euler's constant, is given by

$$\gamma = \sum_{r=2}^{\infty} (-1)^r \frac{g_r}{r}, \qquad g_r = \sum_{k=1}^{\infty} \frac{1}{k^r}, \qquad \gamma = \lim_{n\to\infty}\left(\sum_{i=1}^{n} \frac{1}{i} - \log n\right).$$

Solution by the Proposer. We have

$$\log\left(1+\frac{1}{k}\right) = \frac{1}{k} - \frac{1}{2k^2} + \frac{1}{3k^3} - \cdots \qquad (k \geqq 1).$$

The sum of a finite number of convergent series is convergent, and therefore

$$\sum_{k=1}^{n-1} \log\left(1+\frac{1}{k}\right) = \sum_{k=1}^{n-1} \frac{1}{k} - \frac{1}{2}\sum_{k=1}^{n-1} \frac{1}{k^2} + \frac{1}{3}\sum_{k=1}^{n-1} \frac{1}{k^3} - \cdots .$$

But:

$$\sum_{k=1}^{n-1} \log\left(1+\frac{1}{k}\right) = \log \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3} \cdots \frac{n}{n-1} = \log n.$$

Hence:

$$\frac{1}{2}\sum_{k=1}^{n-1} \frac{1}{k^2} - \frac{1}{3}\sum_{k=1}^{n-1} \frac{1}{k^3} + \cdots = \sum_{k=1}^{n-1} \frac{1}{k} - \log n = -\frac{1}{n} + \left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right).$$

(At this point, we might say let $n \to \infty$ and the theorem is proved. However, this requires justification, which is accomplished as follows):

Now the series:

$$s(n) \equiv \left(\frac{1}{2}\sum_{k=n}^{\infty} \frac{1}{k^2}\right) - \left(\frac{1}{3}\sum_{k=n}^{\infty} \frac{1}{k^3}\right) + \left(\frac{1}{4}\sum_{k=n}^{\infty} \frac{1}{k^4}\right) - \cdots$$

is a series of alternately positive and negative terms which diminish towards zero. Hence $s(n)$ converges, and is less in value than its first term. From which it follows that $\lim_{n\to\infty} s(n) = 0$. Add $s(n)$ to both sides of the last equation above, obtaining:

$$\frac{1}{2}g_2 - \frac{1}{3}g_3 + \frac{1}{4}g_4 - \cdots = s(n) - \frac{1}{n} + \left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right).$$

We may now let $n \to \infty$, thus completing the proof.

Orthocentric Tetrahedron and Euler Lines

4049 [1942, 480]. *Proposed by V. Thébault, San Sebastián, Spain*

In an orthocentric tetrahedron $ABCD$ the straight lines joining the centroid with the orthocenter of the triangles of the faces cut the respective radical planes of the circumsphere and the spheres with the medians of $ABCD$ as diameters in four points of the same plane.

Solution by N. A. Court, University of Oklahoma. I. Consider the general tetrahedron $(T)=ABCD$, and let (Q) be its "quasi-polar" sphere, *i.e.*, the sphere having for center the Monge point M of (T) and for the square of its radius one third of the power of M for the circumsphere (O) of (T).* This sphere is orthogonal to the four spheres having the medians of (T) for diameters.†

THEOREM A. *The radical plane of the circumsphere of a tetrahedron (T) and the sphere having for diameter a median of (T) is the polar plane of the centroid of the respective face of (T) with respect to the quasi-polar sphere of (T).*

Let (AG_a) be the sphere having for diameter the median AG_a of (T). This sphere being orthogonal to (Q), the points A, G_a are conjugate with respect to (Q), hence the polar plane λ of the centroid G_a of the face BCD passes through A and is perpendicular to the line MG_a, say, at the point K, which point lies thus on the sphere (AG_a). Now the orthogonal projection of the mid-point of AG_a upon the plane λ is the mid-point of the segment AK, hence AK is a diameter of the circle along which the plane λ cuts the sphere (AG_a).

If O, G are the circumcenter and the centroid of (T), we have, both in magnitude and in sign, $MG=GO$, $AG=3GG_a$. Applying Menelaus' theorem to the triangle AGO and the transversal MG_a, we find that the point of intersection of MG_a with AO is the diametric opposite of A on the circumsphere (O) of (T).

Hence (O) passes through the point K, and AK is a diameter of the circle along which the sphere (O) is cut by the plane λ. Thus the plane λ cuts the two spheres (AG_a), (O) along the same circle and is therefore their radical plane.

THEOREM B. *The four radical planes of the circumsphere of a tetrahedron (T) and the spheres having the medians of (T) for diameters meet the corresponding faces of the polar reciprocal tetrahedron of (T) with respect to the quasi-polar sphere of (T) in four lines lying in the same plane.*

The radical plane λ of the circumsphere (O) and the sphere (AG_a) being the polar plane of G_a for (Q), the plane λ meets the polar plane α of A for (Q) in the polar line, say, m of AG_a for (Q). Now the median AG_a contains the centroid G of (T), hence the polar plane ϕ of G for the sphere (Q) passes through the line m.

Similarly for the other vertices of (T). Hence the proposition.

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* N. A. Court, On the theory of the tetrahedron, Bulletin of the American Mathematical Society, Vol. 48, 1942, p. 583.

† *Ibid.*, p. 587. Also this MONTHLY, Vol. 39, 1932, pp. 196, 197.

II. If the tetrahedron (T) becomes an orthocentric tetrahedron (T_h), its Monge point coincides with its orthocenter H, and the quasi-polar sphere becomes the polar sphere (H) (see, for instance, the writer's *Modern Pure Solid Geometry*, p. 265, art. 813). Moreover, (T_h) coincides with its polar reciprocal with respect to the sphere (H). The two theorems proved above thus become:

THEOREM A′. *The radical plane of the circumsphere of an orthocentric tetrahedron (T_h) and a sphere having for diameter a median of (T_h) is the polar plane of the centroid of the respective face of (T_h) with respect to the polar sphere of (T_h).*

THEOREM B′. *The four radical planes of the circumsphere of an orthocentric tetrahedron (T_h) with the spheres having for diameters the medians of (T_h) meet the respective faces of (T_h) in four lines lying in the same plane.*

The plane of the four lines is the polar plane of the centroid G of (T_h) with respect to the sphere (H), *i.e.*, the orthic plane of (T_h).

The four points in which the radical planes considered meet the Euler lines of the respective faces of (T_h) obviously lie in the orthic plane of (T_h) which proves the proposed proposition. Moreover, this proposition would remain valid if instead of the Euler lines four arbitrary lines were taken in the four faces of (T_h).

The propositions A' and B' may be proved directly, in the same manner as the propositions A and B were proved.

Note. The corresponding proposition for the triangle is due to V. Thébault (this MONTHLY, vol. 49, 1942, p. 63, question E 467). It may proved be in a similar manner.

Editorial Note. All of the above theorems may be extended to the nondegenerate simplex S in n dimensions with the $n+1$ vertices A_i, the centroid G, and the circumcenter C. The *Monge planes* π_{ij} are defined as the hyperplanes perpendicular to the edges A_iA_j and passing respectively through the centroid G_{ij} of the remaining $n-1$ vertices. With an arbitrarily chosen origin of vectors the equation of π_{ij} is

$$(1) \qquad \pi_{ij}\colon\ (\mathbf{a}_i - \mathbf{a}_j)\cdot[\mathbf{x} - (n+1)\mathbf{g}/(n-1)] + (\mathbf{a}_i^2 - \mathbf{a}_j^2)/(n-1) = 0,$$

where $\mathbf{g}$ is the vector of G and $\mathbf{a}_i$ that of A_i. This system of $n(n+1)/2$ equations has one and only one solution for the vector $\mathbf{x}$ since the system of the same number equations

$$(\mathbf{a}_i - \mathbf{a}_j)\cdot\mathbf{x} - (\mathbf{a}_i^2 - \mathbf{a}_j^2)/2 = 0$$

has one and only one solution which is the vector $\mathbf{c}$ of C. Hence, if we denote by $\mathbf{m}$ the vector solution of (1), we must have

$$(2) \qquad (n+1)\mathbf{g} = 2\mathbf{c} + (n-1)\mathbf{m},$$

and the point whose vector is $\mathbf{m}$ is called the *Monge point M*.

We now take the origin of vectors at M, and it is easily seen that

$$(3)\qquad (n+1)\mathbf{g}\cdot\mathbf{a}_i - \mathbf{a}_i^2 = nm, \qquad 2\sum \mathbf{a}_i\cdot\mathbf{a}_j = n(n+1)m, \qquad 2\mathbf{c} = (n+1)\mathbf{g}.$$

The scalar m may be positive, negative, or zero, but it will be convenient at present to exclude the value zero. Let (M) denote the sphere with center M and radius $\sqrt{m}$, and $[P]$ the polar of G with respect to (M). We easily obtain the following equations

$$(4)\qquad \begin{aligned} &(M)\colon \mathbf{x}^2 - m = 0; \qquad (C')\colon n\mathbf{x}^2 - 2\mathbf{c}\cdot\mathbf{x} + m = 0;\\ &(G)\colon \mathbf{x}^2 - 2\mathbf{g}\cdot\mathbf{x} + m = 0; \qquad (C);\ \mathbf{x}^2 - 2\mathbf{c}\cdot\mathbf{x} + nm = 0;\\ &[P]\colon \mathbf{g}\cdot\mathbf{x} - m = 0; \end{aligned}$$

where (C') is the inverse of (C) with respect to (M), (G) has center G and is orthogonal to (M), and $[P]$ is the common radical plane of the four spheres. These are the same equations as in (4) in the solution of 3963 [1942, 133] with H replaced by M.

If we denote by (A_iG_i) the sphere with diameter A_iG_i where G_i is the centroid of the remaining n vertices, we have for its equation

$$(5)\qquad \begin{aligned} (A_iG_i)\colon \mathbf{x}^2 - [2\mathbf{c} + (n-1)\mathbf{a}_i]\cdot\mathbf{x}/n + m = 0,\\ 4n^2r_i^2 = (n+1)^2(\mathbf{g} - \mathbf{a}_i)^2; \end{aligned}$$

where r_i is its radius. The radical plane of (A_iG_i) and (C) has the equation $\mathbf{g}_i\cdot\mathbf{x} - m = 0$, $n\mathbf{g}_i = 2\mathbf{c} - \mathbf{a}_i$, where $\mathbf{g}_i$ is the vector of G_i. We easily obtain the identity

$$(6)\qquad n\{G_i\} + \{A_i\} \equiv (n+1)\{G\};$$

where $\{A_i\} \equiv \mathbf{a}_i\cdot\mathbf{x} - m$ and $\{A_i\} = 0$ is the equation of the polar of A_i with respect to (M), *etc.* Thus any configuration of points common to the polars of A_i and G_i belongs also to the polar of G. For $n=2$ the point M is in all cases the orthocenter. A necessary and sufficient condition that S be orthocentric with the origin at M and $n>2$ is that $\mathbf{a}_i\cdot(\mathbf{a}_j - \mathbf{a}_k) = 0$, i, j, k distinct; then it follows that $\mathbf{a}_i\cdot\mathbf{a}_j$ is a constant and this constant must be m. In this case S is self-polar with respect to (M).

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Louise Adams has been appointed assistant professor at High Point College, North Carolina.

Assistant Professor C. K. Alexander of Occidental College has been appointed associate professor and head of the department of mathematics.

Assistant Professor Harriet W. Allen of Hollins College has been promoted to an associate professorship.

Dr. H. G. Ayre of Western Illinois State Teachers College has been promoted to an associate professorship.

Assistant Professor H. M. Bacon of Stanford University has been promoted to an associate professorship.

Dr. G. A. Baker of the College of Agriculture of the University of California has been promoted to an assistant professorship.

Assistant Professor Juna L. Beal has been appointed acting head of the department of mathematics at Butler University.

Associate Professor A. C. Berry of Lawrence College, Wisconsin, has been promoted to a professorship.

Dr. B. H. Bissinger and Dr. Fritz Herzog of Cornell University have been appointed instructor and assistant professor, respectively, at Michigan State College.

Assistant Professor J. W. Blincoe of the University of Tennessee has been appointed to a professorship at Randolph-Macon College.

Dr. J. O. Blumberg of the University of Pittsburgh has been promoted to an assistant professorship.

W. H. Bradford of the John McNeese Junior College of the Louisiana State University has been promoted to an assistant professorship and appointed head of the department of mathematics and science.

Assistant Professor A. D. Bradley of Hunter College became a lieutenant (j.g.) in the U.S.N.R. and is now teaching in the Naval Air Station in Dallas, Texas.

R. L. Calvert of the Utah State Agricultural College has been promoted to an assistant professorship.

Dr. Max Dehn of Illinois Institute of Technology has been appointed professor of mathematics at St. John's College, Annapolis.

Professor Ralph Hull of the University of British Columbia has been appointed professor and chairman of the department of mathematics at the University of Nebraska.

Mrs. Phyllis H. Hutchings has been appointed assistant professor of astronomy and mathematics at Whitman College.

Associate Professor Nathan Jacobson of the University of North Carolina has been appointed associate professor at Johns Hopkins University.

Dr. S. A. Jennings of the University of British Columbia has been appointed to an assistant professorship.

Dr. G. W. Mackey of Illinois Institute of Technology has been appointed to an instructorship at Harvard University.

Dr. Haim Reingold, Supervisor of Mathematics Instruction in the Signal Corps Training Schools of the Illinois Institute of Technology, has been appointed assistant professor of mathematics in that institution.

Assistant Professors M. A. Sadowsky and L. R. Wilcox of Illinois Institute of Technology have been promoted to associate professorships.

Dr. C. L. Seebeck, Jr., of the University of Alabama has been promoted to an assistant professorship.

Dr. W. S. Snyder of Illinois Institute of Technology has been appointed assistant professor at the University of Tennessee.

Assistant Professor T. L. Wade, Jr., of the University of Alabama has been appointed head of the department of mathematics at the Florida State College for Women.

The following appointments to instructorships are announced:
Illinois Institute of Technology: Furio Alberti, Dr. Ruth Mason Ballard, P. L. Browne, H. J. Miser
Massachusetts Institute of Technology: G. N. Raney
Whitman College: Mrs. Pearl C. Miller
University of Rochester: H. P. Atkins

Professor Emeritus E. B. Van Vleck of the University of Wisconsin died on June 2, 1943.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

ESSENTIAL MATHEMATICS FOR MINIMUM ARMY NEEDS

In this department of the MONTHLY, August–September, 1943, attention was called to a report on Pre-Induction Courses in Mathematics issued by a committee of the United States Office of Education. Since the writing of that report, the War Department, believing that a supplementary statement of a more specific nature would be helpful in carrying out the philosophy outlined in that basic report, requested the United States Office of Education to cooper-

ate in further study of the subject. The members of the committee receiving the new assignment were Virgil S. Mallory, State Teachers College, Montclair, New Jersey, Chairman; Rolland R. Smith, Public Schools, Springfield, Massachusetts; C. Louis Thiele, Public Schools, Detroit, Michigan; F. L. Wren, George Peabody College for Teachers, Nashville, Tennessee; William A. Brownell, Duke University; and Wells Harrington, representing the Civilian Pre-Induction Training Branch of the War Department. John Lund and Giles M. Ruch represented the U. S. Office of Education.

The complete report of the new Committee appeared in the October issue of the *Mathematics Teacher*. Also a brief statement of the findings appeared in *Education for Victory*, September 1, 1943. In view of the importance of the project, Professor Virgil S. Mallory was invited to write a digest of the work of the Committee. The following paragraphs are due to him.

The procedure employed by the Committee was to confer with Army officers directly in charge of the training of enlisted men, and to observe the basic training process itself during the first thirteen weeks of the inductee's Army life. To systematize both interviews and observations, a check list of one hundred forty-one items was employed. This check list, carefully prepared by the Committee, had the benefit of criticism from more than twenty Army training officers stationed in the Washington area. It was then tried out in the original and in a revised form with a dozen training officers in nearby camps. In its final form its contents provided the basis for informal conferences with another group of ninety-six training officers and for item-by-item checking by one hundred seventy-eight officers who were serving as instructors in basic training in seventy-four different kinds of Army jobs in Replacement Training Centers and unit training centers in eight states. The centers visited are operated by branches which together train seventy-five per cent of the enlisted men of the Army.

The purpose of the Committee's investigation was to determine those items in mathematics which should make up the *minimum* equipment of the inductee. Every inductee might well have more than this minimum, but he cannot have less and successfully meet the demands of basic training. It should be emphasized that many enlisted men go on for specialized training which calls for more mathematics than is outlined in the report. Such men can profit by as much of the sequential work in mathematics as they can master.

The list of essential topics in mathematics for minimum Army needs is not presented by the Committee as a course of study. The responsibility for determining the nature and details of such a course rests with local school authorities. There is no thought that only the items listed should be taught, and that these should be taught only within the limits suggested. Nor is there any thought of asking capable students to substitute a course based on the listed items for the sequential mathematics of the senior high school. For those not taking such courses, instruction in the essential mathematics listed in the report should be carried as far as time permits. Moreover, there are other mathematical concepts

and skills, not found to be part of the necessary minimum equipment, but almost certainly valuable in the Army, on which instruction might well be offered.

It is a fallacy to assume that enrollment in advanced high school courses in mathematics assures proficiency in the minimum essentials listed in the report. Ample evidence in the Army and in civilian life shows that the study of algebra and geometry does not guarantee the maintenance of lower-order mathematical abilities and knowledge. On this account, school officials and teachers should take steps to ascertain whether students taking advanced mathematics possess the needed minimum essentials, and then to teach whatever may be necessary. It should be noted in this connection that the earlier report already referred to suggested changes in the sequence of courses to give time for much of the material of the present report.

The casual reader of the report can easily oversimplify the problem if he thinks of the essentials listed purely in terms of mechanical, paper-and-pencil computation. Proficiency in computation is of course necessary, but it should not be mere mechanical proficiency. Much more is called for than computational competence.

On this last point the testimony of training officers is unequivocal. They say that many enlisted men, even those who seem to be able to obtain correct answers in abstract computation, are unable to think quantitatively; that is to say, they cannot use in practical situations even the limited skills which they possess. The implications of this charge (and it was universal) are unmistakable, and they cannot be disregarded. What is needed is a reorientation, a change of emphasis, in instruction. Computation has too often been stressed, and accurate, skilled thinking in concrete quantitative situations has been minimized. Many students have acquired tricks with numbers which have proved valueless under conditions of use. Meanwhile two aspects of mathematical learning have suffered, namely, understanding and experience in application.

It follows that refresher courses in the high school and elsewhere are not the sole remedy for the present emergency. Such courses may only restore former skills which are inadequate for Army needs and for the demands of civilian life. Instead, young men about to enter the Army must be taught something which heretofore has not been taught often enough, namely, the ability to meet quantitative problems effectively and confidently. They must be able to identify the quantitative aspects of the situations which confront them, to deal with these situations by approximation and estimation when computation is not required, to recognize and use the simpler symbolism of mathematics, to tell when and how mathematical symbolism, concepts, and processes are to be employed, and to compute accurately, quickly, and intelligently when computation is called for. Courses with objectives less ambitious than these are of limited value.

Some additional explanation is desirable in regard to the check list which was used by members of the Committee in their visits to Army camps. The items on the list were concerned with mathematical skills, concepts, and relationships as well as with memorized formulas and rules; they covered the field of arithmetic,

and dealt with the simpler aspects of algebra, geometry, and trigonometry. With many of the items an example was given to set a working level of difficulty. Only the uses of mathematics in the thirteen weeks of basic training given to all inductees were to be recorded.

As already indicated, a total of one hundred seventy-eight instructors actually engaged in teaching inductees filled out the check list. They were asked to do four things: (1) to mark each item as of frequent, occasional, or rare use; (2) to classify the examples given as of less, the same, or greater difficulty than inductees would actually need; (3) to furnish practical examples of how the mathematical items listed are used; and (4) to report on mathematical needs not covered in the list. Each instructor was advised before marking the check list to be concerned only with the mathematical needs in his own particular field of instruction, not with his philosophy about needs in general.

In selecting from the items on the check list those which should be recommended for inclusion in the outline of essential mathematics for Army needs, attention was given first of all to frequency of use. On this basis a number of the items obviously had to be rejected. But it soon became apparent that frequency of use alone was not a sufficient basis for rejecting or including an item. In many cases, an item which was indicated as rarely used might still be of fundamental importance. The conclusions of the Committee appear in the following outline.

Outline of Essential Mathematics for Minimum Army Needs

A. *Reading and Writing Arithmetical Symbols.* Whole numbers (to six places). Common fractions, especially those with denominators which are powers of 2 through 64, and with denominators 3, 5, 6, 10, and 12; fractions with denominators in the ten-thousands are frequently used in map reading. Decimal fractions (to three places); in machine work it is sometimes necessary to read micrometer calipers to ten-thousandths. Per cents (to 100%); in certain instances per cents less than 1% are used to indicate the composition of metals and the proportions of chemicals in solutions.

B. *Counting.* Counting by 1's, 2's, 5's, and 10's (to 500).

C. *Operations with Whole Numbers.* Addition (columns of not more than five addends of three digits each, and shorter columns of not more than six digits). Subtraction (numbers of not more than six digits). Multiplication (numbers of four digits by numbers of two digits). Division (numbers of four digits divided by numbers of two digits).

D. *Operations with Common Fractions.* The denominators are limited as in the second topic of (A).

E. *Operations with Decimal Fractions.* The fractions are restricted as in the third topic of (A).

F. *Part-Whole Relationships,* with comon fractions, decimal fractions, and per cents. Finding part of a quantity. Finding what part one number is of another. Finding a number, given a part and its relative size.

G. *Ratio and Proportion.* A study of the basic idea with problem solving.

H. *Powers and Roots.* Finding powers (squares and cubes). Finding squares and square roots from tables.

I. *Graphs and Maps.* Understanding grids and scales. Determining directions from a map. Interpreting and making maps and graphs.

J. *Tables.* Reading tables.

K. *Formulas and Equations.* Understanding of simple symbolism of algebra. Simple formulas to be memorized: area of circle, triangle, rectangle; volume of cylinder, rectangular solid; distance, rate, and time relationships. Substitution in simple formulas. Solution of simple equations.

L. *Positive and Negative Numbers.* Symbolism and meaning.

M. *Measurement,* including understanding of basic units. Length, weight, area, and volume. Temperature (C and F), angles (degrees, minutes, and seconds), time (24-hour clock). Metric system with simple equivalents. Measuring instruments. Limits of accuracy or tolerances. Estimation.

N. *Geometric Concepts.* Point; straight, curved, horizontal, vertical, oblique, parallel, and perpendicular lines; angle; and slope. Triangle (right, scalene, isosceles, and equilateral); parallelogram (square and rectangle); trapezoid; circle; ellipse; regular polygon; prism; cylinder; cone; and sphere. Similarity.

O. *Drawing and Construction.* Use of ruler graduated in 32ds and in 10ths of an inch, and in millimeters; use of compasses to construct circles; use of protractors to measure and to draw angles. Construction of a perpendicular to a line. Understanding of views of a simple object as given in a scale drawing, blueprint, or sketch. Knowledge of the 3-4-5 relation of the sides of a right triangle.

P. *Miscellaneous.* Averages: mean, median, and mode. Rounding off numbers.

The report then gives, in considerable detail, suggestions for teaching each of the items included in the above list. More space is devoted to topics related to the teaching of arithmetic than to those topics with which high school teachers are most familiar. Throughout this part of the report, emphasis is placed on the following points.

(1) *Meaning and understanding.* "Learning should proceed from the concrete (where meaning is most easily apprehended) to the abstract (where the idea or skill is freed of its particularized content) and back again to the concrete (where the freed idea or skill can be applied usefully)."

(2) *Applications.* From the standpoint of learning, providing ample experiences in application completes the cycle from concrete to abstract and back again to concrete.

(3) *Estimation and approximation.* It is not always necessary and is frequently impracticable to use pencil and paper to find answers in Army situations. In such cases the soldier must estimate and approximate, and he will be successful in so doing only to the extent (a) that he understands meanings, (b) that he has what may be called "mathematical sense," and (c) that he de-

velops an appreciation of relative size both with abstract number and with units of measure.

The following paragraphs typify the summary to the report.

"The items listed previously must be part of the equipment of all Army inductees. On this point there is no distinction among prospective inductees who are in private schools or in public schools, who have left school for no particular reason, or who have positions in industry. In the age population concerned there is no distinction between prospective inductees who are in high school and those who are in the grades, or, within the high school, between those who are taking the four-year mathematics sequence, and those who have had one year or less of mathematics since the eighth grade. This fact carries a number of implications for persons charged with the responsibility for organizing instructional programs.

"Students majoring in mathematics need to be examined carefully to make sure that they do have the needed ideas and skills. To the extent that they do not, provision should be made in each advanced course to correct all deficiencies. (And it should be repeated that this provision is inadequate if it consists only in "refresher" practice, to re-instate useless number tricks.) Furthermore, the content of these advanced courses should be studied carefully with a view (a) to eliminating material which will not function and (b) to including vital applications arising from the emergency. In this connection, mathematics teachers should consult the report of the Office of Education Mathematics Committee, in which many valuable suggestions are offered. Also, they might well make use of the Seventeenth Yearbook of the National Council of Teachers of Mathematics (1942), entitled "A Source Book of Mathematical Applications."

"As for a special course, to provide students in school with the mathematical essentials, school officials must think in terms of the *ages* of students, not in terms of the *grade* level they may have reached. Physically able 16- and 17-year-old prospective inductees will soon be in military service, whether at present they are high school students or are enrolled in the eighth grade. Steps should be taken to bring together in convenient groupings all students who are about to be inducted or are about to leave school for work. It is obvious that the schools offer the readiest means for assembling prospective inductees for instructional purposes.

"Prospective inductees already out of school, for whatever reason, should be given an opportunity to make up their mathematical deficiencies. It is often possible to have special classes in public school buildings late in the afternoon or in the evening. Many business colleges are already offering such courses as are many private industrial concerns. These agencies of pre-induction training should be encouraged and multiplied.

"Let the mathematics course be taught by persons who know mathematics, not alone in the technical sense, but in the sense of concrete, sensible, practical application. Such teachers will be able to tell when their students have acquired the ideas and skills set in the minimum list."

A NEW DEFERMENT PROCEDURE FOR MATHEMATICIANS

In April, 1943, a deferment procedure for mathematicians, involving the National Committee on Physicists and Mathematicians, was authorized under Activity and Occupation Bulletin No. 35. This was explained fully in a memorandum distributed to chairmen of departments of mathematics on May 11, 1943, and reproduced in the June-July issue of this MONTHLY. By this procedure employers were advised to send the original copy of Form 42A to the National Committee for evaluation. From the Committee, the form was returned to the local board with an appropriate statement by the Committee concerning the advisability of the deferment. The National Committee was also empowered to appeal cases of registrants for whom occupational deferment was not granted by the local board. Under this arrangement, the Committee has performed its duties well and effectively.

Activity and Occupation Bulletin No. 35, which authorizes the activities of the National Committee, is to be rescinded in the near future, over the strong protests of representatives of mathematics and physics.* The Committee will therefore cease to exist and will no longer be available to advise local boards on problems connected with the deferment of mathematicians. The new procedure for persons engaged in the occupations defined as critical in Local Board Memorandum No. 115, revised August 16, 1943, is described in Local Board Memorandum 115B. *Only those cases in which occupational deferment is refused by the local board will become involved in this procedure.*

In the August 16 revision of Local Board Memorandum 115, a list of critical occupations, which includes mathematicians, was set up.

The following represents a summary of the occupational deferment procedure for the critical occupations:

1. The employer continues, as in the past, to present his case for deferment to the local board through the use of Form 42A.
2. The local board will arrive at a classification for the registrant on the basis of the evidence submitted to it by the employer. It may, if it so desires, consult with the local United States Employment Service office concerning classification. (LBM 115 and LBM 115-C)
3. The local board will then make its decision. If occupational deferment is granted, no further action is involved during the period covered by the deferment. If, however, deferment is refused, and the registrant is placed in I-A, the case is continued as follows.
4. *If no appeal is taken* within the 10-day period allowed for this purpose, the local board is *directed* to refer the case to the local (local with respect to the registrant's local board) office of the United States Employment Service and a 30-day stay of induction is granted to permit action by this office.

* The essential part of Bulletin No. 35 was rescinded October 21.—Ed.

5. *If an appeal is taken* and the local board's decision is reversed, the registrant will be reclassified, and the case is closed for the duration of the deferment period. If, however, the I-A classification made by the local board is supported by the appeal board, the case *must be* referred to the local U.S.E.S. office.
6. When the local U.S.E.S. office receives a case from the local board either directly (4, above), or after appeal (5, above), it may certify to the local board either that

   (a) the registrant should be deferred in his present position, or
   (b) that they have succeeded in placing the registrant in a new position.

   In the first case the local board is directed to reopen the classification of the registrant, and presumably will grant occupational deferment. In the second case, a further period of 10 days is given the registrant for the filing of a new Form 42A requesting occupational deferment in his new position, and on the basis of which the local board shall reopen the classification.
7. If the U.S.E.S. fails to present any notification to the local board during the period allowed, or if it certifies that the registrant is needed in new employment, but does not succeed in placing the man in such employment, the local board may proceed with the induction of the registrant.
8. The local offices of the United States Employment Service can refer cases reaching them and involving persons in "critical" professional and scientific occupations to the National Roster of Scientific and Specialized Personnel. The details of the relation between the U.S.E.S. offices and the Roster have not been fully defined.

The following suggestions are made to facilitate the proper functioning of the procedure herein outlined.

(1) It becomes of even greater importance than heretofore that an employer's original presentation of a case to the local board for occupational deferment or continuation of deferment be made as strong as possible. No opportunity for strengthening a case should be overlooked.

(2) The employer having once requested occupational deferment should be prepared to appeal every case in which such deferment is refused by the local board. This again implies that the original presentation should be as strong as possible, so as to stand up under an appeal.

(3) If an appeal does not secure deferment, the employer should take it upon himself to see that the case is referred to the local U.S.E.S. office, as directed by Local Board Memorandum 115-B. It is possible that the local board may not be familiar with the requirements that they do this.

(4) The employer should take steps to insure that a case which has reached the U.S.E.S. office is actually referred by it to the National Roster.

(5) It should be noted that there is no clear statement as to whether an appeal may be taken *after* the action of the U.S.E.S. Presumably an appeal could be taken at this time, but it is implied in LBM 115-B that the proper time for an appeal is immediately (within 10 days) after the local board's original I-A classification.

(6) In case the procedures here listed have been followed and misclassification of important personnel nevertheless results, the National Roster should be informed regarding the particulars of the case.

The Secretary of the War Policy Committee will be interested in learning of the experiences of department chairmen with the new procedure. Such information will give our representatives in Washington and the War Policy Committee a basis for judging the effectiveness of the plan.

October 20, 1943.

J. R. KLINE,
Secretary, War Policy Committee

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE TWENTY-SIXTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The twenty-sixth summer meeting of the Mathematical Association of America was held at the New Jersey College for Women, Rutgers University, New Brunswick, New Jersey, on Saturday and Sunday, September 11–12, 1943, in conjunction with the summer meeting and colloquium of the American Mathematical Society and the meeting of the Institute of Mathematical Statistics. Three hundred and fourteen persons were in attendance at the meetings, including the following one hundred seventy-three members of the Association:

C. R. ADAMS, Brown University
LOUISE ADAMS, High Point College
E. B. ALLEN, Rensselaer Polytechnic Institute
C. B. ALLENDOERFER, Haverford College
R. D. ANDERSON, University of Texas
H. E. ARNOLD, Wesleyan University
L. A. AROIAN, Hunter College
W. L. AYRES, Purdue University

A. V. BAEZ, Wagner College
FRANCES E. BAKER, Vassar College
D. H. BALLOU, Middlebury College
J. A. BENNER, Lafayette College
GARRETT BIRKHOFF, Harvard University
G. D. BIRKHOFF, Harvard University
B. H. BISSINGER, Cornell University
E. E. BLANCHE, Curtiss-Wright Corporation
T. A. BOTTS, U. S. N. R.
JULIA W. BOWER, Connecticut College
J. G. BOWKER, Middlebury College
H. W. BRINKMANN, Swarthmore College
B. H. BROWN, Dartmouth College
C. T. BUMER, Kenyon College
L. H. BUNYAN, Rutgers University
HERBERT BUSEMANN, Illinois Institute of Technology

Hobart Bushey, Hunter College
Jewell H. Bushey, Hunter College

S. S. Cairns, Queens College
Mildred E. Carlen, Brown University
W. B. Carver, Cornell University
John Cawley, Lafayette College
F. L. Celauro, Syracuse University
J. O. Chellevold, Wartburg College
Mary D. Clement, Wells College
A. B. Coble, University of Illinois
H. R. Cooley, New York University
T. F. Cope, Queens College
Richard Courant, New York University
C. C. Craig, University of Michigan
E. L. Crow, Bureau of Ordnance, Navy Department
Forrest Cumming, University of Georgia
H. B. Curry, Frankford Arsenal
J. H. Curtiss, Bureau of Ships, Navy Department
E. H. Cutler, Lehigh University

D. R. Davis, State Teachers College, Montclair, N. J.
F. F. Decker, Syracuse University
R. H. Downing, Kaiser Cargo, Inc.
Arnold Dresden, Swarthmore College
W. H. Durfee, Yale University
O. L. Dustheimer, John Carroll University

Benjamin Epstein, Frankford Arsenal
J. P. Everett, Western Michigan College
G. M. Ewing, University of Missouri

W. H. Fagerstrom, College of the City of New York
A. B. Farnell, U. S. Military Academy
F. A. Ficken, University of Tennessee
W. W. Flexner, Cornell University
Tomlinson Fort, Lehigh University
R. M. Foster, Bell Telephone Laboratories
J. S. Frame, Michigan State College
Orrin Frink, Jr., Pennsylvania State College

M. G. Galbraith, Rutgers University
R. E. Gaskell, Brown University
B. P. Gill, College of the City of New York
H. S. Grant, Rutgers University
J. W. Green, Aberdeen Proving Ground
J. A. Greenwood, Bureau of Aeronautics, Navy Department
J. I. Griffin, College of the City of New York
C. C. Grove, College of the City of New York
V. G. Grove, Michigan State College

V. H. Haag, Hershey Junior College
Theodore Hailperin, Cornell University
D. W. Hall, University of Maryland
P. R. Halmos, Syracuse University
G. E. Hay, University of Michigan
G. A. Hedlund, University of Virginia
M. H. Heins, Illinois Institute of Technology
T. H. Hildebrandt, University of Michigan
Einar Hille, Yale University
T. R. Hollcroft, Wells College
L. C. Hutchinson, Brown University
Emma Hyde, Kansas State College

B. W. Jones, Cornell University

Aida Kalish, Brooklyn, N. Y.
Edward Kasner, Columbia University
J. R. Kline, University of Pennsylvania
Helen L. Kutman, Hunter College

W. D. Lambert, U. S. Coast and Geodetic Survey
V. V. Latshaw, Lehigh University
H. L. Lee, University of Tennessee
Solomon Lefschetz, Princeton University
Joseph Lehner, Kellex Corporation
Marguerite Lehr, Bryn Mawr College
D. C. Lewis, Jr., University of New Hampshire
Marie Litzinger, Mount Holyoke College
R. T. Luginbuhl, University of Pennsylvania

C. C. MacDuffee, University of Wisconsin
Saunders Mac Lane, Harvard University
H. F. MacNeish, Brooklyn College
Margaret P. Martin, DeLamar Institute of Public Health
W. T. Martin, Syracuse University
Rev. P. H. McGrath, St. Peter's College
E. J. McShane, University of Virginia
A. E. Meder, Jr., New Jersey College for Women
L. L. Merrill, Rensselaer Polytechnic Institute
F. H. Miller, Cooper Union
E. B. Mode, Boston University
Deane Montgomery, Princeton University
R. K. Morley, Worcester Polytechnic Institute
Richard Morris, Rutgers University
D. S. Morse, Union College
E. J. Moulton, Northwestern University

Rev. Paul Muehlmann, West Baden College
F. D. Murnaghan, Johns Hopkins University
W. R. Murray, Franklin and Marshall College

C. A. Nelson, New Jersey College for Women
P. B. Norman, New York University

C. O. Oakley, Haverford College
F. W. Owens, Pennsylvania State College

Gordon Pall, McGill University
Charles Pflaum, University of Pennsylvania
A. E. Pitcher, Lehigh University
W. G. Pollard, University of Tennessee
Hillel Poritsky, General Electric Company

J. F. Randolph, Cornell University
C. H. Rawlins, Jr., U. S. Naval Academy
G. E. Raynor, Lehigh University
O. H. Rechard, University of Wyoming
C. J. Rees, University of Delaware
Mina S. Rees, Hunter College
C. F. Rehberg, New York University
R. G. D. Richardson, Brown University
J. F. Ritt, Columbia University
R. E. Root, U. S. Naval Academy
Joseph Rosenbaum, Bloomfield, Conn.
J. B. Rosser, Cornell University
S. G. Roth, Washington Square College

J. M. Sachs, U. S. N. R.
Raphael Salem, Massachusetts Institute of Technology
Henry Scheffé, Princeton University
S. H. Schelkunoff, Bell Telephone Laboratories
I. J. Schoenberg, University of Pennsylvania
Abraham Schwartz, Pennsylvania State College
H. M. Schwartz, University of Illinois
J. A. Shohat, University of Pennsylvania
D. T. Sigley, Johns Hopkins University
L. L. Smail, Lehigh University
M. F. Smiley, Lehigh University
W. M. Smith, Lafayette College
Andrew Sobczyk, Massachusetts Institute of Technology
P. I. Speicher, Albright College
E. P. Starke, Rutgers University
H. W. Steinhaus, Equitable Life Assurance Society
Ruth W. Stokes, Winthrop College
R. E. Street, Langley Memorial Aeronautical Laboratory
J. L. Synge, Ohio State University
Otto Szász, University of Cincinnati

J. D. Tamarkin, Brown University
J. I. Tracey, Yale University
A. W. Tucker, Princeton University
J. W. Tukey, Princeton University

H. E. Wahlert, New York University
G. L. Walker, University of Delaware
R. J. Walker, Aberdeen Proving Ground
A. D. Wallace, University of Pennsylvania
R. M. Walter, New Jersey College for Women
K. W. Wegner, Carleton College
Anna Pell Wheeler, Bryn Mawr College
P. M. Whitman, University of Pennsylvania
G. T. Whyburn, University of Virginia
S. S. Wilks, Princeton University

R. C. Yates, U. S. Military Academy
Bertram Yood, U. S. N. R.

Oscar Zariski, Johns Hopkins University

Convenient dormitory rooms were available for mathematicians and their families, and excellent meals were served in Cooper Hall. On Saturday afternoon a very pleasant reception for those attending the meetings was given by Dean Margaret T. Corwin of the New Jersey College for Women at the Dean's residence; and on Sunday evening members of the Department of Music of the College presented an excellent musical program.

The joint dinner for the three organizations was held on Sunday evening at 6:30. Professor Morris acted as toastmaster, and he first introduced Dean Corwin who extended a very gracious welcome to the visitors. Dean Richardson of Brown University was the next speaker. He called attention to the relative

neglect of the field of applied mathematics in America and to the present increase of interest in this direction and the special work which Brown University is now doing in this field. Professor G. D. Birkhoff spoke on the development of interest in higher mathematics in the universities of Mexico and South America. On motion of Professor Ayres, resolutions were adopted by a rising vote extending thanks to President Clothier of Rutgers University, Dean Corwin of the College for Women, Professor Morris, chairman of the Committee on Arrangements, and Mr. Snedecker, in charge of business management of the college, for the many facilities of the college which had been placed at the disposal of the convening organizations, and for the attention to the details of the arrangements that contributed so largely to the enjoyment of the meetings.

The American Mathematical Society held sessions on Sunday and Monday. As the twenty-fifth Colloquium, Professor E. J. McShane of the University of Virginia delivered three lectures on the subject "Existence theorems in the calculus of variations." On Sunday afternoon there were two lectures by invitation, the first by Professor Antoni Zygmund of Mount Holyoke College on "The complex method of the theory of trigonometric series," and the second by Professor F. D. Murnaghan of Johns Hopkins University on "Finite deformations of an elastic solid."

The Institute of Mathematical Statistics held sessions on Sunday morning and afternoon and a joint session with the Society on Monday morning.

The Mathematical Association held sessions on Saturday morning, afternoon and evening. Credit for the very interesting program should go to the program committee, which consisted of Professors G. A. Hedlund, chairman, C. C. MacDuffee, and G. B. Price. The program follows.

First Session of the Association

1. "Introduction of a Riemannian geometry on a manifold" by Professor S. S. Cairns, Queens College.
2. "Mathematical models" by Professor Frances Baker, Vassar College.
3. "Mathematical properties of military maps" by Professor W. W. Flexner, Cornell University.

Second Session of the Association

1. "Mathematics in Mexico and South America" by Professor G. D. Birkhoff, Harvard University.
2. "Curves and Surfaces" by Professor J. W. T. Youngs, Purdue University.
3. "Some neglected work of James Stirling and Karl Schellbach on interpolation" by Professor I. J. Schoenberg, University of Pennsylvania.

Third Session of the Association

1. "The Navy V-12 college training program" by Dean I. C. Crawford, School of Engineering, University of Michigan.

2. "The Navy V-12 training program at Dartmouth" by Professor B. H. BROWN, Dartmouth College.

3. "The Army B and C premeteorology training programs at Brown University" by Professor J. D. TAMARKIN, Brown University.

4. "The application of mathematics in the Army A training program in meteorology" by Professor B. HAURWITZ, Massachusetts Institute of Technology.

MEETING OF THE BOARD OF GOVERNORS

Nine members of the Board, including two regional governors, were present at the meeting held Sunday morning.

The following sixty persons were elected to membership on applications duly certified:

FURIO ALBERTI, B.S.(Chicago) Instr., Illinois Inst. of Tech., Chicago, Ill.

A. H. ANDERSON, M.E.(Marquette) Head of Science Dept., Whitefish Bay Schools, Whitefish Bay, Wis.

C. S. BANWARTH, M.S.(Notre Dame) Instr., Signal Corps Training, Illinois Inst. of Tech., Chicago, Ill.

P. R. BARTRAM, A.B.(Hamilton) Clerk, Bethlehem Steel Co., Buffalo, N. Y.

E. M. BEESLEY, Ph.D.(Brown) Asst. Prof., Univ. of Nevada, Reno, Nevada

GARRETT BIRKHOFF, A.B.(Harvard) Asso. Prof., Harvard Univ., Cambridge, Mass.

T. A. BOTTS, Ph.D.(Virginia) Lt. (j.g.), U.S.N.R., 2009 Wisconsin Ave., Washington-7, D. C.

BENJAMIN BRAVERMAN, A.M.(Columbia) Chm. of Dept., Seward Park High School, New York, N. Y.

P. L. BROWNE, A.M.(Michigan State Coll.) Instr., Illinois Inst. of Tech., Chicago, Ill.

H. D. BRUNK, A.M.(Rice) Fellow, Rice Institute, Houston, Texas

HERBERT BUSEMANN, Ph.D.(Göttingen) Asst. Prof., Illinois Inst. of Tech., Chicago, Ill.

W. H. CAIN, A.M.(Columbia) Asso. Prof., Western Michigan Coll., Kalamazoo, Mich.

PAUL CIVIN, Ph.D.(Duke) Instr., Univ. of Buffalo, Buffalo, N. Y.

PAUL D'ARCO, B.S.(Chicago) Instr., Radio Mechanics, Army Air Force Tech. Training Command, Illinois Inst. of Tech., Chicago, Ill.

M. L. DEMOSS, M.S.(Kansas St. T. C., Pittsburg) Instr., General Motors Inst., Flint, Mich.

FLORA DINKINES, A.M.(Michigan) Instr., Univ. of South Carolina, Columbia, S. C.

REV. W. C. DOYLE, Ph.D.(St. Louis Univ.) Asst. Prof., Rockhurst Coll., Kansas City, Mo.

P. R. DUNLAP, A.B.(Taylor Univ.) Kalkaska, Mich.

R. L. DURHAM, B.S.(Trinity Coll.) President, Southern Sem. and Jr. Coll., Buena Vista, Va.

J. P. ESPOSITO, Ed.M. (DePaul) Teacher, Crane Tech. High School, Chicago, Ill.

A. B. FARNELL, M.S.(Louisiana State) 1st Lt., Instr., U. S. Military Acad., West Point, N. Y.

C. B. GASS, A.M.(Nebraska) Asso. Prof., Nebraska Wesleyan Univ., Lincoln, Nebr.

IRVING GERST, A.M.(Columbia) Instr., Air Force Tech. School, Biloxi, Miss.

MICHAEL GOLOMB, Ph.D.(Berlin) Instr., Purdue Univ., Lafayette, Ind.

THEODORE HAILPERIN, Ph.D.(Cornell) Instr., Cornell Univ., Ithaca, N. Y.

REV. B. A. HAUSMANN, Ph.D.(Yale) Chm. of Dept., Univ. of Detroit, Detroit, Mich.

FRANK HAWTHORNE, B.S. in Educ.(Penn. St. T. C., Edinboro) Instr., Alliance Coll., Cambridge Springs, Pa.

EDWARD HELLY, Ph.D.(Vienna) Visiting Lecturer, Illinois Inst. of Tech., Chicago, Ill.

E. J. Hills, Ph.D.(Southern California) Instr., Los Angeles City Coll., Los Angeles, Calif.
H. J. Jordan. M/Sgt., Weather Forecaster, Army Air Forces, Base Weather Office, Smyrna, Tenn.
G. K. Kalisch, Ph.D.(Chicago) Instr., Univ. of Kansas, Lawrence, Kans.
O. J. Karst, A.B.(Montclair State T. C.) Instr., Newark Coll. of Engineering, Newark, N. J.
F. J. Kerr. Pleasantville, Pa.
R. A. Kliphardt, B.S.(Armour Inst. of Tech.) Grad. Asst., Illinois Inst. of Tech., Chicago, Ill.
J. C. Koken, A.B.(Missouri) Instr., Parks Air Coll., East St. Louis, Ill.
P. J. Kopp, A.M.(Duke) Major, Chem. Warfare Service, Washington, D. C.
H. D. Lipsich, A.B.(Cincinnati) Teaching Fellow, Grad. School, Univ. of Cincinnati, Cincinnati, Ohio
J. D. Maddrill, Ph.D.(California) Consulting Actuary and Engineer, 828 Eaton Road, Drexel Hill, Pa.
Szolem Mandelbrojt, Dr.ès Sc.(Paris) Visiting Prof., Rice Institute, Houston, Texas
E. J. Mickle, Ph.D.(Ohio State) Instr., Ohio State Univ., Columbus, Ohio
H. J. Miser, M.S.(Ill. Inst. of Tech.) Instr., Illinois Inst. of Tech., Chicago, Ill.
W. V. Neisius, B.S.Ch.E.(Georgia Tech.) Instr., Georgia School of Tech., Atlanta, Ga.
George Ogawa, A.M.(State Coll. of Washington) Route 1, Pullman, Wash.
H. A. Palmer, B.S.(Coll. of Idaho) Instr., Coll. of Idaho, Caldwell, Idaho
W. A. Peterhans. Prof., Visual Training, Illinois Inst. of Tech., Chicago, Ill.
C. W. Pflaum, A.B.(Michigan) Instr., Univ. of Pennsylvania, Philadelphia, Pa.
George Piranian, M.S.(Utah State), A.M.(Rice) Instr., Rice Institute, Houston, Texas
A. L. Putnam, Ph.D.(Harvard) Instr., Yale Univ., New Haven, Conn.
Haim Reingold, Ph.D.(Cincinnati) Asst. Prof., Illinois Inst. of Tech., Chicago, Ill.
J. K. Riess, Ph.D.(Brown) Asst. Prof., Physics, Tulane Univ., New Orleans, La.
John Riordan, B.S.(Yale) Tech. Staff, Bell Telephone Labs., Inc., New York, N. Y.
Raphael Salem, Dr. in Math.Sc.(Paris) Asst. Prof., Massachusetts Inst. of Tech., Cambridge, Mass.
Charles Salkind, M.S.(C.C.N.Y.) Teacher, S. J. Tilden High School, Brooklyn, N. Y.
J. C. Smith, A.M.(Buffalo) Instr., Cornell Univ., Ithaca, N. Y.
A. D. Sollins. Asst. Mathematician, U. S. Coast and Geodetic Survey, Washington, D. C.
Peter Thullen, Ph.D.(Münster, Germany) Prof., Matematicas Superiores, Escuela de Artilleria e Ingenieros; Director, Departmento Matematico-Actuarial, Instituto Nacional de Prevision, Ecuador
Mrs. Carol C. Valuckas, A.M.(Cornell) Asst., Univ. of New Mexico, Albuquerque, N. M.
Israel Wallach, M.S.(New York Univ.) Teacher, Thomas Jefferson High School, New York, N. Y.
E. E. Wallick, Ed.M.(Temple Univ.) Teacher, Junior-Senior High School, Lakewood, N. J.
F. M. Wood, M.A.(Queen's) Asso. Prof., Civil Engg., McGill Univ., Montreal, P. Q., Canada

A report of the War Policy Committee was presented by Secretary Dresden. An important feature of this report was the fact that the Rockefeller Foundation had made a grant to cover the expenses of the committee. It was also reported that the Committee on Available Teachers of Collegiate Mathematics had accomplished some worthwhile results in the placing of teachers in positions where they were needed.

The Board discussed the desirability of holding mathematical meetings during the period of the war emergency. It was voted to hold the summer and annual meetings for 1944 in conjunction with the American Mathematical Society at times and places to be determined later. A resolution was passed expressing

the opinion that mathematical meetings should be continued during the emergency at least to the extent of two national meetings and the usual smaller meetings, and should be increased in number if the need should become apparent.

The secretary reported that, in line with an earlier action of the Board authorizing the disposal of the Association library, he plans to discontinue most of the arrangements for exchanging the AMERICAN MATHEMATICAL MONTHLY with other periodicals after the end of the year 1943. The Board approved this procedure.

W. B. CARVER, *Secretary-Treasurer*

JOINT MEETING OF THE ILLINOIS, MICHIGAN, AND INDIANA SECTIONS

The first joint meeting of the Illinois, Michigan, and Indiana Sections of the Mathematical Association of America was held at the University of Notre Dame on April 9–10, 1943. This meeting replaced the regular twenty-fourth annual meeting of the Illinois Section, and the twentieth annual meeting of the Indiana Section. The Michigan Section held its regular spring meeting in March.

Forty-six registered at the meeting, including the following thirty members of the Association: H. M. Ackley, Emil Artin, W. L. Ayres, S. F. Bibb, I. W. Burr, C. H. Butler, C. J. Coe, A. H. Copeland, P. D. Edwards, L. R. Ford, J. W. Givens, Jr., M. R. Hestenes, L. S. Johnston, M. W. Keller, E. C. Kiefer, W. C. Krathwohl, E. D. McCarthy, Karl Menger, G. T. Miller, Paul Muehlman, Ivan Niven, P. M. Pepper, G. W. Petrie, E. W. Ploenges, J. C. Polley, W. T. Reid, R. M. Thrall, W. R. Utz, J. W. Wiley, A. J. Zanolar.

At the business meeting of the Indiana Section on Saturday the following officers were elected for the next year: Chairman, P. M. Pepper, University of Notre Dame; Vice-Chairman, Emil Artin, Indiana University; Secretary, M. W. Keller, Purdue University. The next meeting of the Indiana Section will be held at Indianapolis on October 29–30, 1943, in conjunction with the fall meeting of the Indiana Academy of Science.

The meetings on Friday afternoon and evening and on Saturday morning were devoted to a series of lectures arranged by the department of mathematics of the University of Notre Dame. At these sessions the following papers were presented:

1. *The teaching of the calculus of probabilities*, by Professor A. H. Copeland, University of Michigan.

The speaker discussed certain pedagogical problems which arise in teaching the calculus of probability. A brief outline of the subject was given from which the major portion of a course in probability could be obtained by filling in the details of the indicated proofs. In such a course Stieltjes integration is almost

unavoidable, and most students are unfamiliar with this topic. But it was pointed out that statistical considerations lead naturally and simply to an understanding of this integral. The significance of the relation between probabilities and statistical data was also discussed. The speaker presented an outline by means of which the student could acquire an insight into this relation. The presentation of the subject as outlined was recommended as a simplification of much of the material usually presented in courses in probability.

2. *Mathematics and meteorology*, by Professor W. T. Reid, University of Chicago.

This paper was devoted to a description of the '*A*', '*B*', and '*C*' programs of the Army Air Force for meteorological training. Primary attention was given to the position of mathematics in the curricula of these programs.

3. *Mathematics in navigation*, by Ensign A. L. Whiteman, U. S. N., introduced by Professor Karl Menger.

In this paper Ensign Whiteman discussed the place of mathematics in a course in navigation. He pointed out some of the more important problems that arise in navigation, and indicated what mathematics is needed to solve them.

4. *Nomography*, by Professor L. R. Ford, Illinois Institute of Technology.

The content of Professor Ford's address concerned the development of methods by means of which one could teach nomography with the greatest possible simplicity, and at the same time cover the general theory. It was remarked that the material to be taught falls into three parts, namely: (1) the use of three-rowed determinants; (2) the parametric representation of curves; (3) the transformation theory.

Various examples were used to show how to set up the basic determinant from which a chart is made. The speaker explained that the projective transformation arises naturally as a result of multiplying the basic determinant by a determinant with constant elements. He then solved what he called "the problem of the rectangular page," which is the problem of projecting a desired portion of an alignment chart so as to fill up a given page.

5. *On the theory of complex functions*, by Professor Emil Artin, Indiana University.

The simplification of some of the proofs given in courses on complex variables was effected by the use of certain topological notions. The topics treated were Cauchy's integral theorem, Cauchy's formula, singular points, zeros, and analytic continuation.

6. *On calculus*, by Professor Karl Menger, University of Notre Dame.

Professor Menger emphasized the possibility of dividing the statements which occur in the calculus into three groups. Those of the first group do not require the concept of a limit. This group includes in particular the graphical and numerical methods. Those of the second group use the concept of a limit,

but in a rather mild way. This group includes the formal rules of the calculus of derivatives and the calculus of antiderivatives which can be deduced from two rules and two specific derivatives by means of an "Algebra of Analysis." The third group is based upon the existence of a maximum for each continuous function on a closed interval. This group includes the theorems of Rolle, Taylor, etc., and the theory of maxima and minima.

7. *The teaching of two unrelated topics in freshman algebra*, by Professor P. M Pepper, University of Notre Dame.

The first topic discussed was a schematic diagram for solving mixture problems. The second topic was an innovation in the teaching of logarithms. Professor Pepper proposed that the student be introduced to the theory of logarithms by the computation of a two place table of logarithms of the integers from one to ten by means of the equalities (some of which are approximate) $2^{10}=10^3$, $3^9=2\cdot 10^4$, $4=2^2$, $5=10/2$, $6=2\cdot 3$, $7^4=2^3\cdot 3\cdot 10^2$, $8=2^3$, $9=3^2$.

8. *Averages*, by Professor I. W. Burr, Purdue University.

About twenty averages were listed in this paper. Their applications were discussed with special emphasis upon the following fundamental concepts: (1) an average is that constant which for some purpose can replace the individual members; (2) an average may be found by transforming the variable, finding the average of the results, and applying the inverse transformation. The paper was illustrated by applications in various fields.

E. C. KIEFER, *Secretary of the Illinois Section*
C. J. COE, *Secretary of the Michigan Section*
M. W. KELLER, *Secretary of the Indiana Section*

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-seventh Annual Meeting, Chicago, Illinois, November 27–28, 1943.

The following is a list of the Sections of the Associations with dates of future meetings so far as they have been reported to the Secretary.

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INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
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MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, Jan. 29, 1944
OHIO, Columbus, April 6, 1944
OKLAHOMA
PHILADELPHIA, Philadelphia, Nov. 27, 1943
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 11, 1944
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

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Edited by OTTO DUNKEL, Washington University, H. S. M. COXETER, University of Toronto, and ORRIN FRINK, JR., Pennsylvania State College

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WHAT IS MEASURE?

S. M. ULAM, University of Wisconsin

The concept of measure includes the notions generalizing the old ideas of length, area and volume of figures; all of which are among the oldest in mathematics and, in fact, as basic as the idea of number itself.

1. The elementary approach. The first mathematical approach to the idea of length of a curve, area of a surface, or volume of a solid consisted in assuming them as "evident" or given for some elementary figures, for instance line segments, rectangles, parallelepipeds or prisms. One calculated these numbers for polygons or polyhedra by decomposing them into elementary figures; and for general curved figures, passages to the limit were used. Archimedes followed this procedure for the computation of areas and volumes of a few of the simplest geometrical objects. The question of the consistency of this method was, of course, raised much later. Not until the development of the infinitesimal calculus, which gave the necessary tools for working systematically with the limit processes, was the class of figures whose areas or volumes could be defined and calculated significantly enlarged.

Cauchy, Riemann and Jordan gave rigorous definitions and statements of properties of the definite integral, and thus gave solid foundations for a treatment of the areas and volumes of the objects that formed the domain of study of most of the 19th century mathematics—sets defined by inequalities satisfied by continuous functions on the n-dimensional Euclidean space. [Let us point out here that in general the problem of the integral and the problem of measure are intimately related. A general notion of integral permits one to define a general measure, and vice-versa.] In the second half of the 19th century, in many parts of mathematics, it became necessary to investigate more general sets of points in Euclidean space. In the study of trigonometric series, in function theory, and especially in the investigations of Poincaré in the theory of probability and the general theory of dynamical systems, there appeared sets of points defined by discontinuous functions (obtained through passages to the limit effected on continuous functions).

Cantor's creation of set theory, where the notion of a geometrical figure was generalized into that of an arbitrary subset of points of a given space, introduced also a need for an axiomatic investigation of the problem of measure of sets and, at the same time, made possible a logical analysis of the notion of measure in general.

2. Lebesgue's measure. Borel, and above all Lebesgue [0], applied the ideas of set theory to the problem of measure. Lebesgue's procedure for introducing a measure for sets situated in the Euclidean space was essentially this:

There is given a collection (class) of sets situated in the Euclidean space. This class contains the elementary figures and is large enough to include all sets that can be obtained by the processes usually employed in analysis; in particular the complement of a set that is in the class also belongs to the class, and

the union of any denumerable number of sets in the class yields sets belonging to this class (the denumerable union of sets corresponds to the process of summing infinite series). One has to attach to every set in the class a non-negative, real number, called its measure, so that the following postulates will be satisfied:

I. All sets of a specific subclass should have measure, for example, all sets consisting of a single point.

II. The measure of a set should coincide with its ordinary value in the case when the set is an elementary figure.

III. The postulate of additivity: two forms, a weaker and a stronger form are possible. This requires that the measure of a finite (or in a stronger form, denumerably infinite) sum of mutually disjoint sets should be equal to the numerical sum of the measures of the individual sets.

IV. The invariance or congruence postulate: This requires that sets congruent in the sense of elementary geometry, should have equal measure.

The smallest class of sets for which it is meaningful to discuss this general problem of measure, consists of the so-called Borel sets. These are all sets that one can obtain from intervals (or parallelepipeds in the n-dimensional case) by the two operations of taking a complement of a set and taking infinitely denumerable sums of sets—repeated any number of times. Lebesgue recognized that one can enlarge this class by considering all sets that differ from a Borel set by a subset of any Borel set of measure. He succeeded in giving a constructive definition of a measure of this kind. His measure meets many but, as we shall see, not all the needs of analysis.

We give these well known historical facts because, even in this abbreviated and schematic form, they throw light on the developments that followed Lebesgue.

3. The abstract point of view. If we examine the problem critically, the folowing questions arise at once:

How large can the class of sets be for which a measure, in the sense given above, can be defined? Lebesgue's class is closed under the operations of denumerable addition and intersection of sets, and so meets many but not all of the situations arising in analysis. Using simply the operation of projection of a Borel set from the n-dimensional space into the $n-1$ dimensional space (for example from the plane into the straight line) one can define sets that do not belong to Borel's class. By operating again with complements of such sets and projections, one obtains a wide class of sets, the so-called projective sets. Whether these belong to Lebesgue's class is still an open question. Such sets arise very naturally, especially in function spaces. It seems very natural to include the operation of projection of sets (or what would be the same, taking the images through continuous transformations) in the consideration of the class of measurable sets. Lebesgue's class certainly does not contain all sets. This was shown first by Vitali [1].

The n-dimensional Euclidean space still remains the most important space of mathematics, but it is only the most important special case among the many spaces studied in geometry and analysis. It may be variously looked upon as an example of a general topological space, of a group manifold, of a Riemannian space, or of a finite dimensional vector space. Is a theory of measure for subsets of these spaces possible? The great development of the theory of probability creates the need of a theory of measure for sets in general "phase spaces" some of which are infinitely dimensional function spaces.

In analogy with the property of invariance of Lebesgue's measure for congruent sets, we might have to examine, in spaces more general than the Euclidean, the various meanings of congruence, and the corresponding properties of a measure.

Let us study all these points in greater detail.

4. The problem of the class of measurable sets. Let us make clear from the beginning that this discussion is necessary only because it is impossible to have measure defined for all subsets of a set [2]. To make things precise, we shall examine the situation where all sets for which a measure has to be defined, are subsets of the interval $0 \leqq x \leqq 1$. The class of sets should be such that the sums of denumerably many sets belonging to it also belong to the class, and the complement of a set in the class is also in the class. We seek a real valued function $m(A)$, the measure of the set, having the following properties:

I. $m(E)=1$, where E denotes the entire interval $[0-1]$.

II. $m(p)=0$, when p denotes a set composed of any single point p.

III. The additivity property, that is, the measure of the sum of two disjoint sets, is equal to the numerical sum of the measures: $m(A+B)=m(A)+m(B)$ whenever $A \cdot B=0$.

III′. The additivity for a denumerably infinite number of disjoint sets: $m(\sum_{i=1}^{\infty} A_i)=\sum_{i=1}^{\infty} m(A_i)$ if $A_i \cdot A_j=0$ for $i \neq j$.

It is impossible to have a measure function defined for all sets on the interval with the properties I, II, III′ [2]. But if one requires only I, II, III, one can define a measure with these properties for *all* subsets, and as a matter of fact, one can have an additional property of the measure, namely, that any two sets congruent in the sense of elementary geometry, will have equal measure [3].

Quite generally it is possible to have a finitely additive measure function, in any additive class of sets, the measure assuming only the two values 0 and 1, and being equal to 0 for all sets of a *prescribed* additive subclass of the given class [4].

The fact that it is not possible to have an infinitely additive measure for all sets on the interval, makes it necessary, *even in the case where one does not require invariance or congruence properties* for the measure to consider *subclasses* of measurable sets; we see that the problem is one of algebra and set theory at this stage. The problem of determining for what additive classes of sets there will exist a real valued infinitely additive set function is not yet solved. An answer

can be given in the case when the given class is generated by denumerably many sets (generated by the operations of taking the complement of a set and adding denumerably many sets in the class) [5].

5. Invariant measures. The problem of measure becomes more interesting when one requires that sets that are congruent or equivalent should have the same measure.

We shall now explain the meaning of congruence or equivalence by means of a few examples.

If the space E is Euclidean or more generally metric, the congruence of two sets A and B means that one can map A into B by a one-to-one transformation, so as to preserve distances between pairs of points. (The transformation needs to be defined only on A, and not necessarily on the whole space E.) It is natural to require that any two congruent sets should have equal measure.

Another case: If the space E is Euclidean, we have a group of point transformations of E into itself, for example, the rigid motions. In general, when E is a manifold on which a given group $\mathfrak{G}$ of point transformations is specified, one might call two sets A and B equivalent if there is a transformation T in the group $\mathfrak{G}$ such that $T(A)=B$. In any space E that is also a group manifold a group of transformations is obtained in a natural way, for example, by multiplying all elements x of the space by a fixed element a on the left: $T(x)\equiv a\cdot x$.

Finally, we could conceive in the most general way of equivalence among subsets of a given space E as a given (but otherwise arbitrary) reflexive and transitive relation between sets, thus dividing the sets into classes of mutually "equivalent" ones.

In all these cases one can ask for a measure function which, in addition to the additivity properties, would assign equal values to equivalent sets.

Let us indicate briefly the present state of knowledge concerning the existence of measures with specified invariance properties. Lebesgue's measure is defined only for certain sets, but it has the properties I, II, III′, IV [6]. Banach has constructed a measure function for all sets on the real line, or all sets in the Euclidean plane with properties I, II, III, IV. The fact that a measure of this sort is not possible for all subsets of the Euclidean space of *three dimensions* was first proved by Hausdorff [7]. A later proof was given by Banach and Tarski [8] in their famous paradoxical decomposition of two solid spheres, of different radii. Each sphere can be decomposed into the same finite number of mutually disjoint sets, the sets forming the bigger sphere being respectively congruent to those used in the decomposition of the smaller sphere!

In a very interesting paper [9] Von Neumann has discussed the problem of measure for all subsets of any given group (thus generalizing the problem from the Euclidean vector groups to an arbitrary group) and characterized those groups for which a measure function satisfying I, II, III, IV can be defined for all subsets of the group. Let us note that IV refers to congruence as defined above for a general group.

Since in general we cannot define a measure function for all subsets, we have to consider special classes of subsets. In an important paper [10] A. Haar has established the existence of a denumerably additive and invariant measure function for subsets of any locally compact topological group. The class of sets for which he succeeded in defining his measure includes all Borel sets (we recall that these are sets obtainable from open sets by the operations of complementation and denumerable summation).

The problem of the existence of a measure function was treated for a general notion of equivalence of sets by Tarski [11]. He obtained necessary and sufficient conditions under which, if *one requires finite additivity only*, a measure function satisfying the congruence postulate is possible. But the solution of this general problem for denumerably infinite additivity is yet to be obtained. Special notions of equivalence for sets, and the existence of a measure having equal values for equivalent sets, have been considered by several others [12].

We cannot here enter into the discussion of properties and applications of measure functions. Let us remark, however, that in many cases the properties which we have postulated for measure determine it uniquely. Lebesgue himself showed that a measure function for sets in the Euclidean space satisfying I, II, III, IV, must necessarily coincide with the one he has constructively defined. This also holds for measures in Haar's groups [13].

Limitations of space forbid the discussion of applications of measure theory to the general theory of probability where one deals with some of the most abstract aspects of the theory and constructs measures in composite spaces, using given measures in given spaces [14], or to ergodic theory [15]. For connections between measures and topologies in algebraical structures the reader is referred to the book of A. Weil [10]. Likewise, we cannot discuss here the general point of view under which one would define measures, not necessarily for sets that are subsets of a given space, but rather for elements of a general Boolean algebra [16]. Also one could study measures with other than real values; p-adic values for example seem indicated in certain general situations arising in topology.

6. The general problem. To summarize, we note that measure is a set function associating a non-negative, real number with every set in a certain class of sets, all forming subsets of a given space. Characteristic properties of measure are its additivity for disjoint sets (finite additivity or, if one can obtain it, denumerably infinite additivity). The other characteristic property is equality of measure for sets that are congruent or otherwise considered equivalent, the notion of congruence or equivalence being provided by the geometry or algebra of the given situation. The problem of existence of such set functions has been solved in many special cases, including some most important ones in Euclidean space. However the general problem, that is, to determine whether or not, in a given class of sets with a given equivalence relation, a measure is possible, seems very difficult.

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AN ELEMENTARY PROOF OF THE BUDAN-FOURIER THEOREM

N. B. CONKWRIGHT, University of Iowa

1. Introduction. It is possible that the treatment of the Budan-Fourier theorem outlined here may be of interest to those who teach introductory courses in the theory of equations.

Let

$$f(x) = 0 \tag{1}$$

be a polynomial equation of degree n with real coefficients. Let V_c denote the number of variations of sign in the sequence

$$f(x), f'(x), f''(x), \cdots, f^{(n)}(x), \tag{2}$$

when $x=c$, where c is any real number. The customary statement of the theorem is as follows:

Let a and b be any two real numbers, $a<b$. Then the number of roots of (1) *in the interval $a<x\leqq b$ (an m-fold root being counted m times) is $V_a - V_b - 2k$, where k is a positive integer or zero.*

In elementary text-books the theorem is sometimes stated in a simplified form obtained by imposing the condition that neither $f(a)$ nor $f(b)$ is zero.

It is a familiar fact that Descartes' rule of signs is an immediate consequence of the Budan-Fourier theorem. It is equally well known that Descartes' rule can be established independently by an argument of an elementary nature. The following proof of the Budan-Fourier theorem will be based upon Descartes' rule. The method used is similar to an argument which the writer has previously employed to derive Descartes' rule.*

2. Proof of the theorem. In this discussion the interval $a<x\leqq b$ will be denoted by I. The symbols h_1, h_2, s, and t will be used to denote quantities each of which is either a positive integer or zero. The number of roots of (1) in the interval I will be denoted by p.

It is easily seen that p is equal to the number of positive (non-zero) roots of

$$f(x+a) = f(a) + f'(a)x + \tfrac{1}{2}f''(a)x^2 + \cdots + \frac{1}{\lfloor n}f^{(n)}(a)x^n = 0 \tag{3}$$

minus the number of positive (non-zero) roots of

$$f(x+b) = f(b) + f'(b)x + \tfrac{1}{2}f''(b)x^2 + \cdots + \frac{1}{\lfloor n}f^{(n)}(b)x^n = 0. \tag{4}$$

This follows from the fact that equations (3) and (4) are obtained by diminishing the roots of (1) by a and b respectively.

* See the author's Introduction to the Theory of Equations, pp. 48–49.

By virtue of Descartes' rule it is known that (3) has V_a-2h_1 positive roots, and that (4) has V_b-2h_2 positive roots. Therefore

$$p = (V_a - 2h_1) - (V_b - 2h_2) = V_a - V_b - 2k, \tag{5}$$

where $k=h_1-h_2$. To establish the Budan-Fourier theorem we need only prove that $k\geqq 0$.

The proof that $k\geqq 0$ will be effected by mathematical induction. It can easily be verified that $k\geqq 0$ if $f(x)$ is of the first degree. Hence it will be sufficient to show that this inequality holds for every equation of degree n if the corresponding inequality holds for every equation of degree $n-1$. We shall therefore base the argument to follow upon the assumption that the specified inequality is valid for every equation of degree $n-1$.

Now let q denote the number of roots of the equation $f'(x)=0$ in the interval I, and let V_c' denote the number of variations of sign in the sequence

$$f'(x), f''(x), \cdots, f^{(n)}(x),$$

when $x=c$, where c is any real number. By an argument similar to that which led to (5) it can readily be shown that

$$q = V_a' - V_b' - 2k'$$

where k' is an integer or zero. And by virtue of our assumption it may be affirmed that $k'\geqq 0$.

It follows from Rolle's theorem that $q\geqq p-1$, say $q=p-1+s$. We note also that

$$V_a = V_a' \quad \text{or} \quad V_a = V_a' + 1,$$

and

$$V_b = V_b' \quad \text{or} \quad V_b = V_b' + 1.$$

An examination of the various possibilities will reveal that

$$V_a - V_b \geqq V_a' - V_b' - 1. \tag{6}$$

Consider first the case in which the inequality sign holds in (6). Then

$$V_a - V_b \geqq V_a' - V_b' = q + 2k' = p - 1 + s + 2k'. \tag{7}$$

Now if $s+2k'\neq 0$, we have $V_a-V_b\geqq p$. It follows from (5) that $k\geqq 0$, as was to be proved.

If $s+2k'=0$, we have $V_a-V_b\geqq p-1$. But it follows from (5) that V_a-V_b-p is *even*, and hence $V_a-V_b\neq p-1$. Consequently $V_a-V_b>p-1$, and therefore $V_a-V_b\geqq p$. Again it follows from (5) that $k\geqq 0$.

Consider next the case in which the two members of (6) are equal. This is possible only if $V_a=V_a'$, and

$$V_b = V_b' + 1. \tag{8}$$

It will be shown in the next paragraph that, if (8) holds, then $q \geqq p$. Accepting this fact for the moment, and writing $q = p + t$, we have

$$V_a - V_b = V_a' - V_b' - 1 = q + 2k' - 1 = p + t + 2k' - 1. \tag{9}$$

The first and last members of (9) are essentially the same as in (7). And by an argument similar to that developed in connection with (7) it can be shown that $k \geqq 0$ in the case under consideration.

To complete the proof of the Budan-Fourier theorem it will now suffice to show that $q \geqq p$ if (8) holds. In this case $f(b) \neq 0$, and if $f'(b) \neq 0$, the quantities $f(b)$ and $f'(b)$ must have unlike signs. Now let r be the greatest root of (1) in the interval I. Then $f'(x)$ must vanish at least once in the interval $r < x \leqq b$, for otherwise $f(b)$ and $f'(b)$ would have the same sign. (This is obvious from graphical considerations, and can also be proved by a simple argument based upon the mean value theorem.) Hence the equation $f'(x) = 0$ has in the interval I the $p-1$ roots vouched for by Rolle's theorem, and at least one additional root in the interval $r < x \leqq b$. Therefore $q \geqq p$.

A PLANE REPRESENTATION OF VECTORS AND TENSORS*

F. P. BEER, Williams College

1. The methods of plane representation of space vectors. Two methods are used for the plane representation of a free vector **P** of components X, Y, Z. The first method, introduced by B. Mayor† and developed by R. von Mises‡, associates with **P** a line-bound†† vector **P**$'$ of the plane XOY, whose x- and y-components are equal to X and Y respectively, and whose moment with respect to OZ is equal to cZ, c being a constant. The second method, used by W. Prager§, associates with **P** a line-bound vector **P**$''$ normal to the plane XOY, and such that the z-component of **P**$''$ is equal to Z, and its moments with respect to OX and OY are equal to cX and cY respectively. The vectors **P**$'$ and **P**$''$ both characterize **P** and are called the first and the second images of **P**.

In both cases the system formed by the images of given vectors of the space is equivalent to the image of the sum of the given vectors. Furthermore, using the two methods simultaneously, W. Prager** has established formulas expressing

* This work was carried out at Brown University, Providence, R. I., with suggestions from Professor W. Prager, during the Summer Session of Advanced Instruction and Research in Applied Mechanics, 1942.

† B. Mayor, Introduction à la statique graphique des systèmes de l'espace, Lausanne, 1910

‡ R. von Mises, Graphische Statik räumlicher Kraftesysteme, Zeitschrift für Mathematik und Physik, vol. 64, p. 209, 1916.

†† I.e. one whose line of action is fixed.

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** W. Prager, Die Formänderungen von Raumfachwerken, Zeitschrift für angewandte Mathematik und Mechanik, vol. 7, p. 421, 1927.

the scalar and vector products of two vectors by means of their images. In the following we need only the formula for the scalar product of two vectors $\mathbf{P}_1$ and $\mathbf{P}_2$ represented by their first and second images respectively. Denoting by $\mathbf{r}_1'$ the vector joining O to a point of the line of action of $\mathbf{P}_1'$ and by $\mathbf{r}_2''$ the position vector corresponding to the trace of $\mathbf{P}_2''$ on XOY, we have

$$\mathbf{P}_1\cdot\mathbf{P}_2 = \frac{1}{c}(\mathbf{P}_2'' \times \mathbf{P}_1')\cdot(\mathbf{r}_2'' - \mathbf{r}_1'). \tag{1}$$

In order to obtain a simpler formulation of our results, we shall slightly modify these methods of representation by rotating the image vectors by $-\pi/2$ around the z-axis. This change of orientation does not alter the law of summation, and formula (1) also remains valid. The first image $\mathbf{P}'$ of the vector $\mathbf{P}(X, Y, Z)$ is then characterized by the two components $X'=Y$, $Y'=-X$ and a line of action p'. The equation of p' can be derived from the condition imposed on the moment of $\mathbf{P}'$ with respect to OZ; it has the form

$$xX + yY + cZ = 0, \tag{2}$$

where x and y are the running coordinates. In a similar way, the second image $\mathbf{P}''$, of modulus Z and normal to XOY, is found to pierce that plane at a point p'' of coordinates

$$x'' = c\frac{X}{Z}, \qquad y'' = c\frac{Y}{Z}. \tag{3}$$

As the line of action p' of the first image and the trace p'' of the second image both characterize the direction of $\mathbf{P}$, a relationship must exist between these two elements. Eliminating X, Y, Z between (2) and (3), we find that p' is the antipolar of p'' with respect to the circle defined by

$$x^2 + y^2 - c^2 = 0.$$

The relation between the two images will be completely determined if we recall that their moduli P' and P'' satisfy the equations

$$P'\cdot d' = cZ \quad \text{and} \quad P'' = Z,$$

where d' is the distance of p' from the origin. It should be noticed that the modulus of an image is to be considered as an algebraic quantity, the sign of which depends upon the sense of the image. Denoting by d'' the distance Op'', we have

$$d'\cdot d'' = c^2,$$

and consequently

$$P'' = \frac{d'}{c}P' \quad \text{and} \quad P' = \frac{d''}{c}P''. \tag{4}$$

These equations express the modulus of one image as a function the of elements of the other. The formulae (4) show that the signs of P' and P'' are alike. This means that $\mathbf{P}''$ and the moment vector of $\mathbf{P}'$ with respect to O have the same sense.

2. Plane representation of symmetric tensors. Consider a symmetric tensor*

$$\theta = \begin{Bmatrix} I_x & -D_z & -D_y \\ -D_z & I_y & -D_x \\ -D_y & -D_x & I_z \end{Bmatrix}. \tag{5}$$

The tensor θ serves to define the transformation of a vector $\boldsymbol{\omega}(\omega_x, \omega_y, \omega_z)$ into a vector $\mathbf{S}=\theta\,\boldsymbol{\omega}$, whose components are

$$\begin{aligned} S_x &= I_x\omega_x - D_z\omega_y - D_y\omega_z, \\ S_y &= -D_z\omega_x + I_y\omega_y - D_x\omega_z, \\ S_z &= -D_y\omega_x - D_x\omega_y + I_z\omega_z. \end{aligned} \tag{6}$$

Associate with $\mathbf{S}$ its first image $\mathbf{S}'$ in the plane XOY, and with $\boldsymbol{\omega}$ its second image $\boldsymbol{\omega}''$. The equation of the line of action s' of $\mathbf{S}'$ is obtained by substituting S_x, S_y, S_z for X, Y, Z in (2). Similarly, the coordinates x'', y'' of the trace Ω'' of $\boldsymbol{\omega}''$ on XOY are found by substituting $\omega_x, \omega_y, \omega_z$ for X, Y, Z in (3). Using (6) we obtain the following relation between x'', y'' and the running coordinates x, y of s':

$$I_x xx'' + I_y yy'' - D_z(xy'' + yx'') - D_y c(x + x'') - D_x c(y + y'') + I_z c^2 = 0.$$

Hence s' and Ω'' are polar and pole with respect to the conic q of equation

$$I_x x^2 + I_y y^2 - 2D_z xy - 2D_y cx - 2D_x cy + I_z c^2 = 0. \tag{7}$$

The coefficients of this conic are readily determined from the components of θ, and, as s' and Ω'' characterize the directions of $\mathbf{S}$ and $\boldsymbol{\omega}$ respectively, we see that q determines the tensor θ, as far as the directions are concerned.

Compare q with the conjugate quadrics Q and Q' used by Cauchy in his representation of a symmetric tensor. The equation of these quadrics is

$$I_x X^2 + I_y Y^2 + I_z Z^2 - 2D_x YZ - 2D_y ZX - 2D_z XY = \pm h^2, \tag{8}$$

where h is a constant. The equation of the cone K asymptotic to Q and Q' is obtained by setting $h=0$ and we see that q can be considered as the intersection of K with the plane $Z=c$. If Q and Q' are unparted and biparted hyperboloids respectively, K is real, and q is a real conic (ellipse, hyperbola or parabola). If Q and Q' are real and imaginary ellipsoids respectively, as in the case of the tensor of inertia, K is imaginary, and q is an imaginary ellipse. In order to avoid using imaginary curves, we shall then use instead of q its conjugate $\bar{q}$.

* The choice of notations is due to the fact that later θ will denote the tensor of inertia of a rigid body.

The coordinates x_0, y_0 of the center C of q and $\bar{q}$ are obtained by solving

$$(9) \qquad I_x x_0 - D_z y_0 - D_y c = 0, \qquad - D_z x_0 + I_y y_0 - D_x c = 0.$$

Introduce the new coordinates $\xi = x - x_0$, $\eta = y - y_0$. Denoting by Δ the determinant of the matrix (5) and by δ the minor belonging to I_z, the equations of q and $\bar{q}$ may be written respectively as

$$(10) \qquad I_x\xi^2 - 2D_z\xi\eta + I_y\eta^2 + c^2\frac{\Delta}{\delta} = 0, \quad I_x\xi^2 - 2D_z\xi\eta + I_y\eta^2 - c^2\frac{\Delta}{\delta} = 0.$$

One of these conics at least is real, and we can determine its orientation and axes from the corresponding equation (10). A real conic is thus associated with θ, and can be easily constructed.

We know already how to draw the line of action s' when $\boldsymbol{\omega}''$ and the conic q or $\bar{q}$ have been constructed. We now shall show how the modulus and sense of $\mathbf{S}'$ can be determined from the same elements. The moment of $\mathbf{S}'$ with respect to the center C of the conic, is obtained as the sum of the moment cS_z with respect to O and the moment, with respect to C, of a vector equipollent to $\mathbf{S}'$ and attached to O. Hence

$$g \cdot S' = x_0 S_x + y_0 S_y + cS_z,$$

where S' (whose sign characterizes the sense of the moment of $\mathbf{S}'$ with respect to O) is the modulus we want to determine, and g the algebraic distance of s' from C, considered as positive if O and C are on the same side of s'. Using the relations (6) we find

$$\begin{aligned} gS' = {} & \omega_x(I_x x_0 - D_z y_0 - D_y c) + \omega_y(- D_z x_0 + I_y y_0 - D_x c) \\ & + \omega_z(- D_y x_0 - D_x y_0 + I_z c). \end{aligned}$$

Equations (9) show that the two first parentheses are equal to zero. The third one is equal to $f(x_0, y_0)/c$, where $f(x, y)$ denotes the first member of the equation (7) obtained for q. Writing $f(x, y)$ as in the first of the equations (10) we obtain

$$\frac{1}{c} f(x_0, y_0) = \frac{c\Delta}{\delta} \cdot$$

Finally, recalling that ω_z is equal to the modulus ω'' of $\boldsymbol{\omega}''$, we find

$$(11) \qquad S' = \frac{c\Delta}{\delta} \frac{\omega''}{g} \cdot$$

The first image of $\mathbf{S}$ is thus entirely determined from the second image of $\boldsymbol{\omega}$. The senses of $\boldsymbol{\omega}''$ and of the moment of $\mathbf{S}'$ in C (of modulus gS') are alike or different according as the quotient of Δ and δ is positive or negative. In the case of the tensor of inertia these two determinants are positive, and $\boldsymbol{\omega}''$ and the moment of $\mathbf{S}'$ with respect to C have the same sense.

3. Scalar product of S and ω. Moment of inertia. We shall now form the scalar product of the vectors $\mathbf{S}$ and $\boldsymbol{\omega}$ by means of their images $\mathbf{S}'$ and $\boldsymbol{\omega}''$. Drawing the perpendiculars $\Omega''L$ and CL' on s' (see figure), we denote by $\mathbf{u}$ the vector running from L to Ω'' and by $\mathbf{v}$ that from L' to C. Choosing L as the extremity of the vector $\mathbf{r}_1'$, formula (1) yields

$$\mathbf{S}\cdot\boldsymbol{\omega} = \frac{1}{c}(\boldsymbol{\omega}'' \times \mathbf{S}')\cdot\mathbf{u}. \tag{12}$$

As $\boldsymbol{\omega}''$ and $\mathbf{S}'$ are perpendicular to each other the modulus of their vector product equals $\boldsymbol{\omega}''\cdot S'$ or, according to (11)

$$\frac{c\Delta}{\delta}\frac{\omega''^2}{g}\cdot$$

The direction of the vector product is that of $\mathbf{v}$. The vector product has the sense of $\mathbf{v}$ if the quotient of Δ and δ is positive. Noting that both $|g|$ and $|\mathbf{v}|$ are equal to $L'C$, we get

$$\boldsymbol{\omega}'' \times \mathbf{S}' = \frac{c\Delta}{\delta}\omega''^2\frac{1}{(L'C)^2}\mathbf{v}.$$

The vectors $\mathbf{u}$ and $\mathbf{v}$ being parallel, their scalar product equals $(L\Omega'')(L'C)$, and formula (12) takes the form

$$\mathbf{S}\cdot\boldsymbol{\omega} = \frac{\Delta}{\delta}\omega''^2\frac{L\Omega''}{L'C}\cdot \tag{13}$$

Draw $\Omega''C$, and denote its intersection with s' by N. Then

$$\mathbf{S}\cdot\boldsymbol{\omega} = \frac{\Delta}{\delta}\omega''^2\frac{N\Omega''}{NC}\cdot \tag{14}$$

The two last formulae are valid in the general case of a symmetric tensor. Consider now the particular case of the tensor of inertia. The conic q is then an imaginary ellipse, and s' must be considered as the antipolar of Ω'' with respect to the real ellipse $\bar{q}$. As the center C of $\bar{q}$ is located between N and Ω'',

$$\frac{N\Omega''}{NC} = \frac{NC + C\Omega''}{NC} = 1 + \frac{C\Omega''}{NC}\cdot$$

Multiply numerator and denominator of the last quotient by $C\Omega''$. Now the product of NC by $C\Omega''$ is equal to the square of CM, where M is the intersection of $C\Omega''$ with $\bar{q}$. Hence

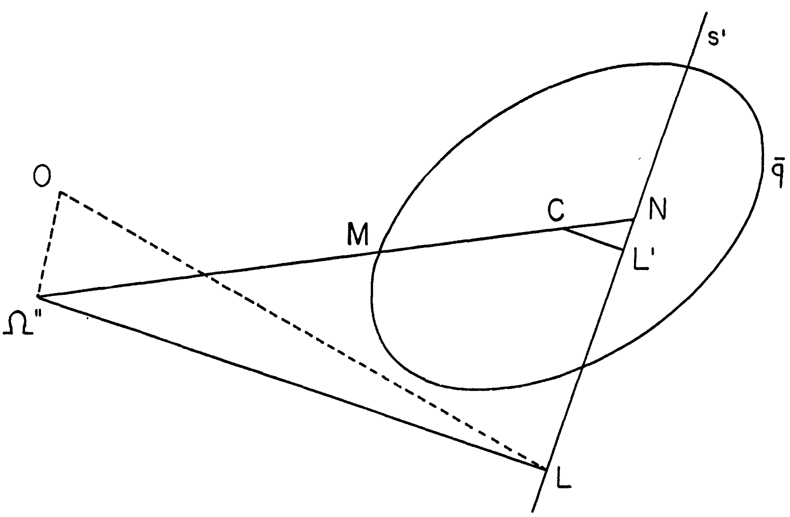
$$\frac{N\Omega''}{NC} = 1 + \frac{(C\Omega'')^2}{(CM)^2} = \frac{(CM)^2 + (C\Omega'')^2}{(CM)^2}\cdot$$

If R and r denote the lengths of the segments CM and $C\Omega''$ respectively, formula (14) takes the form

$$\mathbf{S}\cdot\boldsymbol{\omega} = \frac{\Delta}{\delta}\,\omega''^2\,\frac{R^2 + r^2}{R^2}. \tag{15}$$

This formula requires only the construction of $\boldsymbol{\omega}''$ and $\bar{q}$ for the calculation of the scalar product $\mathbf{S}\cdot\boldsymbol{\omega}$.

If $\boldsymbol{\omega}$ represents the instantaneous rotation of a rigid body with a fixed point O, the kinetic energy, equal to one half of $\mathbf{S}\cdot\boldsymbol{\omega}$, can be graphically determined if the six components of the tensor of inertia attached to O are known. Moreover, dividing this scalar product by ω^2 we obtain the moment of inertia I of the body



with respect to the instantaneous axis of rotation. As the quotient of ω'' by ω is equal to $\cos\gamma$, where γ is the angle formed by $\boldsymbol{\omega}$ and OZ, formula (15) yields

$$I = \frac{\Delta}{\delta}\cos^2\gamma\,\frac{R^2 + r^2}{R^2}. \tag{16}$$

Denoting by ρ^2 the quotient of Δ by $m\delta$ where m is the mass of the body, we have the following expression for the radius of gyration k corresponding to the direction of $\boldsymbol{\omega}$:

$$k = \rho\cos\gamma\,\frac{\sqrt{R^2 + r^2}}{R}.$$

The graphical construction of k is easy, once the ellipse $\bar{q}$ has been drawn.

The graphical methods developed in this paper will be useful in a graphical discussion of the rotation of a rigid body around a fixed point.

A PROJECTIVE CONSTRUCTION FOR PLANE NODAL CUBICS

I. C. FISCHER, University of Minnesota

1. The problem. The problem under consideration is to discover the base elements for the construction of plane nodal cubic curves, with the use of straight edge alone, having given a sufficient number of conditions for the unique generation by a projective method. A few of the many possible cases which may arise will be discussed here; others can easily be formulated from the general construction.

In another paper,* the author presented a method for constructing a plane sextic, with degenerations, in which the base elements were a point conic, two straight lines, and three mutually perspective flat pencils. If the conic is replaced by a straight line, a cubic of the class with which this paper is concerned will result.

The method of construction employed here somewhat resembles Grassmann's,† in that three fixed points and three fixed lines are employed, but the method of generation is entirely different since we use the three fixed points as centers for three mutually perspective flat pencils. Another straight line construction‡ utilizes two pencils of lines in one-two correspondence. The author's solution is obviously a special case of a more general construction referred to above.‡

2. The general construction. Consider three noncollinear points A, B, C, and the lines l, l', l'', which are not concurrent. A ray through A cuts l in point L. The rays BL and CL meet l' and l'' in L' and L'' respectively. Line $L'L''$ is on a point P of AL. The totality of all points P so defined constitutes the required locus. Before proving that this locus is a nodal cubic, a few projective relations will be discussed.

Let $[L]$ be a range of points on l, and $[a]$ a pencil of lines on point A; also let $\overset{O}{\doublebarwedge}$ and $\barwedge$ represent a perspective correspondence with center O, and a projectivity, respectively. The chain of perspectives

$$[L'] \overset{B}{\doublebarwedge} [L] \overset{C}{\doublebarwedge} [L'']$$

defines a projective correspondence Π such that $\Pi[L'] = [L'']$, and the lines $L'L''$ defined by all pairs of homologous points on the two ranges generate a line conic k, of which l' and l'' are elements. Since $[L']$ is perspective with $[b]$ then $[b]$ is perspective with k and

$$[a] \overset{l}{\doublebarwedge} [b] \overset{l'}{\doublebarwedge} k; \quad \text{also} \quad [a] \overset{l}{\doublebarwedge} [c] \overset{l''}{\doublebarwedge} k.$$

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* Projective construction for certain algebraic curves, this MONTHLY, April 1940.

† Plane Curves of the Third Order, H. S. White, p. 108, 1925.

‡ On the straight line construction of unicursal cubics, W. H. Bunch, this MONTHLY, Vol. 42, 1935, pp. 74–80.

Therefore, $[a] \barwedge k$, and since homologous pairs of $[a]$ and k intersect in the points P, then P is the locus of the intersection of homologous lines of a line conic and a projective pencil of lines. That all the points P define a cubic uniquely will be shown by a synthetic and an analytic proof.

2-a. *A synthetic proof.* Consider a point conic K with M any point on it such that the pencil $[m]$ is perspective with K and also projective with another pencil $[a]$ with center A not on K. Then $[a]$ and $[m]$ generate a point conic K' which is projective with K. Thus A is on K' and M is an intersection of K and K'. To each point on K there corresponds one and only one point on K' and such homologous pairs coincide only in the intersections of the two conics. There are at most four points common to K and K', let us say at M, M_1, M_2, M_3. Rays AM_1, AM_2, AM_3, are homologous to rays MM_1, MM_2, MM_3. The ray on M which corresponds to AM on A is a tangent to K' and the point on K homologous to M on K' does not coincide with M. Therefore, there are at most three lines of $[a]$ which pass through their corresponding points on K.

A polarity with respect to K transforms the points of K into a line conic k, and the rays of $[a]$ into a range of points $[A']$ on the polar of A. Thus $[a]$ is projective with k, and $[A']$ is projective with k. It follows that not more than three points of $[A']$ will lie on their associate lines of k and these three points are on the locus of P. Now consider any line of the plane such that its points are in projective correspondence with $[A']$. There are then at most three points of this line on the homologous lines of k and these common points are also on the required locus. Therefore, the locus is a point cubic.* That the curve is nodal is shown by the fact that the polar of A with respect to K cuts K in two distinct real points or in no real points, since A is not on K. These intersections are double points of the polar transformation. Hence two distinct lines of k or no such lines pass through A and are intersected in A by homologous rays of $[a]$. Therefore A is an acnode or crunode and is the only possible singularity of the cubic, since a straight line intersects a cubic in at most three real points.

2-b. *The equation of the cubic.* The analytic proof that the construction yields a cubic is conveniently demonstrated by the use of homogeneous point coordinates (x_1, x_2, x_3), with parameters in non-homogeneous form. In what follows it will be understood that the symbol a_ix_i represents a summation with $i=1, 2, 3$.

Let the vertices of the triangle of reference be on the points A, B, C, and let these points be defined by the homogeneous point coordinates (0, 0, 1), (1, 0, 0), and (0, 1, 0) respectively. The various base elements are given by the following equations:

$$a_ix_i = 0, \text{ the line } l = 0; \tag{1}$$

$$b_ix_i = 0, \text{ the line } l' = 0; \tag{2}$$

$$c_ix_i = 0, \text{ the line } l'' = 0; \tag{3}$$

* See problem 7, p. 217, Projective Geometry, Veblen and Young, Vol. 1.

(4) $x_1 - hx_2 = 0$, the pencil $[a]$ on A;

(5) $x_2 - mx_3 = 0$, the pencil $[b]$ on B;

(6) $x_1 - nx_3 = 0$, the pencil $[c]$ on C.

The ranges $[L']$ and $[L'']$, in terms of the parameters are:

(7) $L'[-(b_2m + b_3), b_1m, b_1]$ from (2) and (5);

(8) $L''[nc_2, -(c_1n + c_3), c_2]$ from (3) and (6).

The equation of the line conic k in parametric point coordinates, may be expressed in the usual determinant form:

(9) $$| x_i, L', L'' | = 0.$$

Now since $[a]$, $[b]$, $[c]$, are mutually perspective on axis l, that is, homologous rays of the three pencils are concurrent, the determinant of the coefficients of (4), (5), (6), vanishes, and this fact gives the relation among the parameters:

(10) $$n = hm.$$

The projective correspondence which brings L' into L'', also transforms m into n. Since for each point P corresponding rays of $[b]$ and $[c]$ meet on l, it follows that

(11) $a_1n + a_2m + a_3 = 0$, from (1), (5), and (6).

(12) $(a_1h + a_2)m + a_3 = 0$, from (10), (11).

Solving for m out of (12), and noting that $h = x_1/x_2$ as given by (4), we obtain

(13) $m = -\dfrac{a_3x_2}{p}$, where p is defined below.

(14) $n = -\dfrac{a_3x_1}{p}$, using (10) and (13).

Now substituting (13) and (14) into (9) and expanding, we obtain, after grouping terms, the equation of a cubic:

(15) $$f(x_1, x_2, x_3) = 0, \text{ or}$$
$$p^2s - a_3pr + aa_3^2t = 0, \text{ where}$$
$$p = a_1x_1 + a_2x_2,$$
$$r = b_1c_1x_1^2 + b_2c_2x_2^2 + 2b_1c_2x_1x_2 + b_3c_1x_1x_3 + b_2c_3x_2x_3,$$
$$s = b_1c_3x_1 + b_3c_2x_2 + b_3c_3x_3,$$
$$t = x_1x_2x_3, \qquad a = b_2c_1 - b_1c_2.$$

The general cubic has nine essential constants, whereas the equation (15) has but six such coefficients. This indicates a cubic with specified node. If the

node is not specified by the particular orientation of the base elements given above, the most general equation of $f=0$ contains eight essential constants, which shows that (15) is a nodal cubic.

2-c. *Some fundamental points on f.* Certain points of f can be easily found from the construction and equation. These will be of use in the problems to follow. See Fig. 1.

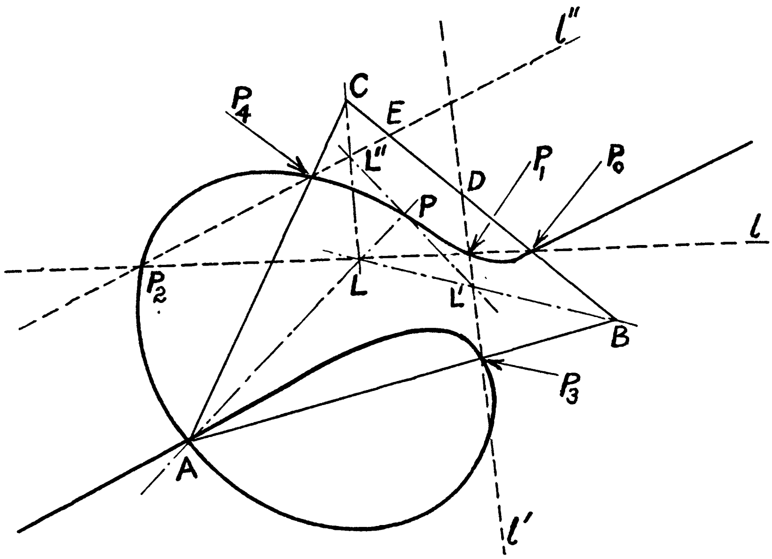


FIG. 1

P_1 is the intersection of l and l', denoted by l, l'; P_2 is on l, l'' and P_3 on l', AB; l'', AC intersect on P_4, and l, BC on P_0. By means of (15) it is easily shown that $p=0$ is the line through A and P_0; $s=0$ is a line P_3P_4; $r=0$ a conic on A, P_3, P_4; $t=0$ is the improper cubic composed of the three axes of the reference triangle.

That A is the node can be shown by the fact that any line through A has two coincident intersections with f. It is not difficult to prove that all the derivatives $\partial f/\partial x_i$ vanish at A, and therefore A is the double point.

3. The inverse of the general construction. The methods for finding the base elements for the construction of f under specified conditions can now be tabulated. To make the procedure as general as possible, the given simple points will be chosen so that no three are collinear, although this restriction is not necessary. The specification of three collinear points in certain order may cause the construction to break down.

Since the cubic is nodal, eight ordinary points or equivalent conditions may be specified to determine it uniquely. If the node is specified, six additional conditions must be imposed. Thus, in addition to the fundamental points men-

tioned, use will be made of P_5 which is located by the ray joining A with l, P_3P_4, and P_6 on a ray through A and P_4P_5, l. P_7 and P_8 are on rays AD and AE, where D and E are the junctions of BC, l' and BC, l'' respectively. The following constructions are readily understood when the general construction is retraced in the reverse order.

3-a. *Given points* P_n ($n=1$–8 *inclusive*). P_1P_2, P_1P_3, P_2P_4 are taken as l, l', l'' respectively. P_3P_4 and P_4P_5 cut l in L_5 and L_6. Then P_5L_5 and P_6L_6 meet in A. AP_7 and AP_8 cut l' and l'' in D and E. DE will intersect AP_3 in B and AP_4 in C, and the construction for any number of points on the curve is uniquely determined.

3-b. *Given* P_n (*with* $n=7$ *deleted*), *and tangent* p_1 *at point* P_1. The tangent p_1 is the line P_1Q_1, where AB and l intersect in L_1. and AC cuts L_1D in Q_1. This result was not obtained synthetically but while working on an analytical phase of the problem. The author had on hand the analytical expressons for many points and lines of the construction figure including the equations of certain tangents. By comparing coefficients, the above construction was arrived at. Thus the method of solution was tentative at best. The analytical verification is a little tedious, but not at all difficult. If P_1 is taken on (1, 1, 1) there will be no loss in generality and we then have

$$a_1 + a_2 + a_3 = 0 \quad \text{and} \quad b_1 + b_2 + b_3 = 0,$$

equations of l and l' may be written

$$\begin{aligned} a_1x_1 - (a_1 + a_3)x_2 + a_3x_3 &= 0, \text{ and} \\ b_1x_1 + b_2x_2 - (b_1 + b_2)x_3 &= 0, \text{ respectively.} \end{aligned}$$

The equation of p_1 as constructed above is then

$$(a_1b_2 + a_3b_1)x_1 - a_1b_2x_2 - a_3b_1x_3 = 0.$$

That this is the equation of the tangent to f at P_1 may be shown by the general equation for a tangent:

$$x_i \frac{\partial f}{\partial x_i} = 0,$$

where the partial derivatives are evaluated for the point P_1, or (1, 1, 1). This will be relatively simple if (15) is differentiated as given in its unexpanded form and subsituting coefficients $a_2=-(a_1+a_3)$ and $b_3=-(b_1+b_2)$, in the final reduction.

Base lines l, l', l'', and point A are located as in (3-a). AP_3 and AP_4 meet l and p_1 in L_1 and Q_1 respectively. Q_1L_1 cuts l' in D and AP_8 cuts l'' in E. Then DE meets AL_1 in B and AQ_1 in C, and the construction is complete. The process, however, may be simplified by the fact that l' is the tangent at P_1 if C is on l'. In this case C coincides with D.

3-c. *Given P_n ($n=1$–6 inclusive), and p_1, p_2 the tangents on P_1 and P_2 respectively.* The tangent p_2 was arrived at directly by associating P_2 with P_1 in the projective relations involved. By a process similar to that used in 3-b it may be shown that p_2 is the line Q_2P_2, where l cuts AC in L_2 and EL_2 meets AB in Q_2.

A, l, l', l'', are located as in 3-a and 3-b. Q_1L_1 and Q_2L_2 meet l' and l'' in D and E respectively. ED then determines B and C. This construction may also be specialized by taking B and C on l'' and l', in which case p_1 and p_2 will coincide with l' and l'' respectively.

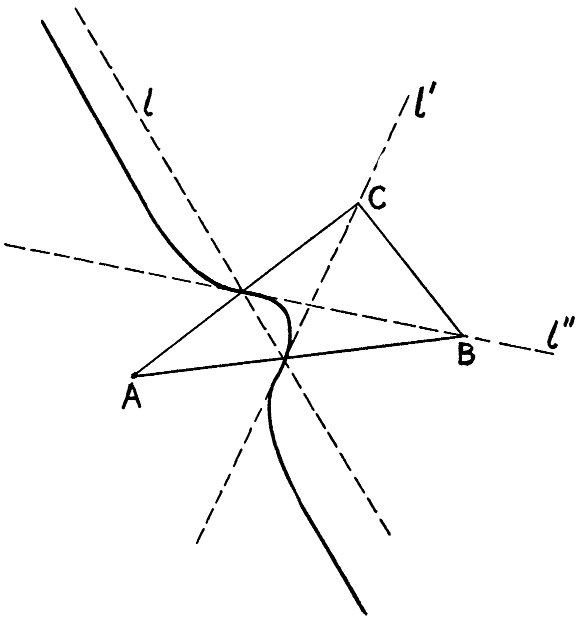


FIG. 2

3-d. *Given the node A and six simple points P_n ($n=1, 2, 3, 4, 7, 8$).* P_1P_2, P_1P_3, P_2P_4, will determine l, l', l'', respectively as before. AP_7 cuts l' in D and AP_8 meets l'' in E. DE then will intersect AP_3 in B and AP_4 in C.

3-e. *Given the node A with real tangents a' and a'', and four ordinary points P_1, P_2, P_3, P_4.* It will be noted that the two lines of the conic k which pass through A determine the real tangents on the node. Let the points P_1, P_2, P_3, P_4, locate the lines l, l', l'', as in the previous cases. Let A be the intersection of any two lines of k, these cutting l' in L_1' and L_1'', and l'' in L_2' and L_2'' respectively. The given tangents a' and a'' cut l in M' and M''. The junction of $L_1'M'$ with $L_1''M''$ is point B. In a similar manner $L_2'M'$ and $L_2''M''$ meet in point C.

It is easily shown that if l' and l'' intersect on A, then AB and AC are real nodal tangents. This fact makes it possible to simplify the construction. In this case let the distinct given points be P_1, P_2, P', P'', where P' and P'' are any two points other than those designated as fundamental. P_1P_2, AP_1, AP_2, deter-

mine l, l', l'' respectively. $P'A$ cuts l in Q'; $Q'P''$ meets l' in L' and l'' in L''. $P''A$ cuts l in Q''; then $Q''L'$ meets a' on B and $Q''L''$ intersects a'' on C.

This case is evidently subject to variation, since when l' is on A, then AB is a tangent on A. Also if l'' is on A, then AC is a tangent on A.

4. Miscellaneous. The node may be made an isolated point by the specification of two real inflections, or by special orientation of the base elements. If l, l', and AB are concurrent, then l' is a tangent on P_1; similarly, if l, l'', and AC are concurrent, l'' is tangent on P_2. If both of these conditions are imposed, the double point A will be, in general, isolated (see Fig. 2). Now if to these restrictions we add the conditions that l' is on C and l'' on B, it will be found that the Hessian of f vanishes at P_1 and at P_2. Thus, P_1 and P_2 are inflections and l', l'', the corresponding inflectional tangents. Many other specializations, exist.

Constructions for conics will arise from the foregoing problems if the cubic is degenerate. If the base lines are chosen so that l, l', AB; l, l'', AC; and l', l'', BC, are three points, the cubic will break off into the line $p=0$ and a conic through A, P_1, P_2, with tangents l' and l'' on P_1 and P_2 respectively. Thus point l', l'' will be a pole and the line $s=0$ its polar with respect to the conic. The equation of this conic is $ps-a_3r=0$.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A NOTE ON THE RIEMANN INTEGRAL

H. E. ROBBINS, Post Graduate School, U. S. Naval Academy

The following theorem, which arose during an investigation of integration by change of variable, is believed to provide an interesting generalization of the ordinary definition of the Riemann integral of a continuous function.

THEOREM. *Let $f(x)$ be a continuous function and let C be a constant given in advance. Choose a sequence of numbers (not necessarily increasing)*

$$a = x_0,\ x_1,\ x_2,\ \cdots,\ x_n = b$$

such that

$$\sum \left| x_i - x_{i-1} \right| < C.$$

Set $\max \left| x_i - x_{i-1} \right| = \delta_n$, *and form the sum*

$$S_n = \sum f(\bar{x}_i)(x_i - x_{i-1}),$$

where $\bar{x}_i$ is any point in the closed interval whose endpoints are x_i and x_{i-1}. Then for

any sequence of such sums in which $\delta_n \to 0$, *we have*

$$\lim S_n = \int_a^b f(x)dx,$$

where the integral on the right is the ordinary Riemann integral.

Remark. In this theorem C may be arbitrarily large but must be fixed for all sums in the sequence. Without this restriction the theorem is false, as may easily be shown.

We proceed to prove the theorem. Let the maximum oscillation of $f(x)$ in any interval of the subdivision be equal to ϵ. Consider the set of all points

$$a = y_0, y_1, y_2, \cdots, y_m = b,$$

consisting of the x_i arranged in increasing magnitude without repetition. In each interval $[y_{k-1}, y_k]$ choose a definite point $\bar{y}_k$. Suppose the interval $[x_{i-1}, x_i]$ contains the points $y_{k-1}, y_k, y_{k+1}, \cdots, y_{k+r}$ as endpoints or interior points. Replace the term

$$f(\bar{x}_i)(x_i - x_{i-1}) \tag{1}$$

by the expression

$$f(\bar{y}_k)(y_k - y_{k-1}) + \cdots + f(\bar{y}_{k+r})(y_{k+r} - y_{k+r-1}) \tag{2}$$

if $x_i > x_{i-1}$, and by the negative of this expression if $x_i < x_{i-1}$. The difference between the term (1) and the corresponding expression (2) will be less in magnitude than $\epsilon|x_i - x_{i-1}|$. Carrying out the replacement for every term of S_n, we obtain a new sum S_n' differing from S_n by less than

$$\epsilon \sum | x_i - x_{i-1} | < C\epsilon.$$

In S_n' collect all terms

$$\pm f(\bar{y}_k)(y_k - y_{k-1})$$

for a fixed k. The result will be simply

$$f(\bar{y}_k)(y_k - y_{k-1}),$$

since the interval $[y_{k-1}, y_k]$ will be traversed some number p times from left to right and $(p-1)$ times from right to left. Since this is true for all k, S_r' is an ordinary Riemann sum for $f(x)$, and hence

$$\lim S_n' = \int_a^b f(x)dx.$$

But

$$\lim (S_n - S_n') = 0,$$

since $|S_n - S_n'| < C\epsilon$ and $\lim \epsilon = 0$. This completes the proof.

AN EXAMPLE ON DOUBLE SERIES

J. E. BROCK, Washington University

It is known that the convergence of a double series $\sum_{i=1,j=1}^{\infty,\infty} a_{ij}$ *i.e.*, the existence of the double limit, $\lim_{m\to\infty,n\to\infty} S_{mn}$ (where $S_{mn}=\sum_{i=1,j=1}^{m,n} a_{ij}$), does not imply the convergence of the series $\sum_{i=1}^{\infty} a_{ij}$ or $\sum_{j=1}^{\infty} a_{ij}$. Bromwich* cites an example of a convergent double series, a finite number of whose rows and columns diverge. It might be of interest, particularly for instructional purposes, to have a simple example in which the series $\sum_{i=1}^{\infty} a_{ij}$ and $\sum_{j=1}^{\infty} a_{ij}$ diverge for *all* j and i.

Consider the double series $\sum_{i=1,j=1}^{\infty,\infty} a_{ij}$ where $a_{ij}=(-1)^j b_i+(-1)^i b_j$ and $\sum_{n=1}^{\infty} b_n$ converges and has the sum 0. It may easily be seen that the double limit exists, for the partial sum $\sum_{i=1,j=1}^{m,n} a_{ij}=\sum_{i=1,j=1}^{m,n}(-1)^j b_i+\sum_{i=1,j=1}^{m,n}(-1)^i b_j$ and the sums on the right are either equal to zero or to $-\sum_{i=1}^{m} b_i$ or to $-\sum_{j=1}^{n} b_j$. Since $\sum_{n=1}^{\infty} b_n=0$, these latter sums approach zero as a limit as $m\to\infty$, $n\to\infty$.

However, $\sum_{i=1}^{\infty} a_{ij}$ and $\sum_{j=1}^{\infty} a_{ij}$ both diverge, since the general term a_{ij} does not approach zero as a limit, when for fixed i, $j\to\infty$, or when for fixed j, $i\to\infty$.

FUNCTIONS NOT FORMULAS FOR PRIMES

IRVING REINER, Brooklyn, New York

A function $F(x)$ is said to be a formula for primes if $F(x)$ is a prime for every positive integral value of x. We shall show that certain types of functions cannot be formulas for primes.

LEMMA. *If $f(x)$ and $g(x)$ are polynomials with integral coefficients and positive leading coefficients, then for all sufficiently large integral values of k*

$$f(a+kp(p-1))^{g(a+kp(p-1))} \equiv f(a)^{g(a)} \pmod{p},$$

where p is any prime and a is any integer such that $g(a)>0$.

Proof. We have $f(a+kp(p-1))\equiv f(a) \pmod{p}$, and $g(a+kp(p-1))\equiv g(a) \pmod{p-1}$. Therefore we may write

$$f(a+kp(p-1))^{g(a+kp(p-1))} \equiv f(a)^{g(a)+M(p-1)} \pmod{p},$$

where M is positive for sufficiently large k. However, if $f(a)\equiv 0 \pmod{p}$, then $f(a)^{M(p-1)}\equiv 0 \pmod{p}$, and if $f(a)\not\equiv 0 \pmod{p}$, then $f(a)^{M(p-1)}\equiv 1 \pmod{p}$ by Fermat's theorem. In either case

$$f(a+kp(p-1))^{g(a)+M(p-1)} \equiv f(a)^{g(a)} \pmod{p},$$

and the lemma is proved.

THEOREM 1. *If $f_i(x)$, $g_i(x)$ $(i=1,\cdots,n)$ are polynomials with integral coefficients and positive leading coefficients, the following is not a formula for primes:*

$$F(x)=\sum_{i=1}^{n} f_i(x)^{g_i(x)}.$$

* Bromwich, T. J. I'A., Theory of Infinite Series, Macmillan (1908), p. 90.

Proof. Otherwise, suppose that $F(x)$ is a formula for primes; let a be an integer such that all of the $g_i(a)$ are positive. Then we have for sufficiently large integral k (by the Lemma)

$$f_i(a + kp(p - 1))^{g_i(a+kp(p-1))} \equiv f_i(a)^{g_i(a)} \pmod{p},$$

where $p = F(a)$. Hence we have

$$F(a + kp(p - 1)) \equiv \sum_{i=1}^{n} f_i(a)^{g_i(a)} \pmod{p}.$$

Therefore $F(a+kp(p-1))$ is divisible by p; but we assumed that $F(x)$ is prime for all positive integral x. Hence $F(a+kp(p-1))$ is either p or $-p$ for all sufficiently great k. This contradicts the fact that $\lim_{x\to\infty} F(x) = \infty$, which is true because the leading coefficients are positive. Hence, $F(x)$ is not a formula for primes.

Example. If a and b are integers, $a>1$, then a^x+b is not a formula for primes.

THEOREM 2. *Let $F(x)$ be defined as*

$$F(x) = \sum_{i=1}^{n} f_i(x)^{[g_i(x)^{h_i(x)}]},$$

where $f_i(x)$, $g_i(x)$, $h_i(x)$ $(i=1, \cdots, n)$ are polynomials with integral coefficients and positive leading coefficients. Then $F(x)$ is not a formula for primes provided there exists an integer a such that:

(1) *All the $g_i(a)$ and $h_i(a)$ are positive;*

(2) *If any $g_i(a)$ has a factor r_i in common with $q = F(a)-1 = p-1$, then any prime divisor of r_i occurs as a factor of q to a power less than or equal to $h_i(a)$.*

Proof. Assume that $F(x)$ is a formula for primes, and let a be an integer whose existence is demanded by the hypothesis. We shall use the above notation. Let the prime factorizations of $q = q_1^{u_1}q_2^{u_2} \cdots q_s^{u_s}$ and $r_i = q_1^{v_1}q_2^{v_2} \cdots q_t^{v_t}$, where $t \leqq s$ and each $v_i > 0$. Clearly this places no restriction on the q_i. Choose $m_i = M_i q\phi(q)$, where $\phi(q)$ is the Euler ϕ-function, such that $g_i(a+m_i)$ and $h_i(a+m_i)$ are positive. Now

$$g_i(a + m_i)^{h_i(a+m_i)} \equiv g_i(a)^{h_i(a)+M'\phi(q)} \pmod{q},$$

where M' is positive for proper choice of M_i. Let $g_i(a) = r_i t_i$, where t_i is prime to q. Then since $h_i(a) \geqq \max\cdot(u_1, u_2, \cdots, u_t)$ and since the Euler function is $\phi(q) = \phi(q_1^{u_1} \cdots q_t^{u_t})\ \phi(q_{t+1}^{u_{t+1}} \cdots q_s^{u_s})$, we have

$$r_i^{M'\phi(q)+h_i(a)} \equiv r_i^{h_i(a)} \pmod{q}.$$

Also $t_i^{\phi(q)} \equiv 1 \pmod{q}$. Therefore

$$g_i(a + m_i)^{h_i(a+m_i)} \equiv g_i(a)^{h_i(a)} \pmod{q}.$$

Now let $m = Mpq\phi(q)$. Then by the reasoning of Theorem 1, for all sufficiently

large values of M, $F(a+m)$ equals p or $-p$. Hence, we again have a contradiction as in Theorem 1, and $F(x)$ is not a formula for primes.

Example 1. $2^{2^x}+k$ is not a formula for primes for integral k.

Example 2. $F(x)=(x+1)^{x^x}+1$ is not a formula for primes. Proof, let $a=1$.

Note. This method is easily extended to functions of the type

$$f(x)^{g(x)^{h(x)^{k(x)}}}$$

under conditions similar to those of Theorem 2.

ON P_r-MATRICES

H. SCHWERDTFEGER, University of Adelaide

The notion of P_r-matrix which has been introduced in a preceding note* can further be restricted usefully in the following way when it still covers all ϕ-symmetric matrices: Let again ϕ be an involutory automorphism or antimorphism of the infinite field $\mathfrak{F}$ containing the matrix elements. For any α in $\mathfrak{F}$ the element $\phi(1)\cdot\phi(\alpha)$ may be called the ϕ-conjugate of α. An n-rowed square matrix A with elements in $\mathfrak{F}$ is said to be a ϕP_r-matrix if it is a P_r-matrix ($r\leqq n$) and the linear relations existing between its rows arise from those between the equally numerated columns by taking the ϕ-conjugate elements of the coefficients of the column relations in corresponding places. Thus when $a^{(i_1)}, \cdots, a^{(i_r)}$ are linearly independent columns of A, then $a'_{(i_1)}, \cdots, a'_{(i_r)}$ are linearly independent rows of A, and from

$$a^{(\nu)} = \sum_{\rho=1}^{r} \lambda_\rho^\nu a^{(i_\rho)} \qquad (\nu = 1, \cdots, n)$$

follows

$$a'_{(\nu)} = \phi(1)\cdot\sum_{\rho=1}^{r}\phi(\lambda_\rho^\nu)a'_{(i_\rho)}.$$

If $r=n$ all λ_ρ^ν are zero except for $i_\rho=\nu$. Thus every regular n-matrix in $\mathfrak{F}$ is a ϕP_n-matrix. A singular ϕP_r-matrix ($r<n$), however, is ϕ-congruent with

$$\begin{pmatrix} A_r & 0 \\ 0 & 0 \end{pmatrix} \tag{1}$$

where A_r is an r-rowed principal submatrix of rank r in A. Here A_r can be any regular r-matrix. Hence the ϕP_r-matrices form the most general class of matrices A of rank r which are ϕ-congruent with a matrix of type (1) where A_r is any regular principal submatrix of A.

* H. Schwerdtfeger, On generalized hermitian matrices, this MONTHLY, vol. 49, 1942, pp. 181–184. The content of §§1–2 of this note is supposed to be known here. (On p. 182, in the last formula, A_ϕ^2 has to be replaced by $A_{\phi 2}$.)

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

SOLVING A RIGHT TRIANGLE WITHOUT TABLES

J. S. FRAME, Michigan State College

Given the three sides a, b, c of a right triangle ABC, in which $C=90°$ and $a<b$, it is possible to express the smallest angle A in terms of the three sides by the approximate formula

$$A \cong \left(\frac{a}{b+2c}\right)(172°), \tag{1}$$

in which the error is less than one minute for angles less than 37°, and is still fairly small for all angles less than 45°, as the following table shows:

(2)

A in degrees	0°	10°	20°	30°	33°	36°	39°	42°	45°
Error in minutes	0′	+.4′	+.7′	+.4′	+.04′	−.5′	−1.4′	−2.7′	−4.4′

For example, to approximate the smallest angle in the familiar 3, 4, 5 triangle, we should have

$$A = \left(\frac{3}{14}\right)(172°) = \frac{258°}{7} = 36°51.4'. \tag{3}$$

The true value from tables is $A=36°52.2'$. Thus the accuracy is better than could be obtained from a slide rule, and does not require trigonometric scales.

The following considerations led to the discovery of this approximate formula. It can be seen, either from tables or from a consideration of power series, that the radian measure of a small angle lies approximately one-third of the way from the sine to the tangent. However, this approximation is obviously not valid for angles near 90°, since the tangent becomes infinite. Suppose therefore that we try to use not the arithmetic but the harmonic average of $\sin\theta$ and $\tan\theta$, weighted two to one, and see if it is a good approximation to the radian measure of an angle θ. We then have

$$\frac{3}{\theta\ (\text{radians})} = \frac{1}{\sin\theta} + \frac{1}{\sin\theta} + \frac{1}{\tan\theta}. \tag{4}$$

For $\theta=\pi/6$ radians, we have the approximation

(4a) $\qquad 18/\pi = 5.732$, instead of the true value 5.730,

whereas for $\theta=\pi/2$ radians, we have

(4b) $6/\pi = 2$, instead of the true value 1.910.

If we let $A = (\theta/\pi)\ 180°$, so that A is measured in degrees, and if we write $\sin A = a/c$, $\tan A = a/b$, then formula (4) becomes

$$\frac{540°}{\pi A \text{ (degrees)}} = \frac{c}{a} + \frac{c}{a} + \frac{b}{a} = \frac{b + 2c}{a} \tag{5}$$

or

$$A \text{ (degrees)} = \left(\frac{a}{b + 2c}\right)\frac{540°}{\pi} = \left(\frac{a}{b + 2c}\right)(171.88°). \tag{6}$$

This formula is extremely accurate for angles less than 20°, but always gives too small a value. We therefore increase the numerical coefficient slightly from 171.88° to 172°. This is easier to work with numerically, and gives a better fit above 30°, without greatly increasing the error for angles less than 30°. Thus we arrive at formula (1).

The correction for formula (1) which is the negative of the error in table (2) is very closely given by the expression:

(7) **Correction for formula (1) in minutes** $= -1.2'(A/30°) + 0.8'(A/30°)$.[5]

It is suggested that this formula (1) might be used to advantage on slide rules having no trigonometric scales, for the purpose of solving right triangles when two sides are known, if the third side is first obtained by the Pythagorean theorem. (See this MONTHLY, vol. 50, p. 55.)

CLUB REPORTS 1942-43

Harvard Mathematical Club, Harvard University

The following talks were presented at the meetings of the club in 1942–43:

The reversibility paradox of hydrodynamics, by Professor Garrett Birkhoff.
Picturing surfaces in four dimensions, by Professor Hassler Whitney.
Mathematics in Mexico and South America, by Professor G. D. Birkhoff.
A proof of Stone's representation theorem, by Leon Kruger.
Construction of measures, by Dr. Edwin Hewitt.
Carrolean Sorites, by Robert Buck.
A random series, by Dr. Harry Pollard.
Projective geometry, by Lindley Burton.
Graduating statistical series, by Robert Hoskins.
Groups as algebras of a single operation, by Jeremiah Certaine.
Hilbert's seventh problem, by Michael Norris.
Lebesgue integration, by Matthew Gaffney, Jr.

Recipients of the Robert Fletcher Rogers Prizes for the best student talks were Mr. Gaffney ($35) and Mr. Burton ($15). Officers for 1942–43 were: President, Michael J. Norris; Vice-President, Robert H. Hoskins; Secretary, Joseph A. Zilber; Treasurer, Lindley Burton; Faculty Adviser, Professor G. D. Birkhoff.

Mathematics Club, University of Kansas

The Club's meetings for the year 1942–43 were divided into two parts. During the first semester the meetings featured talks by faculty members, while during the second semester the speeches were given by undergraduates. The annual fall picnic opened the Club's program, which ended with the annual spring picnic at which prizes for the best speeches were awarded and the officers for the coming year were announced. Meetings were held every two weeks during the school year. The faculty speeches were:

Navigation, by Professor N. W. Storer.

Cryptography, by Professor G. W. Smith.

Probability in artillery fire, by Captain E. E. Baker, CAC.

Some strange phenomena of quantum mechanics, by Dr. L. N. Liebermann.

A problem in convex sets, by Professor G. B. Price.

The undergraduate speeches included the following:

Some problems in naval gunnery, by Harwood Kolsky.

Squaring a circle, doubling a cube, trisecting an angle, by Jean Bartz.

Electricity and imaginary numbers, by Howard Barnett.

Curve fitting, by John Yarnell.

Ancient mathematical puzzles, by Mary Steel.

A mathematical approach to physics, by Rachel Ragel.

Prizes were offered by members of the mathematics department to the three best student speakers. To Harwood Kolsky was awarded the first prize: a copy of *Men of Mathematics* by E. T. Bell, donated by Dean E. B. Stouffer. To Howard Barnett went the second prize: a copy of *What is Mathematics* by Courant and Robbins, donated by Professor J. J. Wheeler. To Mary Steel was given the third prize: a copy of *Amusements in Mathematics* by H. E. Dudeney, donated by Professor P. O. Bell. The officers for this past year have been: President, Harwood Kolsky; Vice-President, Jean Bartz; Secretary-Treasurer, John Yarnell; Social Chairman, Virginia Stephenson and Rachel Ragel; Faculty Adviser, Professor P. O. Bell.

Mathematics Club, Iowa State College

The Club held seven meetings during the school year.

Integral Calculus, a movie, showed pictorially how volume enclosed by surfaces is found by integration, and was a valuable aid in clarifying the integration process for the student of integral calculus.

The operation of statistical machines was the subject of a demonstration meeting which focused attention on the place of statistics in a nation at war. At several of the meetings word problems were given which illustrated the topics discussed. Those present participated actively in solving these problems. The meetings were well attended. The Program Committee consisted of Glee Barth, Joe Elliot, William Slaichert, Eleanor Hoeflin and Marian Carlin. The Faculty Adviser was Mr. Fred Robertson.

Pi Mu Epsilon, University of Nebraska

Less leisure time than usual for most of the students resulted in fewer meetings of our chapter this year. Activities for the year included three program meetings, a social meeting, the annual initiation banquet, and a spring picnic and initiation. Following the usual custom, competitive examinations in analytical geometry and in calculus were held. The winners, Walter Koenig and Norman Zabel, were each awarded ten dollars.

At the beginning of the second semester, a new plan was started to stimulate interest in mathematics. Each week five problems of varying difficulty were proposed for solution. These problems were posted on the bulletin board under their proper heading: *i.e.*, calculus, analytics, trigonometry, algebra, or general. The solutions were discussed at the regular meetings.

Programs included the following topics:

La Place transformation, by Lloyd Jackson.

The solution of problems in electrical circuits and transmission lines by the La Place transformation, by J. R. Parker.

The Hatchet planimeter, by Kotaro Murai.

Officers were as follows: Director, Herman Krueger; Secretary, Marcia Beckman; Treasurer, William Ruyle.

Mathematics Club, Wayne University

The Mathematics Club at Wayne University held regular meetings throughout the year on the average of one every two weeks. Its aim is to assemble all students who are interested in mathematics so that they can get acquainted. This was accomplished by our meetings which consisted of academic papers and social affairs. The outstanding topics discussed were:

The impossibility of the trisection of an angle, by Jane Cronin.

Courses in mathematics, by Dr. M. Coral.

Mathematics in map making, by Robert Walton.

Casting out nines, by Aline Hochberg.

The cube root of a number, by Leonard Antel.

On May 23, 1943, we held our annual picnic to which the mathematics professors and their families were invited. A sack race was held with professors and students competing for a pound of coffee—a student won. With too much to eat and lots of fun, the picnic ended with everyone sitting around a campfire singing college songs. Besides the annual picnic, the Club had a Christmas tea, an informal evening at the home of the faculty adviser, and an Easter party at the home of the new president. The officers for 1942–43 were: President, Evelyn Kurz; Vice-President, Robert Walton; Secretary-Treasurer, Myrtle Keryluk. Those for 1943 are: President, Myrtle Keryluk; Vice-President, Dick Kuba; Secretary-Treasurer, Aline Hochberg; Publicity Chairman, Natalie Costrell; Faculty Adviser, Dr. K. W. Folley.

Pi Mu Epsilon, University of Wisconsin

A lantern demonstration of plane and twisted curves in space was presented by Dr. Florence E. Allen and Professor Paul L. Trump at the opening meeting. Later meetings included the following talks:

Use of matrices in solving linear equations, by Dr. R. H. Brucks.

Mathematical problems pertaining to the unsteady state of heat flow, by Mr. George Thodos.

Boolean algebra, by Mr. Robert Simpson.

The theory of counting, by Professor L. W. Cohen.

Mathematics in European universities, by Professor S. M. Ulam.

A Christmas party was held, and in January fifty new members were admitted at an initiation banquet. A very interesting paper was read at this meeting, entitled

Sir Isaac Newton, by Professor R. E. Langer.

The winners of the competitive mathematical examination sponsored by the chapter were Warren Young, first prize, and David Herwitz, second. Officers for the year were: President, Arne V. Larson; Vice-President, Anne Braun; Treasurer, Henry Rogers; Secretary, Betty Lohr; Faculty Adviser, Professor J. F. Kenney.

Kappa Mu Epsilon, Upsala College

Nine meetings were held throughout the year. At seven of these meetings papers and reports were given by the members and at two meetings the chapter had outside speakers. The topics discussed by the members were:

Why teach mathematics? by Phyllis Gustafson, the president.

A proof from Weierstrass, by Joseph Prieto.

The divine proportion in classic art, by Elizabeth Ebel.

Sources of Euclid, by Marjorie Wolfe.

The history of pi, by Zelda Meisel.

Statistical research, by Hirsch Geller.

At the special midyear meeting the chapter heard an instructive lecture with many visual illustrations from the field of nature and art, entitled

Dynamic symmetry, by Professor William O'Brien of Newark, New Jersey.

On June 3 the society had its annual banquet and initiation of new members. The guest speaker was Mr. Michael McGreal, Principal of the Barringer High School, Newark, who discussed certain problems of teaching high school mathematics. Also at this meeting the officers for the year 1943–44 were installed. They are: President Thales, Zelda Meisel; Vice-President Apollonius, Joseph Prieto; Secretary Abel, Elizabeth Ebel; Treasurer Fibonacci, Lillian Meisel; Historian Gauss, Marjorie Wolfe; Corresponding Secretary Descartes, M. A. Nordgaard.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Mathematics Dictionary. By Glenn James and R. C. James. Van Nuys, California, The Digest Press, 1942. 5+259+22 pages. $3.00.

Mathematics Dictionary. Revised edition., 1943, 8+274+46 pages. $3.00.

The *Mathematics Dictionary* is a handy reference book of definitions, formulas, and tables, covering mathematical topics in arithmetic, algebra, Euclidean and projective geometry, plane and spherical trigonometry, mathematics of finance, statistics, analytic geometry and calculus. Definitions are couched in language suited to the mathematical maturity of the reader who might require them. No attempt is made to cover mathematical terms which arise in advanced mathematical studies beyond the college level, although a few of these words are included. Related concepts are defined together, and illustrative examples are given which are very helpful in clarifying the meaning of technical terms. Many of the definitions in analytic geometry are illustrated by figures which, with one or two exceptions (*e.g.*, the cubic curve $y=kx^3$), are accurately and clearly drawn. The appendix includes tables of logarithms, trigonometric functions, compound interest and annuities, mortality, squares and cubes (and a partial list of square roots and cube roots), denominate numbers, differential and integral formulas, and a list of mathematical symbols.

The revised edition shows a marked improvement over the first edition, because of its clearer typography, because of its more adequate tables in the appendix, because of many improved definitions and illustrative examples, and because of the elimination of many errors. In the first edition the reviewer noted over a hundred errors, varying from misprints to inadequate or faulty definitions. About half of these have been corrected in the revised edition.

Of the errors in the revised edition, the following appear to be the most serious.

(1) The word *harmonic*, which has so many connotations in mathematics, seems to be unfortunate in the treatment it has received. Although La Place's equation is defined, no mention is made of the fact that its solutions are called *harmonic functions*. Instead, a *harmonic function* is said to be the same as a *compound harmonic function* and is defined to be a trigonometric polynomial. Furthermore, *harmonic conjugates with respect to two points* are defined to be "*the two points which divide the line through the two points internally and externally in the same ratio*", rather than "any two points which divide . . . ."

(2) *Stirling's formula, or series*, is defined to be the *same as Maclaurin's*

series. No mention is made of the important approximation $n! \sim (2\pi n)^{1/2}(n/e)^n$, which is commonly called *Stirling's formula*.

(3) The definitions given for certain terms used in navigation, such as *knot*, *middle latitude sailing*, *triangle of plane sailing* are questionable. Although certain text book writers may perhaps be found who would refer to *knots per hour*, the reviewer feels that a knot is a unit of speed, and should never be defined to be "the same as a nautical mile," as is done in this *Dictionary*. In discussing the *triangle of plane sailing* it should be the *departure*, rather than the *difference in longitude*, which is taken as the east-west side of the right triangle. *Middle latitude sailing* does not consist in "sailing on the middle latitude of two places," as the *Dictionary* avers, but in computing the departure along the middle latitude parallel between the meridians of the two places.

(4) *Unit density* is defined to be "the density of some volumetric unit; same as the density of the entire body when the density is constant" . . . rather than "the density of a substance whose mass is numerically equal to its volume."

(5) The tractrix is defined to be: "*the evolute of the catenary; a curve the lengths of whose tangents are equal.*" However, the evolute of the catenary does not have this stated property, and is not called a tractrix, so far as the reviewer is aware.

(6) In several places throughout the *Dictionary*, reference is made to the inverse trigonometric functions, but in most of these places no mention is made of radian measure except in referring to the graphs. However, under the topic: *Value—principal value of an inverse trigonometric function*, we find the statement that "the principal value of the arc-sine is between $-\pi/2$ and $\pi/2$," and then "*e.g.*, the principal value of $\sin^{-1}(1/2)$ is 30°." The reviewer feels that radian measure should be used throughout in the principal definitions of these functions, and that $\sin^{-1}(1/2)=\pi/6$.

Certain matters on which there may be a difference of opinion seem worthy of mention.

(7) Although under *circular*, a *circular cylinder* is defined to be a cylinder whose right section is a circle, a *circular cone* is defined to be a cone whose base is a circle. Granted that the latter definition was used by the ancients, certain modern authors prefer to define a circular cone to be a cone of revolution, and the reviewer feels that this definition should also be mentioned, if only to be consistent with the definition given for the circular cylinder.

(8) The *matrix* seems to receive inadequate treatment as compared with the *determinant*. Multiplication of matrices is a more fundamental concept than multiplication of determinants, and skew-symmetric matrices than skew-symmetric determinants, yet multiplication and skew-symmetric are here defined for determinants and not for matrices. The reviewer feels, even though many authors of texts may disagree, that a square array of elements in rows and columns should be called a *matrix* and not a *determinant*, and that the determinant should be defined to be a certain polynomial formed from the elements of a square matrix, rather than the "symbolic representation" thereof.

(9) The reviewer agrees with the editors of the *Dictionary* that the right-handed coordinate system in space is preferable to the left-handed system, but feels that in defining the trirectangular trihedral it should not be stated that the determinant of the three sets of direction cosines is $+1$ when the trihedral is *right-handed*, but rather when it *has the same orientation as the axes*. The same criticism applies to the definition of the *vector product* of two vectors.

(10) In the definitions of *inequality* and of *greater* and *less*, the expressions *positive number not zero* occur. It would seem that *not zero* is superfluous in referring to positive numbers.

(11) In defining a *power of a number*, we read "*power is sometimes used in the same sense as exponent.*" The reviewer agrees that the word is sometimes used carelessly in this sense, but feels that the *Dictionary* should not sanction this usage, nor use it in the definition, under *laws of integral exponents*, where it says: "*to raise to a power multiply the exponent by the power*".

(12) Although a good table of Denominate Numbers is given in the Appendix, there is no way by which the reader can find in this *Dictionary* the relationship between avoirdupois and troy weights, each of which uses the grain, the ounce, and the pound, but obviously not all with the same meaning in both systems. It would be of interest, perhaps, to point out that a pound of feathers weighs more than a pound of gold, but an ounce of gold weighs more than an ounce of feathers, since avoirdupois weight is used for feathers, but troy weight is used for gold.

Attention might be called to some misprints and errors found in the revised edition. (Numbers refer to pages.)

(22) Bessel functions: the exponent $x(t-1/t)$ should read $x(t-1/t)/2$.

(58) Cubical: the cubic graph $y=kx^3$ should be tangent to the x-axis.

(77) Significant digits: the last digit to the right is not necessarily significant when it is a 0 preceding the decimal point.

(85) Double root: $(x-r)$ should read $(x-r)^2$.

(95) Irrational equation: $\sqrt{x+1}=\sqrt{x+2}$ is a poor illustration, since it has no roots.

(103) Fermat's theorem: read a^{p-1} in the formula $a^{n-1}\equiv 1 \pmod{p}$.

(105) Half-angle formulas: the formula $\tan \frac{1}{2}A=(1-\cos A)/\sin A$ should be included.

(116) Gravitational constant: the letter g is more commonly used to mean the acceleration of gravity than to mean the constant in the equation $F=gm_1m_2/r^2$, as here defined.

(118) Half-angle formulas of spherical trigonometry: in the given formula $\tan \frac{1}{2}\alpha=r/\sin (s-a)$, it would be preferable to use $\tan r$ instead of r, so that r would then be the inradius, as it is in the analogous plane formulas. Similarly, if R were replaced by $\tan R$ in the half-side formulas, the new R would be the circumradius and would have geometric significance.

(119) Simple hexagon: co-linear should read collinear, to be consistent with the spelling elsewhere.

(130) N-scribed: should read inscribed.

(138) Closed interval: Usually denoted by $[a, b]$ rather than (a, b). The latter refers to an open interval.

(142) Kepler's laws: *area* should read *areas*.

(148) Limaçon: read a instead of $2a$ in second line.

(148) Limit: read $0<|x-a|<\delta$ instead of $|x-a|<\delta$.

(157) Mathematical induction: last line of definition should read "true for all positive integral values of n."

(158) Maximum: the slope may change sign without being zero or infinite if the function has a discontinuous derivative.

(158) Maximum, and (166) Multiple: instead of "first derivative which is not zero" or "first derivative of which it is not a root," read "derivative of lowest order which is . . ."

(160) Mean value theorem or the law of the mean: continuity of derivative should be explicitly required.

(168) Napier's analogies: Letters α, β, γ are used here for angles, whereas A, B, C are used in Gauss' formulas (112). Uniformity of notation would be preferable.

(188) Planimeter: Not synonymous with polar planimeter, since there are planimeters of other types.

(199) Probability curve: exponent $-h^2(y-A)^2$ should read $-h^2(x-A)^2$.

(209) Ratio: the six values of the cross ratio of four points are not all distinct when the values are complex cube roots of unity.

(210) Rationalize: the given example $\sqrt{x-1}=x+2$ has only an extraneous root, and is therefore an unfortunate choice.

(213) Reduction to normal form: to be consistent with earlier definitions, use ω in place of θ in the equation of the line.

(216) Repeated: "a dice" should read "a die."

(220) Rule: Descarte's should read Descartes'.

(222) Second: not "one three hundred sixtieth part of a degree." (1/3600).

(225) Bounded sequence: last word should be *terms*, not *term*.

(237) Solution of trigonometric equation: poor definition. The equation $\theta=\cos\theta$ cannot be solved according to the method given.

(237) Species: a right angle is sometimes considered to be a third species in addition to acute and obtuse angles.

(245) Surface integral: the factor sec β is omitted in the displayed integral.

(250) Tangent to a circle: read "a straight line," not "the straight line."

(252) Taylor's theorem: Read $|y-b|$ instead of $|x-b|$ in the fourth line of the page.

(261) Trigonometric curves: the cotangent is omitted by mistake.

(265) Uniform scale: reference to logarithmic paper may be misleading.

(269) Vector components: the word parallelepiped is misspelled.

(273) Zero of a function: the sign $+$ is used for $=$ in two places.

APPENDIX:
(30) 6,085 feet is not a nautical mile.
(31) 69 miles is not exactly one degree of latitude.
Write 1 U. S. gallon = 231 cu. in., since the imperial gallon is different.
Include "equinox year" and "nickel" among definitions.
(43) The three cube roots of unity are often written 1, ω, ω^2.
Read $\bar{z}$ for z.
Read "modulo p" instead of "modulus p."
(44) $\sinh^{-1}x$ is not an angle.

The reviewer would like to see the following words or phrases listed in the *Dictionary* if and when another revised edition is printed:

Average rate, *characteristic equation* (of a matrix), *coset* (of a subgroup) *dummy index*, *equation of time*, *quire*.

The misprints and errors which have been mentioned above are regrettable but they do not alter the fact that the *Mathematics Dictionary* is a useful reference book for terms used in high school or college mathematics. The *Dictionary* does not pretend to serve the needs of the mathematician engaged in graduate work or research. The editors are to be congratulated for having taken the first difficult steps towards a standardization of mathematical terminology.

J. S. FRAME

A First Course in Mathematics for Students of Engineering and the Physical Sciences. By Edward Baker. New York, D. Van Nostrand Company, 1942. 13+295 pages.

This text presents the topics ordinarily studied in a freshman course of "general mathematics" as offered by most colleges. The treatment, however, is not the purely conventional one; an effort is made, by a shift of emphasis, to meet the needs of students intending to specialize in engineering or in the physical sciences.

Simple vector notions in the plane are explained at the start, and are later used, to a limited extent, in deriving formulas of trigonometry and of plane analytic geometry. The importance of the formulas of analytic trigonometry in transforming and simplifying expressions and equations, especially those arising in the sciences, is very neatly and commendably emphasized; but more exercises might well have been provided to illustrate such transformations. The sections dealing with triangulation are merely adequate, and not, as is often the case, overexpanded; quite enough is presented to meet all practical requirements. The analytic geometry of the plane receives somewhat less attention, in certain respects, than is usually accorded to it in over-all survey courses normally taken by freshmen not majoring in mathematics or the sciences. One wonders why space is provided for even a cursory analysis of the general equation of a conic section, rectangular Cartesian coordinates, in a text which does not discuss tangent lines to any plane curve at all—not even in the too brief and sketchy penultimate chapter devoted to the most rudimentary ideas and operations of

the calculus. In the reviewer's opinion, the determining of plane curves which represent geometric loci, and the sketching of such curves from their equations, should not have been relegated to the last of the chapters dealing with plane analytic geometry, and should have been handled much more expansively. The author appears to regard as unimportant, for first-year science students, the eccentricity and the directrices of the ellipse and the hyperbola; and the reviewer is inclined to concur in this judgment.

The problems listed for solution constitute an outstanding and very attractive feature of the book. They are drawn, in large proportion, from various fields of physical and engineering science, and illuminate in a highly stimulating manner the practical significance and scope of even a first course in college mathematics. Numerous problems were evidently selected with the definite object of developing greater dexterity in arithmetic and algebraic manipulations than most of our non-engineering freshmen attain, or are capable of attaining in their abbreviated courses, in these days of "progressive" hop-skip-jump-and-skid elementary and secondary training. Consider, for illustration, the following problem, which is not at all among the most interesting proposed but would certainly have paralyzed most of the freshmen whom this reviewer has entertained in the classroom in recent years: "An empirical formula for the drop in temperature of a gas passing through a flue is

$$ln\ ln\left(\frac{T_1}{t_1}\right) - ln\ ln\left(\frac{T_2}{t_1}\right) = ML.$$

Show that

$$T_1 = t_1\left(\frac{T_2}{t_1}\right)^{e^{ML}}.$$

Calculate T_1 for $t_1 = 1000$ degrees absolute, $T_2 = 800$ degrees absolute, and $ML = 1.5$."

An instructor not versed in engineering lore would probably appreciate amplified information concerning the physical interpretations of all constants and variables occurring in problems of this type. In the problem quoted, he might guess the meanings of the mysterious M and L; but in many of the other problems stated such guessing would be futile.

The typography and the set-up of pages are attractive, and make this text exceptionally readable. A first edition may be expected to show misprints, but the only really unfortunate and confusing one which the reviewer has observed is found in the last equation on page 158. The instructor will undoubtedly have to come to the student's rescue at that spot. He may also be called upon to supplement the argument there presented by explaining an elementary point of number theory which is assumed without comment. In the statement of the Remainder Theorem for polynomials, as given on page 156, the word "numerically" is misleading and should be deleted. It is inadvertently asserted, on page

226, that "any equation of the second degree (in two variables) represents an ellipse, a hyperbola, or a parabola." This assertion, and that made in the last sentence of the same paragraph, will of course receive, in a second edition, the necessary reupholstering and repair.

The omissions noted, and one or two others, are very satisfactorily offset by several unusual inclusions which are of direct interest to the prospective student of science. Moreover, the author's style is fluent, clear, and pleasing. This book may be recommended for use with classes in which smooth and rapid progress is sought—with no "bogging down" anywhere. It is really a good contribution to the current war effort in education. L. S. HILL

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 596. *Proposed by W. C. Rufus, Observatory of the University of Michigan*

A square phalanx of less than five million soldiers was reinforced by ten equal squares and then formed a square of over ten million which could be divided into 400 equal squares. The reinforced army was less than three times the original. How many soldiers were in each of the 400 squares?

E 597. *Proposed by V. Thébault, San Sebastián, Spain*

Let P be any point in the plane of a triangle ABC. Let (U), (V), (W) denote the circles BCP, CAP, ABP, and (U'), (V'), (W') their images by reflection in the respective sides BC, CA, AB. Also let u, u' be the powers of A with respect to (U), (U'), and let v, v', w, w' be defined analogously. Show that the circles (U'), (V'), (W') are concurrent, and that

$$u + u' + v + v' + w + w' = a^2 + b^2 + c^2.$$

E 598. *Proposed by H. S. Wall, Northwestern University*

Let $g_1, g_2, g_3, \cdots$ be any numbers such that

$$0 < g_p < 1, \qquad (1 - g_p)g_{p+1} > \tfrac{1}{4} \qquad (p = 1, 2, 3, \cdots).$$

Prove that

$$\lim_{p\to\infty} g_p = \tfrac{1}{2}.$$

E 599. *Proposed by U. P. Davis, University of Florida*

In how many different ways can $3n$ objects be arranged in a circle, if there are n different kinds, 3 of each kind?

E 600. *Proposed by J. H. Butchart, Grinnell College*

If the radii of the fixed and rolling circles are a and b respectively, the length of one arch of an epicycloid is $8(a+b)b/a$, and the area bounded by one arch and the fixed circle is

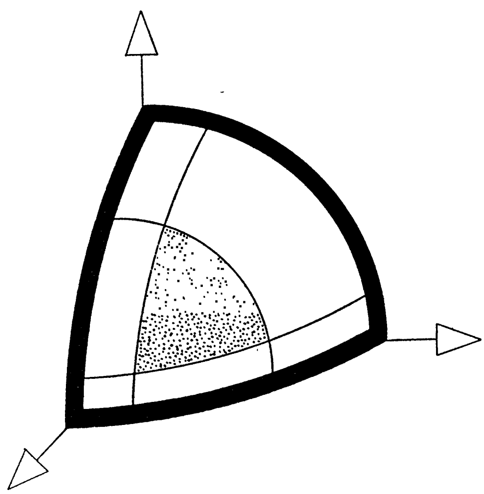
$$\pi(3a^2 + 8ab + 4b^2)b^2/a(a + 2b).$$

Corresponding formulas for the hypocycloid are obtained by changing the sign of b. Prove these results synthetically.

SOLUTIONS

A Triangle of Small-circular Arcs

E 559 [1943, 120]. *Proposed by M. A. Sadowsky, Illinois Institute of Technology*



Let a, b, c be non-negative numbers satisfying $a^2+b^2+c^2<1$. Prove that the three planes $x=a$, $y=b$, $z=c$ will cut out from the first octant of the unit sphere $x^2+y^2+z^2=1$ a curvilinear triangle of area

$$\begin{aligned}\tfrac{1}{2}\pi(1 - a - b - c) &+ a(\arcsin b_a + \arcsin c_a)\\ &+ b(\arcsin c_b + \arcsin a_b)\\ &+ c(\arcsin a_c + \arcsin b_c)\\ - \arcsin b_c c_b &- \arcsin c_a a_c - \arcsin a_b b_a),\end{aligned}$$

where a_b stands for $a/\sqrt{1-b^2}$, and so on.

Solution by E. P. Starke, Rutgers University. The standard method for espressing the area of the given curvilinear triangle as a double integral gives

$$\int_a^{\sqrt{1-b^2-c^2}}\int_b^{\sqrt{1-c^2-x^2}} \frac{dydx}{\sqrt{1-x^2-y^2}}.$$

Performing the first integration, we have

$$A = \int_a^{\sqrt{1-b^2-c^2}}\left(\text{arc cos}\,\frac{c}{\sqrt{1-x^2}} - \text{arc sin}\,\frac{b}{\sqrt{1-x^2}}\right)dx. \tag{1}$$

The final integration is effected by use of the formula

$$\int \text{arc cos}\,\frac{c}{\sqrt{1-x^2}}\,dx = x\,\text{arc cos}\,\frac{c}{\sqrt{1-x^2}} - c\,\text{arc sin}\,\frac{x}{\sqrt{1-c^2}} + \text{arc sin}\,\frac{cx}{\sqrt{1-c^2}\sqrt{1-x^2}}$$

and an analogous formula for the second term of (1). Using these and substituting the limits in (1), we find, after rearranging terms,

$$\begin{aligned} A = {} & \text{arc sin}\,\frac{c\sqrt{1-b^2-c^2}}{\sqrt{b^2+c^2}\sqrt{1-c^2}} + \text{arc sin}\,\frac{b\sqrt{1-b^2-c^2}}{\sqrt{b^2+c^2}\sqrt{1-b^2}} \\ & - c\,\text{arc sin}\,\frac{\sqrt{1-b^2-c^2}}{\sqrt{1-c^2}} - b\,\text{arc sin}\,\frac{\sqrt{1-b^2-c^2}}{\sqrt{1-b^2}} \\ & - a\,\text{arc cos}\,c_a + c\,\text{arc sin}\,a_c + a\,\text{arc sin}\,b_a \\ & + b\,\text{arc sin}\,a_b - \text{arc sin}\,a_c c_a - \text{arc sin}\,a_b b_a. \end{aligned} \tag{2}$$

If the first two terms of (2) are combined as $(\frac{1}{2}\pi - \text{arc sin}\,b_c c_b)$, and the next three terms are written as

$$-c(\tfrac{1}{2}\pi - \text{arc sin}\,b_c) - b(\tfrac{1}{2}\pi - \text{arc sin}\,c_b) - a(\tfrac{1}{2}\pi - \text{arc sin}\,c_a),$$

we have the proposed form for the area A.

Also solved by W. E. Buker and Eugene Sherwood (together).

Parallel Simson Lines

E 561 [1943, 200]. *Proposed by Howard Eves, Syracuse University*

Given two triangles inscribed in the same circle and such that the Simson lines with respect to one triangle of the vertices of the other are concurrent (as in E 535), prove that the Simson lines with respect to the two triangles of a point on the common circumcircle are parallel.

Solution by the Proposer. From the lemma of E 535 [1943, 261] we see that the theorem is true for the point A. The theorem then follows for any other point

on the circumcircle by the first corollary of art. 326 in R. A. Johnson's *Modern Geometry*.

A Perfect Square with Nine Different Digits

E 562 [1943, 200]. *Proposed by V. Thébault, San Sebastián, Spain*

Find a number of the form *ab0cd* whose square contains the nine digits 1, 2, 3, 4, 5, 6, 7, 8, 9.

Solution by M. L. Constable, Philadelphia. Since the square is less than 987654321, the number lies between 10234 and 31098. Since the sum of the digits is 45, the square is divisible by 9, and the number by 3. Therefore, of the 126 possible *combinations* of these nine digits taken four at a time. we select the 42 whose digit-sums are divisible by 3. We form the *permutations* of these combinations, reject those greater than 3198, and insert a zero in the third place of each one remaining. Now we note that, because of the zero in *ab0cd*, the last three digits of its square will be found in the last three digits of the first hundred squares. Therefore a table will show that *cd* cannot be any of

12, 15, 21, 26, 32, 35, 38, 45, 46, 47, 48, 49, 51, 52, 53,
62, 63, 64, 65, 71, 76, 78, 83, 84, 85, 92, 95, 97, 98.

We square the remaining numbers, and find that the squares of

18072, 19023, 23019, 29034

are respectively 326597184, 361874529, 529874361, 842973156.

Also solved by W. E. Buker, C. W. Emmons, Alexander Russ, E. P. Starke, Alan Wayne, and the proposer. Wayne points out that the second and third solutions show an interesting "inversion" in the digits of the numbers and of the squares.

A Hyperbolic Group of Lines

E 563 [1943, 200]. *Proposed by N. A. Court, University of Oklahoma*

Let A', B', C', D' be the antipodes of the circumcenter O of a tetrahedron $ABCD$ on the respective spheres $OBCD$, $OCDA$, $ODAB$, $OABC$. Show that the lines AA', BB', CC', DD' are generators of a quadric. May this quadric be a cone?

Solution by the Proposer. The antipodes A' of O on the sphere $OBCD$ is the point common to the three planes perpendicular to the lines OB, OC, OD at the respective points B, C, D. But these three planes are the tangent planes at B, C, D to the circumsphere of $ABCD$; hence A' is a vertex of the tangential tetrahedron of $ABCD$. Thus the line AA' and the three analogous lines BB', CC', DD' join the vertices of $ABCD$ to the corresponding vertices of its tangential tetrahedron. The proposition now follows from art. 322 of the proposer's *Modern Pure Solid Geometry*.

The four lines will be concurrent if the tetrahedron $ABCD$ is isodynamic (*ibid.*, pp. 280, 281).

A Question of Divisibility

E 564 [1943, 200]. *Proposed by Ivan Niven, Purdue University*

Let a, b, and n be any positive integers such that n divides a^n-b^n. Prove that n divides $(a^n-b^n)/(a-b)$.

Solution by the Proposer. Denote $a-b$ by c. Let p^m be the highest power of a prime p which divides n. If p does not divide c, we can ignore it. If p divides c, we write

$$\frac{a^n - b^n}{c} = \frac{(b+c)^n - b^n}{c} = \sum_{i=1}^{n} \binom{n}{i} b^{n-i} c^{i-1},$$

and show that p^m divides each term of this sum. Since $\binom{n}{i}$ is a multiple of n divided by $i!$, we need prove merely that p divides c^{i-1} to as high a power as it divides $i!$. Now the number of times p divides $i!$ is evidently

$$\sum_{j=1}^{\infty} \left[\frac{i}{p^j}\right] < \sum_{j=1}^{\infty} \frac{i}{p^j} = \frac{i}{p-1} \leqq i.$$

Noting the strict inequality in the first step, we find that our proposition is proved, because p^{i-1} divides c^{i-1}, by hypothesis.

Also solved by E. P. Starke and (incompletely) by R. K. Allen, D. H. Browne, Thor Eriksson, and Joseph Rosenbaum.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4101. *Proposed by R. C. Buck, Harvard University*

Show that

$$D(n) = \begin{vmatrix} (1,1)^\lambda, & (1,2)^\lambda, & \cdots, & (1,n)^\lambda \\ (2,1)^\lambda, & (2,2)^\lambda, & \cdots, & (2,n)^\lambda \\ \vdots & \vdots & & \vdots \\ (n,1)^\lambda, & (n,2)^\lambda, & \cdots, & (n,n)^\lambda \end{vmatrix}$$

$$= (n!)^\lambda \left(1 - \frac{1}{2^\lambda}\right)^{[n/2]} \left(1 - \frac{1}{3^\lambda}\right)^{[n/3]} \left(1 - \frac{1}{5^\lambda}\right)^{[n/5]} \cdots,$$

where (i, j) means the greatest common divisor of the integers i, j.

4102. *Proposed by Hüseyin Demir, Columbia University*

Let O and I be respectively the circumcenter and incenter of a given triangle ABC. Let A_0, B_0, C_0 be points taken respectively on BC, CA, AB so that the sums of the algebraic distances of each point to two other sides are equal to a given length l. Prove synthetically that: (1) The points A_0, B_0, C_0 are collinear; (2) The sum of distances to the sides of ABC of points on $A_0B_0C_0$ is the constant l; (3) the line $A_0B_0C_0$ is perpendicular to the line OI.

4103. *Proposed by V. Thébault, San Sebastián, Spain*

In the system of base $n+1$ the product $P=N\cdot L$ is formed where the number N of $n-1$ digits in descending order is $n(n-2)(n-3)\cdots 321$, and L is less than n and prime to n. If we have $L<n/2$, then the product P has n distinct digits chosen in suitable order from $0, 1, 2, \cdots, n$, and the missing digit is $n-L$. If $L>n/2$, the digit $n-1$ appears twice in P, and the missing digits are n and $(n-1-L)$, the remaining digits being distinct.

Dedicated to E. P. Starke.

SOLUTIONS

Perfect Squares

4048 [1942, 479]. *Proposed by V. Thébault, San Sebastián, Spain*

Find a number of six digits $N=abcdef$ such that the product NN' is a perfect square, where $N'=defabc$.

Solution by Daniel Finkel, Mt. Rainier, Md. Let x and y be three digit numbers such that $(10^3x+y)(10^3y+x)=z^2$ where z is an integer. Solving for x we have

$$2000x = -(10^6+1)y + [(10^6-1)^2y^2 + 4000z^2]^{1/2}. \tag{1}$$

In order to make x rational, set

$$[az + (10^6-1)y]^2 = 4000z^2 + (10^6-1)^2y^2;$$

and we then find that

$$z = \frac{2a(10^6-1)y}{4000-a^2}, \qquad x = \frac{az-2y}{2000}, \qquad 0 < a \leqq 63, \tag{2}$$
$$10^6 - 1 = 27\cdot 37\cdot 7\cdot 11\cdot 13.$$

We now investigate the conditions under which $4000-a^2$, with a an integer, may have for a factor one of 7, 11, 13, 37. Since $4000\equiv 3 \bmod 7$ and 3 is not a quadratic residue mod 7, the number 7 cannot be a factor for any value of a; and similarly 11 cannot be a factor. We then find that 13, 37 will be factors respectively for

$$a \equiv \pm 3 \bmod 13; \qquad a \equiv \pm 2 \bmod 37.$$

Hence the possible values for a are

$$a = 2, 3, 10, 16, 23, 29, 35, 36, 39, 42, 49, 55, 62.$$

Upon trial a solution is found for $a=16$ giving $z=8547y$, $x=547y/8$; and then setting $y=8$ we have the solution

$$N = 547008, \qquad N' = 8547, \qquad z = 68376.$$

Editorial Note The trivial solution $x=y$ is given by $a=2$. For $a=3$ we get $x=692$, $y=307$, $z=461538$, which is the solution given by the proposer without indication of his method of deduction. For $a=10$ we find that $x=641$, $y=25$, $z=128205$.

Identities in Cotangents

4050 [1942, 549]. *Proposed by Arnold Dresden, Swarthmore College*

If $a_1, a_2, \cdots, a_n$ are n distinct complex numbers, $n>1$, such that no two differ by a multiple of π, prove that

$$\sum_{k=1}^{n} \prod_{i=1, i\neq k}^{n} \cot (a_k - a_i) = \sin \frac{n\pi}{2} \cdot$$

Solution by the Proposer. For $n=2$ the formula is true, since $\cot (a_1-a_2) + \cot (a_2-a_1) = 0 = \sin \pi$. Suppose that it is true for $n=2, 3, \cdots, k$; we shall prove that then it is true also for $n=k+1$. For we have then

$$(1) \qquad \sum_{j=1}^{k} \cot (a_1 - a_{k+1}) \prod_{i=1, i\neq j}^{k} \cot (a_j - a_i) = \sin \frac{k\pi}{2} \cot (a_1 - a_{k+1}),$$

$$(2) \qquad \prod_{i=2}^{k+1} \cot (a_1 - a_i) + \sum_{j=2}^{k} \cot (a_j - a_1) \cot (a_1 - a_{k+1}) \prod_{i=2, i\neq j}^{k} \cot (a_j - a_i) = \sin \frac{k\pi}{2} \cot (a_1 - a_{k+1}).$$

Since $\cot P \cot Q = 1 + \cot (P+Q) [\cot P + \cot Q]$, the second term of (2) becomes after setting $P=a_j-a_1$, $Q=a_1-a_{k+1}$,

$$(3) \qquad \sum_{j=2}^{k} \prod_{i=2, i\neq j}^{k} \cot (a_j - a_i) + \sum_{j=2}^{k} \prod_{i=1, i\neq j}^{k+1} \cot (a_j - a_i) + \cot (a_1 - a_{k+1}) \sum_{j=2}^{k} \prod_{i=2, i\neq j}^{k+1} \cot (a_j - a_i).$$

The first term of (3) is equal to $\sin (k-1)\pi/2$, and the third term is equal to

$$\cot (a_1 - a_{k+1}) \left[\sin \frac{k\pi}{2} - \prod_{i=2}^{k} \cot (a_{k+1} - a_i) \right].$$

Hence we have after collection of terms

$$\sum_{j=1}^{k} \prod_{i=1, i\neq j}^{k+1} \cot(a_j - a_i) + \sin\frac{(k-1)\pi}{2}$$
$$+ \cot(a_1 - a_{k+1})\left[-\sin\frac{k\pi}{2} + \sin\frac{k\pi}{2} - \prod_{i=2}^{k} \cot(a_{k+1} - a_i)\right];$$
$$\sum_{j=1}^{k+1} \prod_{i=1, i\neq j}^{k+1} \cot(a_j - a_i) = -\sin\frac{(k-1)\pi}{2} = \sin\frac{(k+1)\pi}{2};$$

and the induction proof is complete.

Identities in Cotangents

4051 [1942, 549]. *Proposed by Arnold Dresden, Swarthmore College*

If $a_1, a_2, \cdots, a_n$ are n distinct complex numbers, $n>1$, such that none of them and none of the differences is a multiple of π, show that

$$\sum_{j=1}^{n} \cot a_j \prod_{i=1, i\neq j}^{n} \cot(a_j - a_i) + (-1)^n \prod_{i=1}^{n} \cot a_i = \sin\frac{(n+1)\pi}{2}.$$

Solution by the Proposer. Since $\cot(a_1-a_2)[\cot a_2 - \cot a_1] = 1 + \cot a_1 \cot a_2$, the formula is true for $n=2$. Suppose that the formula holds for $n=2, 3, \cdots, k$, then we have from the formula in 4050

$$(1) \qquad \sum_{j=1}^{k+1} \cot a_1 \prod_{i=1, i\neq j}^{k+1} \cot(a_j - a_i) = \sin\frac{(k+1)\pi}{2} \cot a_1,$$

$$(2) \qquad \cot a_1 \prod_{i=2}^{k+1} \cot(a_1 - a_i) + \sum_{j=2}^{k+1} \cot a_1 \cot(a_j - a_1) \prod_{i=2, i\neq j}^{k+1} \cot(a_j - a_i) = \sin\frac{(k+1)\pi}{2} \cot a_1.$$

Since $\cot(a_j - a_1)\cot a_1 = 1 + \cot a_j\,[\cot(a_j - a_1) + \cot a_1]$, the second term of (2) becomes

$$(3) \qquad \sum_{j=2}^{k+1} \prod_{i=2, i\neq j}^{k+1} \cot(a_j - a_i) + \sum_{j=2}^{k+1} \cot a_j \prod_{i=1, i\neq j}^{k+1} \cot(a_j - a_i) + \cot a_1 \sum_{j=2}^{k+1} \cot a_j \prod_{i=2, i\neq j}^{k+1} \cot(a_j - a_i).$$

The first term of (3) is equal to $\sin k\pi/2$ from the formula of 4050, and by the induction assumption the third term is equal to

$$\cot a_1\left[\sin\frac{(k+1)\pi}{2} - (-1)^k \prod_{i=2}^{k+1} \cot a_i\right].$$

Hence we have after collection of parts and a cancellation

$$\sum_{j=1}^{k+1} \cot a_j \prod_{i=1, i\neq j}^{k+1} \cot (a_j - a_i) + (-1)^{k+1} \prod_{i=1}^{k+1} \cot a_i = -\sin\frac{k\pi}{2} = \sin\frac{(k+2)\pi}{2},$$

and the induction proof is complete.

Ellipse and Extremal Triangles

4053 [1942, 549]. *Proposed by E. P. Starke, Rutgers University*

Show that all triangles inscribed in an ellipse and having their centroids at the center of the ellipse have the same area, which is the greatest possible area for an inscribed triangle.

Show that all triangles circumscribed about an ellipse and having their centroids at the center of the ellipse have the same area, which is the least possible area for a circumscribed triangle.

Solution by P. C. Hammer, Oregon State College. To solve the two given problems we shall prove a more general theorem.

THEOREM *Let* $(a \cos \theta_i, b \sin \theta_i)$, $i = 1, 2, \cdots, n \geqq 3$, *be the cartesian coordinates of the vertices of a convex polygon inscribed in the ellipse* $x = a \cos \theta$, $y = b \sin \theta$, *and also the coordinates of the points of tangency of a circumscribed polygon, where* $0 < \theta_1 < \theta_2 < \cdots < \theta_n \leqq 2\pi$. *The necessary and sufficient conditions that the inscribed polygon have a maximum and that the circumscribed polygon have a minimum area are*

$$(1) \qquad \theta_2 - \theta_1 = \theta_3 - \theta_2 = \cdots = \theta_n - \theta_{n-1} = 2\pi + \theta_1 - \theta_n = 2\pi/n.$$

Proof. The transformation $x = ax'$, $y = by'$ takes the ellipse into the unit circle, straight into straight lines, and changes all areas by the constant factor $1/ab$. Hence the problem is reduced to one of maximizing the areas of polygons inscribed in the unit circle and of minimizing the areas of circumscribed polygons. The transformed inscribed polygon has the vertices $(\cos \theta_i, \sin \theta_i)$ and its area K_1 is given by

$$(2) \qquad 2K_1 = \sin (\theta_2 - \theta_1) + \cdots + \sin (\theta_n - \theta_{n-1}) + \sin (\theta_1 - \theta_n).$$

Now we may place $\theta_n = 2\pi$ due to the symmetry of the circle, and using partial derivatives, we find readily that conditions (1) are necessary and sufficient to maximize K_1. The area of the corresponding polygon inscribed in the ellipse is

$$\tfrac{1}{2} nab \sin \frac{2\pi}{n} \cdot$$

The area of a polygon circumscribed about the unit circle is

$$K_2 = \cot \phi_1 + \cot \phi_2 + \cdots + \cot \phi_n,$$

where the ϕ_i's are halves of the vertex angles of the polygon. The minimum of K_2 is readily obtained using partial derivatives and leads to $\phi_1 = \phi_2 = \cdots = \phi_n$. The

maximal polygon is thus regular and conditions (1) are fulfilled. The area of the corresponding circumscribed polygon containing the ellipse is

$$nab \tan \frac{\pi}{n} \cdot$$

To apply the theorem to the specific problems we observe that, since all regular polygons have centroids at their centers and as (0, 0) is fixed under the transformation, it is necessary for the extremizing polygons to have centroids at the center of the ellipse. The converse, however, is true only if $n=3$. If the centroid of an inscribed triangle is at (0, 0) then we have the centroid of the transformed triangle is at (0, 0). That such a triangle is equilateral is an immediate consequence of the fact that, if a median of a triangle passes through the circumcenter, the triangle is isosceles. Likewise, if a triangle circumscribed about a circle has its centroid at the center of the circle, it is equilateral due to the fact that, if a median and a vertex angle bisector in a triangle coincide, the triangle is isosceles. From the theorem we then have that in case $n=3$ coincidence of centroid with the center of the ellipse is necessary and sufficient to extremize the areas.

Solved also by J. A. Bullard, H. S. M. Coxeter, H. Eves, H. Scheffé, and the proposer.

Editorial Note. The remaining solvers considered only triangles; Bullard and the proposer made no use of the projection theorem and used partial derivatives in the consideration of the extremal properties, and the proposer referred to the solution of E 191 [1936, 433]. The other solvers used the projection theorem and considered as known the extremal properties for the circle and triangle. The theorems that the regular polygons have respectively the maximum and minimum areas of all convex polygons of the same number of sides inscribed and properly circumscribed about the unit circle is a consequence of the following elementary theorem.

Given the real, continuous, and finite function $f(x)$ of the real variable x which is everywhere concave, or convex, toward the x axis for an interval I for x; then, for an arbitrarily chosen set of n values x_i of I, not all equal, we have

$$\sum_{i=1}^{n} f(x_i) \lesseqgtr nf\left[\sum_{i=1}^{n} x_i/n\right],$$

according as $f(x)$ is always concave or convex toward the x axis.

The proof is obvious. The m points $P_1, P_2, \cdots, P_m$ on the curve $y=f(x)$ are the vertices of a convex polygon P, and the center of mass of the positive integral masses $p_1, p_2, \cdots, p_m$ at these vertices, whose sum is n, lies within this polygon, or the chord if $m=2$. For the polygon of n sides inscribed in the unit circle we need to consider only such as have the center of the circle in their interior. For, if the inscribed polygon $A_1A_2 \cdots A_n$ does not have the center in its interior, there is a vertex, say A_1, such that no vertex lies within say the lower

semicircumference arc for the diameter $A_1'OA_1$. Draw the diameter $A_{n-1}OA'_{n-1}$ and take the point B_n within the arc $A_1''A'_{n-1}$ which continues to A_1. Then, since $n\geqq 3$, arc $A_{n-1}A_n\leqq$arc $A'_{n-1}A_1<$arc B_nA_1, and chord $A_{n-1}A_n<$chord B_nA_1. The polygon $A_1A_2\cdots A_{n-1}B_n$ has an area which exceeds that of the original polygon by the excess of the area of triangle KB_nA_1 over that of the similar triangle $KA_{n-1}A_n$, where K is the intersection of $A_{n-1}B_n$ and A_nA_1. The second polygon has the center O in its interior. Now let x_i be the angle at the center subtended by the ith side of the inscribed polygon of n sides with O in its interior where $0<x_i<\pi$, and the sum of the x_i's is 2π. We have to consider the maximum of the sum of the sines of these n angles. By the above theorem the area of such a polygon if its angles are not all equal is less than the area of the regular polygon. For the properly circumscribed polygon, *i.e.*, containing the center in its interior, we take x_i as half the angle at the center subtended by the ith chord of contact so that $0<x_i<\pi/2$ and the sum of the x_i's is π. The proof follows in a similar manner by considering the sum of the tangents of the angles x_i.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Associate Professor H. F. Archibald of Keuka College, Keuka Park, New York, has been promoted to a professorship.

Professor W. C. Brenke of the University of Nebraska has retired from the chairmanship of the department of mathematics but will continue his other duties during the current academic year.

Assistant Professors C. T. Male and A. H. Fox of Union College have been promoted to associate professorships.

Assistant Professor C. E. Clark of Bradley Polytechnic Institute has been appointed to an associate professorship at Emory University.

Dr. J. B. Coleman has been appointed to a professorship at the University of Richmond.

Dr. L. M. Garrison of Alabama Polytechnic Institute has been appointed to an assistant professorship at Louisiana Polytechnic Institute.

Dr. R. W. Gibson of the University of Illinois has been appointed to an assistant professorship at Kansas State College.

Assistant Professor E. H. C. Hildebrandt of the State Teachers College, Upper Montclair, New Jersey, has been appointed to an assistant professorship at Northwestern University.

Dr. Lois Kiefer of the University of Illinois has been appointed head of the department of mathematics at State Teachers College, Silver City, New Mexico.

Associate Professor F. H. Miller of the Cooper Union School of Engineering has been promoted to a professorship and has been appointed head of the department of mathematics.

Professor M. G. Moore of Tri-State College, Angola, Indiana, has been appointed to an assistant professorship at Bradley Polytechnic Institute.

Drs. Vladimir Morkovin and W. H. Pell of Brown University have accepted positions as research engineers in aerodynamic design at the Bell Aircraft Corporation, Buffalo.

J. D. Novak of the University of Minnesota has been appointed head of the department of mathematics and physics at MacMurray College, Jacksonville, Illinois.

Dr. D. H. Rock of Brown University has been appointed assistant professor of mathematics at Rhode Island State College.

Dr. Robert Schatten has been appointed to an assistant professorship at the University of Vermont.

Dr. G. E. Schweigert of the University of Missouri has been appointed to an assistant professorship at Purdue University.

Dr. C. J. Thorne of Washington University has been appointed to an assistant professorship at Louisiana State University.

The following appointments to instructorships are announced:
Bard College: Dr. H. B. Mann
Brown University: P. T. Miekle, W. E. Barnes
Cooper Union School of Engineering: S. G. Roth
Indiana University: Dr. Marion D. Wetzel
Purdue University: Dr. Fred Kiokemeister
University of Minnesota: Dr. Louis Garfin, Dr. C. M. Jensen
University of Pennsylvania: R. T. Luginbuhl
University of Rochester: Dr. H. S. Kieval
University of Washington: Dr. J. M. Kingston

Professor Edward Helly of Illinois Institute of Technology died on November 28, 1943.

Mr. L. L. Locke, who retired in 1942, died August 28, 1943. He was a charter member of the Mathematical Association.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

THE NAVY COLLEGE TRAINING PROGRAM

The Navy College Training Program (V-12) was inaugurated upon July 1, 1943, in nearly 200 colleges and universities. This new program superseded the previous V-1 and V-7. Its purpose, as outlined by the Navy Department, is to produce Naval officers chosen from high school seniors, high school graduates, and college students who appear to have the potentialities for ultimate selection as officers after carefully prescribed college training. The plan provides that this college training shall be carried on while the men are on active duty, in uniform, receiving pay, and under general military discipline.

The selection of the educational institutions to participate in both the Army and the Navy Programs is made by a joint committee consisting of representatives of the Armed Forces and the War Manpower Commission. In the event of failure on the part of the members of the committee to agree, the final decision is made by the chairman of the War Manpower Commission. Institutions are chosen not only on the basis of educational facilities, but also after a study of available housing and messing accommodations. To great extent, institutions accepted have been designated as Navy institutions or as Army institutions; that is, only a few institutions have been awarded both Army and Navy programs.

The Navy has insisted that all colleges and universities holding Navy contracts offer three 16-week terms in each calendar year, the first term to begin on or about July 1, 1943, the next term to start approximately November 1, the next one about March 1, and so on. Such a uniform calendar is regarded as essential in view of the fact that the transfer of students from one college to another will frequently be necessary.

Of the nearly 70,000 students assigned to various institutions on July 1, approximately 80 per cent were students already in college who were previously enlisted in class V-1 or V-7 or who held probationary commissions in the U. S. Naval Reserve or were enlisted in the Marine Corps or Coast Guard Reserves. Consequently, colleges holding Navy contracts received transfers at all levels, from second-term freshmen to second-term seniors. None of these transfer students were required to enter the new, fully prescribed V-12 curriculum, but certain minimum requirements, including mathematics, were specified, and, within the limits of a student's major interest, it was hoped that he would wish to include as many of the V-12 subjects as possible.

Freshman students entering the rigid V-12 Program were drawn from two sources. More than 300,000 young men took the first Army-Navy test given last

April 2; many who successfully passed this test and who designated the Navy as their preference started their V-12 studies upon July 1; other students of this group have been assigned to institutions for other semesters. The second test, open to male high school seniors in their last semester and to graduates, who will reach their seventeenth birthday but not their twenty-second birthday by March 1, 1944, was given upon November 9, 1943. Ultimately, most men entering the V-12 Program will come from those who have successfully passed the Army-Navy test. In addition to this group, a limited number of Navy men who have seen duty with the fleet have also been assigned to the V-12 Program; these men are high school graduates, and were chosen upon the basis of their past record with the Navy. Among this latter group are many students who have a deficient background in mathematics and science, and a large number of institutions have found it necessary to set up special courses to accommodate them. It is probable that more attention will be paid in the future to the academic qualifications of this group of assignees.

The Navy has specified that each institution will give examinations in the various V-12 courses according to its own practices. It should be noted, however, that the Navy Department will give a qualifying examination of its own to all first year college students near the end of their second term. The usual scholastic standards of the institution are to prevail, and classes are to be of normal size. It is expected that standard textbooks will be used and that the instructor will select the text, provided the one designated is generally recognized as standard. Each institution will determine whether or not credit toward a degree shall be given for the completion of courses in the various curricula. Inasmuch as the content of these courses is practically equivalent to that of standard college courses in the same subjects, it is hoped that credit will be given quite generally.

The several types of officer candidates in the V-12 Program will complete training as follows: Aviation Candidates, 2 terms; Deck Candidates, and Supply Corps Candidates, 4 terms; Pre-Medical and Pre-Dental Corps Candidates, 5 terms; Engineer Candidates, 6 terms; Engineer Specialist Candidates, including Physics Majors, Civil Engineer Corps Candidates, Construction Corps Candidates, Pre-Chaplain Corps Candidates, Aerology Specialist Candidates, 8 terms. All curricula contain two terms of elementary mathematical analysis, and all technical programs also involve two terms of calculus. Descriptions of the various courses in mathematics follow.

Mathematical Analysis I. 5 lecture-recitation periods per week.

This course is designed for the student with a limited high school background in mathematics. Within the limits of this background, the following areas will be considered. Elementary College Algebra: fundamental concepts, variable, constant, function; review of axioms, elementary operations; factoring; fractions; formulas; the graph; linear equations; simultaneous linear equations in two unknowns; quadratic equations; exponents and radicals; variation. Trigonometry: angles and their measures, trigonometric functions, linear interpola-

tion and use of tables; right triangles; fundamental identities; logarithms, including introduction to use of slide rule; functions of multiple angles, addition formulas; identities; inverse trigonometric functions; trigonometric equations; laws of sines, cosines, and tangents; oblique triangles.

Mathematical Analysis II. 5 lecture-recitation periods per week. Prerequisite: Mathematical Analysis I.

A continuation of Mathematical Analysis I. The following areas will be considered. Trigonometry: introduction to spherical trigonometry. Analytic Geometry: points in rectangular and polar coordinate systems; analytic equivalents of distance, slope, *etc.*; loci; straight line; circle; conic sections; polar and parametric equations; introduction to solid analytic geometry, including cylindrical and spherical coordinate systems. College Algebra: determinants and solution of systems of equations; simultaneous quadratics (with graphical solution); theory of equations; complex numbers.

Mathematical Analysis III. 5 lecture-recitation periods per week. Prerequisite: $2\frac{1}{2}$ or more units of high school algebra.

College Algebra: fundamental concepts, laws, operations; review of factoring, fractions, linear equations, the graph, quadratic equations and simultaneous quadratic equations, binomial theorem; variation; progressions; determinants and systems of linear equations; exponents, radicals; logarithms. Trigonometry: angles and their measures; the trigonometric functions; significant figures and approximate computation; linear interpolations and use of tables; right triangles; identities; functions of multiple angles and addition formulas; inverse functions; trigonometric equations; oblique triangles; right and oblique triangles in spherical trigonometry.

Mathematical Analysis IV. 5 lecture-recitation periods per week. Prerequisite: Mathematical Analysis III.

A continuation of Mathematical Analysis III. Analytic Geometry: points in rectangular and polar coordinate systems; distance, slope, angle between lines; loci; straight line; circle; conic sections; polar and parametric equations; tangents and normals; curve tracing in various systems; translation and rotation; empirical determinations (curve fitting); direction cosines and numbers; the plane and line; quadric surfaces and sections; cylindrical and spherical coordinates. College Algebra: permutations, combinations, and probability; theory of equations; complex numbers and DeMoivre's Theorem.

Calculus I. 4 lecture-recitation periods per week. Prerequisite: first year mathematics.

Functions: limits and limit theorems (without proof); the derivative and its interpretations; derivatives of algebraic functions; maxima and minima; rates; derivatives of transcendental functions; applications including Newton's method of approximation, and tangents and normals; derivatives of higher order; the differential with applications; definite integral with applications such as length, area, surfaces, and volume, moments, centroids, moments of inertia; improper integrals.

An early introduction of the integral calculus into Calculus I is desired so that Analytical Mechanics I may be taught from that background.

Calculus II. 4 lecture-recitation periods per week. For Engineer Specialist Candidates *c* and *d* (Electrical Engineering), Calculus II will be given in 6 lecture-recitation periods per week for 8 weeks, and will be immediately followed by Calculus III meeting a like number of times per week for the last 8 weeks of the term.

Applications of the definite integral such as work and attraction; curvature; curve tracing; indeterminate forms; series of constant terms; power series with Taylor's and Maclaurin's theorems with remainder term, and with applications in integration; partial differentiation with applications; multiple integrals with applications.

In the case where the Calculus is divided into 5-period and 3-period courses, some of the material in Calculus II will be advanced to Calculus I.

Calculus III—Differential Equations. 6 lecture-recitation periods per week. Prerequisite: Calculus II.

To be given in the second term as a continuation of Calculus II. Differential Equations will then be given 6 hours per week for 8 weeks, and will cover ordinary equations of the first order and simple ordinary equations of the second order; singular solutions; linear equations with constant coefficients and applications to the physical sciences, with practice in setting up and solving the differential equations; approximate methods; systems of differential equations; an introduction to partial differential equations.

Navigation and Nautical Astronomy I. 3 lecture-recitation periods per week. Prerequisite: first year mathematics.

Brief review of plane and spherical trigonometry and use of logarithms; instruments; compass corrections; terrestrial factors; the sailings; dead reckoning and transverse sailings; piloting; seamanship; weather.

Navigation and Nautical Astronomy II. 3 lecture-recitation periods per week. Prerequisite: Navigation and Nautical Astronomy I.

Celestial factors; the astronomical triangle; altitude; time; azimuth and amplitude; interval to L.A.N.; latitude; longitude and time sights; identification of celestial bodies; position of single sight; Sumner line; Marc St. Hilaire; lines of position; day's work; use of H.O. 203, 208, 211, 214, Ageton (1942).

The Navy has emphasized that institutions should expand and amplify the above outlines within the scope of their facilities and the time limits imposed. Captain S. P. Fullinwider, Jr., Head of the Department of Mathematics at the United States Naval Academy, has suggested the following additional topics which might be treated. In Mathematical Analysis I and II, include the binomial theorem and the rudiments of probability; also, emphasize the solution of numerical equations by graphical and numerical methods. Where Spherical Trigonometry is introduced, include Napier's rules of circular parts and the law of cosines. In the Calculus, it would be profitable to devote two or three lessons

to Simpson's rule and its applications. In Differential Equations, considerable time could be spent upon integration in series and other approximate methods of solving differential equations.

Since the inauguration of the V-12 Program, several bulletins have been issued by the Bureau of Naval Personnel pertaining to the work in mathematics. Excerpts from these bulletins follow.

From Bulletin No. 48. "Navigation and Nautical Astronomy I and II are to be basic courses and are not designed to give any technical skill in the science of navigation. Practical navigation training is to be reserved for Naval Reserve Officer Training Corps and Midshipmen Schools.

"The purpose of Navigation I is to give the basic mathematics with emphasis on the elements of spherical trigonometry and elementary vector mathematics necessary for the student to comprehend the navigation courses he will be given later under Navy instruction. Problems may be drawn from nautical situations to keep the work alive but the emphasis shall be upon mathematical principles only. Any elementary text in Spherical Trigonometry would be an adequate guide to what is desired.

"The purpose of Navigation II is to give the student the astronomical background necessary to an understanding of the principles of celestial navigation and inherent in the practice of celestial navigation. In essence this course shall be a course in elementary nautical astronomy. Any elementary text in nautical astronomy would be an adequate guide to what is desired.

"Since neither Navigation and Nautical Astronomy I or II are courses in practical navigation, equipment as charts, navigational instruments, and other devices for the study of practical navigation will not be necessary."

From Bulletin No. 66. "It has been recognized from the beginning that there would be considerable differences in the amount and quality of mathematical preparation among students entering the V-12 Program from high school. For that reason, two separate courses in mathematics were prescribed as alternatives for the first two college terms, Mathematics I and II for those students with two or less units of high school mathematics and Mathematics III and IV for those with $2\frac{1}{2}$ or more units of high school mathematics. It was contemplated that college and university faculties would put into Mathematics I those students not qualified to attempt Mathematics III. This interpretation is hereby established as Navy policy, ... Colleges and universities are authorized to transfer V-12 freshmen enrolled in Mathematics III to Mathematics I whenever it is evident that such freshmen have inadequate preparation for Mathematics III.

"It appears that some students have entered the V-12 Program whose preparation in mathematics is inadequate even for Mathematics I. Colleges and universities are hereby authorized to establish the necessary refresher or makeup classes for such students, and to make necessary readjustments in the content and hours of subsequent classes in mathematics during the first two terms. It is required, however, that such students complete the substantial equivalent of Mathematics II by the end of their second term in order that they may be pre-

pared for the qualifying examination on the work of the first two terms which all V-12 freshmen will be required to take at the end of the second term.

"The college or university shall be sole judge as to whether or not academic credit shall be given for any refresher classes or any other special classes set up under paragraphs . . . above. Such classes, whether or not given for academic credit, may be counted in the number of hours for academic work prescribed for V-12 students.

"It is not contemplated that any V-12 student shall be permitted to remain in college more than the prescribed number of terms because of deficiency in high school preparation. If he cannot, with necessary refresher classes and other adjustments in the curriculum . . . make up his deficiencies and complete all prescribed courses by the end of his second term in college, he will be subject to separation from the program."

From Bulletin No. 67. In the case of former V-1 and V-7 students, "it is not required that any student attempt to complete in one term both halves of a mathematics or physics course which normally runs for two terms, when successful completion of the first half is a prerequisite to an understanding of the work of the second."

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE TWENTIETH ANNUAL MEETING OF THE NEBRASKA SECTION

The twentieth annual meeting of the Nebraska Section of the Mathematical Association of America was held at the University of Nebraska in Lincoln, Nebraska, on Saturday, May 1, 1943. Professor W. C. Brenke presided in the absence of the chairman, Professor M. A. Basoco.

There were twenty people in attendance, including the following thirteen members of the Association: A. K. Bettinger, W. C. Brenke, C. C. Camp, A. R. Congdon, H. M. Cox, W. A. Dwyer, J. M. Earl, M. G. Gaba, F. S. Harper, Ralph Hull, H. L. Rice, Lulu L. Runge, and Charles Saltzer.

The following officers were elected for the coming year: Chairman, W. A. Dwyer, Creighton University; Secretary, Lulu L. Runge, University of Nebraska; Member of Executive Committee, J. M. Earl, University of Omaha.

The following papers were presented:

1. *P-adic numbers and their arithmetics*, by Dr. Albert Neuhaus, University of Nebraska, introduced by Professor W. C. Brenke.

This paper dealt with the non-archimedian valuations of the rational numbers, the derived p-adic number fields, and the fundamental operations in these fields.

2. *Mileposts of instructional research*, by Professor H. M. Cox, University of Nebraska.

The speaker discussed the mileposts of instructional research as selected from the point of view of the classroom teacher. He advanced the opinion that the progress of instructional research represents expanding areas of consciousness of responsibility. It was also remarked that a course can be described in terms of: (1) identification of subject matter (2) measurement of student achievement; (3) evaluation of student achievement in the light of parallel or overlapping courses and in the light of all-college points of view; (4) evaluation and re-evaluation of subject-matter.

3. *The present norm of the University of Nebraska grading system*, by Professor C. C. Camp, University of Nebraska.

In this investigation Professor Camp used records of grades from 1938 to 1942, and found large fluctuations from year to year, and at different course levels. Only when he considered the percentage frequencies in regrouped data for the entire university did he find what could be called a norm. Grades in the nineties, eighties, seventies, sixties, and those below sixty exhibited percentage frequencies of 11, 41, 30, 10, and 8 respectively. The distribution showed a skewness of -0.7, and was fitted to a Pearson type III and a Charlier type A curve.

4. *The Gaussian imagery of spherical objects*, by Professor O. C. Collins, University of Nebraska, introduced by Professor W. C. Brenke.

Professor Collins discussed the screen representation of a spherical surface as projected by a system of thin lenses. He remarked that every type of perspective projection is possible by this means. A simplified construction was used for the consideration of space images of solid objects.

5. *Certain applications of Postel's projection*, by Professor O. C. Collins, University of Nebraska, introduced by Professor W. C. Brenke.

The speaker discussed the uses of a certain flat network representing the parallels and meridians of a sphere. The network was obtained by projecting the surface of the sphere azimuthally from the zenith of a point on the equator. It was pointed out that the device could be employed for the construction of a universal planisphere, as a template for the construction of world maps centered on any locality, and for the representation and solution of spherical triangles.

6. *Some remarks on the theorem of Blasius in theoretical fluid mechanics*, by Professor M. A. Basoco, University of Nebraska. (Read by title.)

The content of this paper consisted of a simple derivation of a theorem of Blasius (Zeits. f. Math. u. Phys., 1909, 10) which affords an expression for the lift and moment experienced by a cylinder placed in a moving ideal fluid. The lift and moment are expressed as contour integrals involving the complex potential describing the motion.

7. *On the flow about an elliptic cylinder*, by Charles Saltzer, University of Nebraska.

The speaker considered the application of conformal mapping to hydrodynamics in order to obtain the complex potential of the non-circulatory flow about circular and elliptic cylinders, and in order to compute the lift and moment on the cylinders as determined by Blasius' theorem

8. *Arithmetics of hypercomplex number systems*, by Professor Ralph Hull, University of British Columbia.

LULU L. RUNGE, *Secretary*

THE ELEVENTH ANNUAL MEETING OF THE WISCONSIN SECTION

The eleventh annual meeting of the Wisconsin Section of the Mathematical Association of America was held at the University of Wisconsin in Milwaukee on Saturday, May 15, 1943. Sessions were held in the morning and in the afternoon, with the chairman of the Section, Professor R. H. Bardell, presiding. An address of welcome was delivered by Dean Frank O. Holt of the Extension Division of the University of Wisconsin. Members and their guests assembled for lunch at the Hotel Schroeder.

There were forty-eight in attendance, including the following seventeen members of the Association: R. H. Bardell, Leon Battig, Ethelwynn R. Beckwith, May M. Beenken, F. A. Butter, Jr., L. A. V. DeCleene, Fannie Hopkins, R. C. Huffer, M. L. Jautz, R. E. Langer, Mary Felice, R. E. Norris, Irene Price, G. A. Sedlak, P. L. Trump, J. I. Vass, Louise A. Wolf.

The following officers were elected for the coming year: Chairman, May M. Beenken, State Teachers College, Oshkosh; Secretary-Treasurer, P. L. Trump, University of Wisconsin; Program Committee, Leon Battig, University of Wisconsin Extension Division, Sheboygan, Louise A. Wolf, University of Wisconsin in Milwaukee. It was decided to hold the next meeting at Milwaukee-Downer College in Milwaukee in May, 1944. The choice of the exact date of the meeting was left to the host institution.

The following papers were presented:

1. *Geometrical location of the roots of certain composite equations*, by Professor Morris Marden, University of Wisconsin in Milwaukee.

Professor Marden discussed the location of the roots of the equation

$$C(z) = a_0B(0) + a_1B(1)z + \cdots + a_mB(m)z^m = 0$$

relative to the location of the roots of the equations

$$A(z) = a_0 + a_1z + \cdots + a_mz^m = 0$$

and

$$B(z) = b_0 + b_1z + \cdots + b_nz^n = 0.$$

The roots of the equation $A(z)=0$ were assumed to lie either in a ring shaped region, or in a sector bounded by two rays intersecting at an angle not greater than π radians. The roots of $B(z)=0$ were assumed to lie either between two

circles consisting of points the ratio of whose distances from the points $z=0$ and $z=m$ was constant, or between two circles consisting of points at which the segment from $z=0$ to $z=m$ subtended a constant angle. The roots of $C(z)=0$ were then found to lie in a region obtained by expanding, in a suitable manner, the region containing the roots of $A(z)=0$.

2. *The characteristics and applications of photo-cells*, by Walther Richter, American Institute of Electrical Engineers, introduced by Professor R. H. Bardell.

3. *Mathematics training for the armed forces now in progress at the University of Wisconsin*, by Professor R. E. Langer, University of Wisconsin.

4. *Mathematics in the signal corps*, by Haym Kruglak, Milwaukee Vocational School, introduced by Professor H. P. Evans.

5. *Applications of mathematics to industry*, by J. A. Deubel, Perfex Corporation, introduced by Professor Ethelwynn R. Beckwith.

P. L. TRUMP, *Secretary*

THE SECOND ANNUAL MEETING OF THE METROPOLITAN NEW YORK SECTION

The second annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at Brooklyn College, Brooklyn, New York, on Saturday, May 8, 1943. Professor F. H. Miller presided at the morning session. At the afternoon session, Professor H. F. Mac Neish, Chairman of the Section, acted as general chairman, and Dr. Edna E. Kramer-Lassar, Vice-Chairman of the Section, acted as program chairman.

The attendance was about one hundred and twenty-three, including the following fifty-eight members of the Association: R. G. Archibald, L. A. Aroian, A. V. Baez, Brother Bernard Alfred (Welch), Frank Boehm, C. B. Boyer, A. B. Brown, Jewell Hughes Bushey, J. H. Bushey, Louise M. Comer, T. F. Cope, W. H. H. Cowles, Jesse Douglas, W. H. Fagerstrom, J. M. Feld, Edward Fleisher, R. M. Foster, Etta Greenberg, Harriet M. Griffin, C. C. Grove, R. A. Harrison, Solomon Hurwitz, R. A. Johnson, Sidney Kaplan, Herman Karnow, E. H. Koch, Jr., Edna E. Kramer-Lassar, Nathan Lazar, C. H. Lehmann, Herman Levy, C. C. MacDuffee, H. F. Mac Neish, P. H. Mc Grath, May Hickey Maria, A. E. Meder, Jr., Joseph Milkman, F. H. Miller, E. C. Molina, L. T. Moore, M. A. Nordgaard, Max Peters, Mina S. Rees, Moses Richardson, S. G. Roth, Arthur Sard, S. A. Schelkunoff, Edna C. Schnefel, James Singer, E. R. Stabler, J. E. Thompson, H. E. Wahlert, Israel Wallach, Alan Wayne, John Williamson, H. P. Wirth, Jack Wolfe, Margaret Y. Woodbridge, R. C. Yates.

At the beginning of the afternoon session President Harry D. Gideonse of Brooklyn College welcomed the Section to Brooklyn College. At the close of the afternoon session the following officers were elected for the coming year: Chair-

man, C. C. MacDuffee, Hunter College; Vice-Chairman, Max Peters, New Utrecht High School; Secretary, H. E. Wahlert, New York University; Treasurer, F. H. Miller, Cooper Union

The following papers were presented:

1. *Mathematics in navigation*, by Lieutenant Commander Delwyn Hyatt, USN, U. S. Merchant Marine Academy, introduced by Professor Mac Neish.

It was pointed out in this address that problems in navigation involve a considerable amount of mathematics up to and including spherical trigonometry, but that the actual practice of navigation has been reduced to the use of tables so that the navigator needs only the ability to copy numbers correctly from the tables, and to perform additions and subtractions. It was stated that the armed forces have learned that the average youth inducted in recent years has not been able to perform the simple mathematical operations required in practice.

2. *Some applications of mathematical statistics in the war effort*, by Professor Harold Hotelling, Columbia University, introduced by Dr. Kramer-Lassar.

The speaker described a number of war time applications of mathematical statistics. Among the applications mentioned were the following: determination of probabilities of hits and associated probabilities, quality control of manufactured articles by sampling inspection, cryptography, personnel placement and psychological researches, meteorology, medical and agricultural research, statistical designs in physical, chemical, and engineering research, and economic and administrative statistics. It was predicted that those who make fundamental studies of mathematical statistics with a view of its immediate usefulness will be able to utilize their knowledge after the war.

3. *Secret communications*, by Major D. D. Millikin, New York University, introduced by Mr. Wahlert.

Major Millikin gave a loosely chronological discussion of codes and ciphers, chosen for their human interest or humor, but which illustrated the devices most frequently employed for secret communications. Included in the presentation were many examples of historic interest, with special emphasis on the methods used during the period from the Revolutionary War through World War I. Applications of cryptography in the fields of literature, recreation, business, and sports were also described.

4. *Pure mathematics as a war course*, by Professor F. J. Murray, Columbia University, introduced by Professor Jewell Hughes Bushey.

The speaker called attention to the fact that our civilization is based upon techniques which are fundamentally mathematical. It was remarked that the content and sequential arrangement of the standard mathematics courses has been developed through many centuries for the purpose of solving technical problems, and in order to obtain a precise description and understanding of natural phenomena. Emphasis was placed upon the thesis that to cut or weaken the standard mathematics curriculum would be a very harmful procedure.

5. *Should the present practical trend in secondary mathematics be extended to the college mathematics curriculum?* by Dr. Nathan Lazar, Midwood High School.

6. *Should the related mathematics course be the required mathematics course for all ninth year pupils?* by Max Peters, New Utrecht High School.

In February, 1943, a special one-year course in related mathematics was introduced in the New York City high schools to give non-academic students the essential mathematics needed for success in pre-induction and post-induction training. The course included the following topics: a review of arithmetic with emphasis on applied problems; the simple properties of plane and solid geometric forms; the use of instruments such as the protractor, micrometer, vernier, and the slide rule; a unit of algebra including the equation, the formula, ratio and proportion, variation, and graphs; indirect measurement, including scale drawing, the Pythagorean theorem, and trigonometry of the right triangle; a unit on vectors. The speaker expressed the opinion that such a course with its emphasis on applications gives the student a much richer insight into the role that mathematics plays in our civilization, and permits greater flexibility in adapting the ninth year students, both academic and non-academic.

H. E. WAHLERT, *Secretary*

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., April, 1944
ILLINOIS
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, New York, April 22, 1944
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, Jan. 29, 1944
OHIO, Columbus, April 6, 1944
OKLAHOMA
PHILADELPHIA, Philadelphia, November, 1944
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, MARCH 11, 1944
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1944

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN